University of California Davis **Name (Print):** ______________________
Euclidean Geometry MAT 141 **Student ID (Print):** ______________________

Sample Final Examination III **March 17 2020**
Time Limit: 120 Minutes

This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) (**Geometry in the 2-torus T^2**). Let us consider $\Gamma = \langle t_{(0,1)}, t_{(1,0)} \rangle$, the 2-torus $T^2 = \mathbb{R}^2/\Gamma$ with coordinates (x, y), and $P = (0, 0), Q = (0.9, 0.8) \in \mathbb{R}^2$.

   (a) (5 points) Draw the two Γ-orbits $\Gamma P, \Gamma Q$ of the two points $P, Q \in \mathbb{R}^2$.

   (b) (5 points) Compute the distance $d_{T^2}(\Gamma P, \Gamma Q)$.

   (c) (10 points) Find the number $|L_1 \cap L_2|$ of intersection points between the two lines $L_1, L_2 \subseteq T^2$, where $L_1 = \{(x, y) \in T^2 : y = 0\} \subseteq T^2$ and $L_2 \subseteq T^2$ is the image in $\mathbb{R}^2/\Gamma$ of the unique line containing the two points $P, Q \in \mathbb{R}^2$.

2. (20 points) (**Isometries in the 2-sphere** S^2) Consider the 2-sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^2 : x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3,$$

and the three lines $E_1 = \{(x, y, z) \in \mathbb{R}^3 : x = 0\} \cap S^2$, $E_2 = \{(x, y, z) \in \mathbb{R}^3 : y = 0\} \cap S^2$ and $E_3 = \{(x, y, z) \in \mathbb{R}^3 : z = 0\} \cap S^2$.

(a) (5 points) Find the images $(\overline{r}_{E_2} \circ \overline{r}_{E_1})(N)$ and $(\overline{r}_{E_2} \circ \overline{r}_{E_1})(S)$ of both the North pole $N = (0, 0, 1)$ and the South Pole $S = (0, 0, -1)$ under the isometry

$$\overline{r}_{E_2} \circ \overline{r}_{E_1} : S^2 \longrightarrow S^2.$$

(b) (10 points) Show that the composition of isometries $\overline{r}_{E_3} \circ \overline{r}_{E_2} \circ \overline{r}_{E_1} \in \mathrm{Iso}(S^2)$ is neither a rotation nor a reflection.

(c) (5 points) Is it true that $\overline{r}_{E_3} \circ \overline{r}_{E_2} \circ \overline{r}_{E_1} = \overline{r}_{E_1} \circ \overline{r}_{E_2} \circ \overline{r}_{E_3}$?

3. (20 points) (**Stereographic Projection in the 2-sphere** S^2) Consider the 2-sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^2 : x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3,$$

the stereographic projection $\pi_N : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$ and the two points $P = (3/5, 4/5, 0)$ and $S = (0, 0, -1)$.

(a) (5 points) Find the images $\pi_N(P)$ and $\pi_N(S)$.

(b) (10 points) Let $L_{P,S} \subseteq S^2$ be the unique line containing $P, S \in S^2$. Show that the image $\pi_N(L) \subseteq \mathbb{R}^2$ is a line in the Euclidean 2-plane $\mathbb{R}^2$.

(c) (5 points) Give an example of a line $M \subseteq S^2$ such that its image $\pi_N(M)$ is *not* a line in $\mathbb{R}^2$.

4. (20 points) (**Lines in the Hyperbolic Upper-Half Plane** $\mathbb{H}^2$) Let $P, Q \in \mathbb{H}^2$ be the points $P = (0,1) = i, Q = (0,2) = 2i$, and consider the line

$$L = \{z \in \mathbb{H}^2 : |z| = 1\} = \{(x,y) \in \mathbb{H}^2 : x^2 + y^2 = 1\}.$$

(a) (5 points) Show that $M = \{z \in \mathbb{H}^2 : |z + 3/2| = 5/2\}$ is a hyperbolic line which contains $Q = 2i$.

(b) (5 points) Show that M is parallel to L.

(c) (10 points) Find a hyperbolic line $N \subseteq \mathbb{H}^2$ which is distinct from M, parallel to L and contains Q, i.e. $N \neq M$, $L \cap N = \emptyset$ and $Q \in N$.

5. (20 points) For each of the five sentences below, circle the **unique** correct answer. You do *not* need to justify your answer.

(a) (2 points) Euclid's Fifth Postulate "Given a line and a point not on it, at most one line parallel to the given line can be drawn through the point." does not hold in the:

(1) Hyperbolic Plane $\mathbb{H}^2$, (2) The 2-sphere S^2,

(3) The complement $S^2 \setminus \{(0,0,1)\}$, (4) None of these three.

(b) (2 points) Two hyperbolic lines in $(\mathbb{H}^2, d_{\mathbb{H}^2})$ cannot:

(1) Be parallel, (2) Intersect in more than one point,

(3) Be Euclidean lines, (4) None of the other answers.

(c) (2 points) Every isometry $\varphi \in \mathrm{Iso}(S^2)$ must:

(1) Be a reflection or a rotation, (2) Have a fixed point,

(3) Be a product of one or two reflections, (4) None of the other answers.

(d) (2 points) The stereographic projection $\pi_N : S^2 \setminus \{(0,0,1)\} \longrightarrow \mathbb{R}^2$:

(1) Sends lines to lines, (2) Is an isometry,

(3) Maps lines to circles, (4) None of the other answers.

(e) (2 points) Let $\Gamma \subseteq \mathrm{Iso}(\mathbb{R}^2)$ be a discontinuous fixed-point free subgroup which contains a glide reflection. Then $\mathbb{R}^2/\Gamma$ must be

(1) The Euclidean Klein bottle,

(2) The Euclidean Möbius band,

(3) The Hyperbolic Plane $\mathbb{H}^2$,

(4) None of the three answers above is correct.