University of California Davis **Name (Print):** ________________
Euclidean Geometry MAT 141 **Student ID (Print):** ________________

Solutions to Sample Final Examination II

March 17 2020

Time Limit: 120 Minutes

This examination document contains 7 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

Do not write in the table to the right.

1. (20 points) (**Isometries of Euclidean Plane** $\mathbb{R}^2$). Let $L = \{(x, y) \in \mathbb{R}^2 : x = y\} \subseteq \mathbb{R}^2$ be a line, and $f = t_{(1,1)} \circ \overline{r}_L$ a glide reflection along L.

   (a) (5 points) Show that $f(P) \neq P$ for any $P \in \mathbb{R}^2$, i.e. f has no fixed points.

   In coordinates, the map f is given by

   $$f(x, y) = t_{1,1}(\overline{r}_L(x, y)) = t_{1,1}(y, x) = (y + 1, x + 1).$$

   If a point $(x, y) \in \mathbb{R}^2$ is fixed by f, then

   $$(x, y) = f(x, y) = (y + 1, x + 1),$$

   so $x = y + 1$ and $y = x + 1$. Substituting the second relation into the first, we have $x = x + 2$, which has no solution. Therefore, f has no fixed points.

   (b) (10 points) Let $M = \{(x, y) \in \mathbb{R}^2 : y = 0\} \subseteq \mathbb{R}^2$ be the horizontal line. Show that the isometry $f \circ \overline{r}_M$ is a rotation and find its center.

   We can write the translation $t_{(1,1)}$ as the composition $\overline{r}_{N_2} \circ \overline{r}_{N_1}$, where N_1 and N_2 are the lines

   $$N_1 = \{(x, y) \in \mathbb{R}^2 : y = -x\} \qquad \text{and} \qquad N_2 = \{(x, y) \in \mathbb{R}^2 : y = -x + 1\}.$$

   Our isometry is then

   $$f \circ \overline{r}_M = \overline{r}_{N_2} \circ \overline{r}_{N_1} \circ \overline{r}_L \circ \overline{r}_M.$$

   The lines N_1 and L meet at $(0, 0)$, so rotate these lines together by $-\pi/4$. The line L then rotates to M, and N_1 rotates to the y-axis $N_1' := \{x = 0\}$. This gives us

   $$f \circ \overline{r}_M = \overline{r}_{N_2} \circ \overline{r}_{N_1'} \circ \overline{r}_M \circ \overline{r}_M = \overline{r}_{N_2} \circ \overline{r}_{N_1'},$$

   which is a rotation because it is the composition of two reflections along non-parallel lines. The center of this rotation is the common point of N_2 and N_1', which is $(0, 1)$.

   (c) (5 points) Is it true that $f \circ \overline{r}_M = \overline{r}_M \circ f$?

   No. We argued in (b) that $(0, 1)$ is the center of the rotation $f \circ \overline{r}_M$. Indeed,

   $$f \circ \overline{r}_M(0, 1) = f(0, -1) = (0, 1).$$

   On the other hand,

   $$\overline{r}_M \circ f(0, 1) = \overline{r}_M(2, 1) = (2, -1).$$

   Since $f \circ \overline{r}_M$ and $\overline{r}_M \circ f$ disagree on at least this point, we conclude that they are different maps.

2. (20 points) (**Isometries and Triangles in the 2-sphere S^2**) Consider the 2-sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^2 : x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3,$$

and the three lines $L_1 = \{(x, y, z) \in \mathbb{R}^3 : y = 0\} \cap S^2$, $L_2 = \{(x, y, z) \in \mathbb{R}^3 : x = y\} \cap S^2$ and $L_3 = \{(x, y, z) \in \mathbb{R}^3 : z = 0\} \cap S^2$.

(a) (5 points) Let $T \subseteq S^2$ be the triangle given by L_1, L_2, L_3 with vertices

$$(1, 0, 0), \frac{1}{\sqrt{2}}(1, 1, 0), (0, 0, 1).$$

Show that the angle α of T at $N = (0, 0, 1)$ is $\pi/4$.

Label the points $P = (1, 0, 0)$, $Q = \frac{1}{\sqrt{2}}(1, 1, 0)$ and $N = (0, 0, 1)$. The side NP of T is part of line L_1, and the side NQ is part of the line L_2. Finding the angle between the relevant planes Π_{L_1} and Π_{L_2} that cut out these lines would only tell us the angle α up to supplement. That is, the result could be α or $\pi - \alpha$, so we have to do better than this.

To do so, we'll pick out the exact tangent vectors $\mathbf{v}_P$ and $\mathbf{v}_Q$ which are based at N and point in the directions of P and Q respectively. By drawing the picture, you can see that $\mathbf{v}_P = \langle 1, 0, 0\rangle$ and $\mathbf{v}_Q = \frac{1}{\sqrt{2}}\langle 1, 1, 0\rangle$, essentially the points P and Q themselves. Since these are unit vectors, they make the angle

$$\alpha = \arccos\big(\mathbf{v}_P \cdot \mathbf{v}_Q\big) = \arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4},$$

as desired.

(b) (5 points) Compute the area of the triangle T.

Repeat the calculations from (a) for the angles at P and Q too, or just draw a picture. The angles found are both $\pi/2$, so

$$\text{Area}(T) = \left(\frac{\pi}{4} + \frac{\pi}{2} + \frac{\pi}{2}\right) - \pi = \frac{\pi}{4}.$$

This triangle cuts out one eighth of the upper hemisphere, which itself has area 2π.

(c) (10 points) Find the area of the image $\pi_N(T)$ of $T \subseteq S^2$ under the stereographic projection $\pi_N : S^2 \setminus \{(0, 0, 1)\} \longrightarrow \mathbb{R}^2$.

The image $\pi_N(T)$ has infinite area. Notice that $\pi_1(L_3) = L_3$ is the unit circle, so one line segment stays put under stereographic projection. But L_1 and L_2 both contain the North pole N, so they map under π_N to distinct lines in $\mathbb{R}^2$ which pass to the origin. Specifically, $\pi_N(L_1)$ is the x-axis, and $\pi_N(L_2)$ is the line $x = y$. The image $\pi_N(T)$ of our triangle T is the unbounded region of the lower part of the first quadrant in $\mathbb{R}^2$ cut out by $\pi_N(L_1$, $\pi_N(L_2$, and $\pi_N(L_3$. This certainly has infinite area.

3. (20 points) (**Distances and Lines in the 2-sphere** S^2) Consider the 2-sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^2 : x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3.$$

(a) (5 points) Compute the spherical distance between

$$P = (1, 0, 0) \in S^2 \text{ and } Q = \frac{1}{\sqrt{3}}(1, 1, 1) \in S^2.$$

Using the formula $d_{S^2}(P_1, P_2) = 2\arcsin\big(\frac{1}{2}d_{\mathbb{R}^3}(P_1, P_2)\big)$, we find

$$d_{S^2}(P, Q) = 2\arcsin\sqrt{\frac{3-\sqrt{3}}{6}}.$$

(b) (10 points) Find an implicit equation for the line $L_{P,Q} \subseteq S^2$ defined as the set of equidistant points to $P, Q \in S^2$, i.e. $L_{P,Q} = \{R \in S^2 : d_{S^2}(R, P) = d_{S^2}(R, Q)\}$.

The line $L_{P,Q}$ is the intersection of S^2 with some plane Π through the origin. Reflection in the line $L_{P,Q}$ will exchange P and Q, so Π has normal vector

$$Q - P = \frac{1}{\sqrt{3}}(1 - \sqrt{3}, 1, 1).$$

Therefore, $L_{P,Q}$ is the subset of S^2 cut out by the equation

$$(1 - \sqrt{3})x + y + z = 0.$$

(c) (5 points) Find an isometry $\varphi : S^2 \longrightarrow S^2$ such that $\varphi(P) = Q$.

As described in the solution to (b), the reflection $\varphi = \overline{r}_{L_{P,Q}}$ exchanges P and Q.

4. (20 points) (**Distances in the Hyperbolic Upper-Half Plane** $\mathbb{H}^2$) Let $P, Q \in \mathbb{H}^2$ be the points $P = (0, 2) = 2i$ and $Q = (2, 2) = 2 + 2i$.

(a) (5 points) Compute the hyperbolic distance $d_{\mathbb{H}^2}(P, Q)$.

One of the formulas for $d_{\mathbb{H}^2}(z_1, z_2)$ (where $z_1, z_2 \in \mathbb{C}^2$ are complex numbers) is

$$d_{\mathbb{H}^2}(z_1, z_2) = \log \frac{|z_1 - \overline{z}_2| + |z_1 - z_2|}{|z_1 - \overline{z}_2| - |z_1 - z_2|}.$$

Here $|P - Q| = |2| = 2$ and $|P - \overline{Q}| = |-2 - 4i| = 2\sqrt{5}$, so

$$d_{\mathbb{H}^2}(P, Q) = \log \frac{2\sqrt{5} + 2}{2\sqrt{5} - 2} = 4 \log \varphi,$$

where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio.

(b) (10 points) Find an implicit equation for the unique line $L \subseteq \mathbb{H}^2$ which contains $P, Q \in \mathbb{H}^2$ and draw a picture for it.

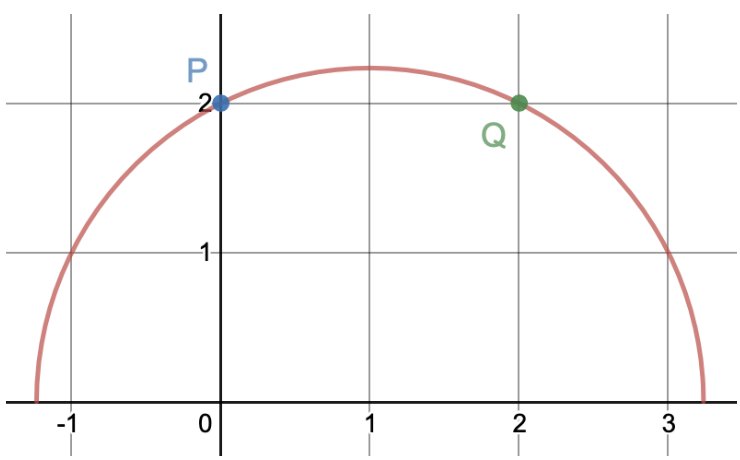
Lines in the hyperbolic planis e are exactly the vertical Euclidean lines and the half circles with center on the x-axis. Since P and Q are not vertically separated, L must be a half circle containing both. By symmetry, the center of this circle is halfway between the x-coordinates of P and Q, at the point $(1, 0)$. Finally, the radius r of L is the distance from P (or (Q)) to this center, which is

$$r^2 = (0 - 1)^2 + (2 - 0)^2 = 5.$$

Therefore, the equation for L is

$$(x - 1)^2 + y^2 = 5.$$

Here is the graph, with P and Q indicated.



(c) (5 points) Find an isometry $\phi : (\mathbb{H}^2, d_{\mathbb{H}^2}) \longrightarrow (\mathbb{H}^2, d_{\mathbb{H}^2})$ such that $\phi(P) = Q$.

Horizontal translations are hyperbolic isometries, so we can take $\phi(z) = z + 2$ in complex coordinates, which is $\phi(x, y) = (x + 2, y)$ in Cartesian coordinates.

5. (20 points) For each of the ten sentences below, circle whether they are **true** or **false**. You do *not* need to justify your answer.

(a) (2 points) Given a line $L \subseteq \mathbb{H}^2$ and a point $P \in \mathbb{H}^2$ not in the line, there exists a unique parallel line $M \subseteq \mathbb{H}^2$ to L containing P.

(1) True. (2) **False**.

(b) (2 points) Any Euclidean line $M \subseteq \mathbb{H}^2$ is a hyperbolic line.

(1) True. (2) **False**.

(c) (2 points) There exists a unique isometry $\varphi : S^2 \longrightarrow S^2$ which fixes both the North and the South pole.

(1) True. (2) **False**.

(d) (2 points) If two triangles $T_1, T_2 \subseteq \mathbb{R}^2$ have the same interior angles, then Area(T_1)=Area(T_2).

(1) True. (2) **False**.

(e) (2 points) If two triangles $T_1, T_2 \subseteq S^2$ have the same interior angles, then Area(T_1)=Area(T_2).

(1) **True**. (2) False.

(f) (2 points) For each $n \in \mathbb{N}$, there exists two lines $L_1, L_2 \subseteq T^2$ such that $|L_1 \cap L_2| = n$.

(1) **True**. (2) False.

(g) (2 points) There exists lines $L_1, L_2 \subseteq C$ in the cylinder C such that $|L_1 \cap L_2| = 5$.

(1) True. (2) **False**.

(h) (2 points) The Klein bottle K is locally isometric to the Möbius band.

(1) **True**. (2) False.

(i) (2 points) For any $P \in S^2$, there exists a disk $D \subseteq S^2$ such that D is isometric to an Euclidean disk in $\mathbb{R}^2$.

(1) True. (2) **False**.

(j) (2 points) The stereographic projection $\pi_N : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$ preserves distances.

(1) True. (2) **False**.