This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. Fill in all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may not use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this and explain why the theorem may be applied.

(B) Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) Mysterious or unsupported answers will not receive full credit. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.
1. (20 points) (Geometry in the 2-torus $T^2$). Let us consider $\Gamma = \langle t_{(0,1)}, t_{(1,0)} \rangle$, the 2-torus $T^2 = \mathbb{R}^2/\Gamma$ with coordinates $(x, y)$, and $P = (0, 0), Q = (0.9, 0.8) \in \mathbb{R}^2$.

(a) (5 points) Draw the two $\Gamma$-orbits $\Gamma P, \Gamma Q$ of the two points $P, Q \in \mathbb{R}^2$.

(b) (5 points) Compute the distance $d_{T^2}(\Gamma P, \Gamma Q)$.

(c) (10 points) Find the number $|L_1 \cap L_2|$ of intersection points between the two lines $L_1, L_2 \subseteq T^2$, where $L_1 = \{(x, y) \in T^2 : y = 0\} \subseteq T^2$ and $L_2 \subseteq T^2$ is the image in $\mathbb{R}^2/\Gamma$ of the unique line containing the two points $P, Q \in \mathbb{R}^2$. 
2. (20 points) (Isometries in the 2-sphere $S^2$) Consider the 2-sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^2 : x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3,$$

and the three lines $E_1 = \{(x, y, z) \in \mathbb{R}^3 : x = 0\} \cap S^2$, $E_2 = \{(x, y, z) \in \mathbb{R}^3 : y = 0\} \cap S^2$ and $E_3 = \{(x, y, z) \in \mathbb{R}^3 : z = 0\} \cap S^2$.

(a) (5 points) Find the images $(\tau_{E_2} \circ \tau_{E_1})(N)$ and $(\tau_{E_2} \circ \tau_{E_1})(S)$ of both the North pole $N = (0, 0, 1)$ and the South Pole $S = (0, 0, -1)$ under the isometry

$$\tau_{E_2} \circ \tau_{E_1} : S^2 \rightarrow S^2.$$ 

(b) (10 points) Show that the composition of isometries $\tau_{E_3} \circ \tau_{E_2} \circ \tau_{E_1} \in \text{Iso}(S^2)$ is neither a rotation nor a reflection.

(c) (5 points) Is it true that $\tau_{E_3} \circ \tau_{E_2} \circ \tau_{E_1} = \tau_{E_1} \circ \tau_{E_2} \circ \tau_{E_3}$?
3. (20 points) (Stereographic Projection in the 2-sphere $S^2$) Consider the 2-sphere
\[ S^2 = \{ (x, y, z) \in \mathbb{R}^2 : x^2 + y^2 + z^2 = 1 \} \subseteq \mathbb{R}^3, \]
the stereographic projection $\pi_N : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$ and the two points $P = (3/5, 4/5, 0)$ and $S = (0, 0, -1)$.
(a) (5 points) Find the images $\pi_N(P)$ and $\pi_N(S)$.

(b) (10 points) Let $L_{P,S} \subseteq S^2$ be the unique line containing $P, S \in S^2$. Show that the image $\pi_N(L) \subseteq \mathbb{R}^2$ is a line in the Euclidean 2-plane $\mathbb{R}^2$.

(c) (5 points) Give an example of a line $M \subseteq S^2$ such that its image $\pi_N(M)$ is not a line in $\mathbb{R}^2$. 
4. (20 points) (Lines in the Hyperbolic Upper-Half Plane $\mathbb{H}^2$) Let $P, Q \in \mathbb{H}^2$ be the points $P = (0, 1) = i, Q = (0, 2) = 2i,$ and consider the line

$$L = \{z \in \mathbb{H}^2 : |z| = 1\} = \{(x, y) \in \mathbb{H}^2 : x^2 + y^2 = 1\}.$$ 

(a) (5 points) Show that $M = \{z \in \mathbb{H}^2 : |z + 3/2| = 5/2\}$ is a hyperbolic line which contains $Q = 2i.$

(b) (5 points) Show that $M$ is parallel to $L.$

(c) (10 points) Find a hyperbolic line $N \subseteq \mathbb{H}^2$ which is distinct from $M,$ parallel to $L$ and contains $Q,$ i.e. $N \neq M,$ $L \cap N = \emptyset$ and $Q \in N.$
5. (20 points) For each of the five sentences below, circle the unique correct answer. You do not need to justify your answer.

(a) (2 points) Euclid’s Fifth Postulate “Given a line and a point not on it, at most one line parallel to the given line can be drawn through the point.” does not hold in the:

(1) Hyperbolic Plane \( \mathbb{H}^2 \), (2) The 2-sphere \( S^2 \),

(3) The complement \( S^2 \setminus \{(0,0,1)\} \), (4) None of these three.

(b) (2 points) Two hyperbolic lines in \((\mathbb{H}^2, d_{\mathbb{H}^2})\) cannot:

(1) Be parallel, (2) Intersect in more than one point,

(3) Be Euclidean lines, (4) None of the other answers.

(c) (2 points) Every isometry \( \varphi \in \text{Iso}(S^2) \) must:

(1) Be a reflection or a rotation, (2) Have a fixed point,

(3) Be a product of one or two reflections, (4) None of the other answers.

(d) (2 points) The stereographic projection \( \pi_N : S^2 \setminus \{(0,0,1)\} \rightarrow \mathbb{R}^2 \):

(1) Sends lines to lines, (2) Is an isometry,

(3) Maps lines to circles, (4) None of the other answers.

(e) (2 points) Let \( \Gamma \subseteq \text{Iso}(\mathbb{R}^2) \) be a discontinuous fixed-point free subgroup which contains a glide reflection. Then \( \mathbb{R}^2/\Gamma \) must be

(1) The Euclidean Klein bottle,

(2) The Euclidean Möbius band,

(3) The Hyperbolic Plane \( \mathbb{H}^2 \),

(4) None of the three answers above is correct.