University of California Davis **Name (Print):** ____________________

Euclidean Geometry MAT 141 **Student ID (Print):** ____________________

Solutions to Sample Final Examination III **March 17 2020**

Time Limit: 120 Minutes

This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) (**Geometry in the 2-torus** T^2). Let us consider $\Gamma = \langle t_{(0,1)}, t_{(1,0)} \rangle$, the 2-torus $T^2 = \mathbb{R}^2/\Gamma$ with coordinates (x, y), and $P = (0,0), Q = (0.9, 0.8) \in \mathbb{R}^2$.

   (a) (5 points) Draw the two Γ-orbits $\Gamma P, \Gamma Q$ of the two points $P, Q \in \mathbb{R}^2$.

   You should have the square lattices

   $$\Gamma P = \{(n,m) : n, m \in \mathbb{Z}\} \qquad \text{and} \qquad \Gamma Q = \{(0.9+n, 0.8+m) : n, m \in \mathbb{Z}\}.$$

   (b) (5 points) Compute the distance $d_{T^2}(\Gamma P, \Gamma Q)$.

   The points $(1,1) \in \Gamma P$ and $(0.9, 0.8) \in \Gamma Q$ achieve the minimal distance between points in the orbits ΓP and ΓQ. So

   $$d_{T^2}(\Gamma P, \Gamma Q) = d_{\mathbb{R}^2}((1,1), (0.9, 0.8)) = \sqrt{0.1^2 + 0.2^2} = \frac{1}{2\sqrt{5}}.$$

   (c) (10 points) Find the number $|L_1 \cap L_2|$ of intersection points between the two lines $L_1, L_2 \subseteq T^2$, where $L_1 = \{(x,y) \in T^2 : y = 0\} \subseteq T^2$ and $L_2 \subseteq T^2$ is the image in $\mathbb{R}^2/\Gamma$ of the unique line containing the two points $P, Q \in \mathbb{R}^2$.

   We think of T^2 as the standard fundamental domain, the square with side length 1 and corner $(0,0)$. Then L_1 is the horizontal sides of T^2 and L_2 is the line with slope $8/9$ that connects ΓP and ΓQ. Starting at $(0,0)$, as L_2 goes up a distance 1, it has moved horizontally by $9/8$, placing it at $(1/8, 0)$ in the square. Repeating this 8 times, we get back to the starting $(0,0)$. Since L_2 crossed L_1 a total of 8 times in this process, we conclude that $|L_1 \cap L_2| = 8$.

2. (20 points) (**Isometries in the 2-sphere** S^2) Consider the 2-sphere

$$S^2 = \{(x,y,z) \in \mathbb{R}^2 : x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3,$$

and the three lines $E_1 = \{(x,y,z) \in \mathbb{R}^3 : x = 0\} \cap S^2$, $E_2 = \{(x,y,z) \in \mathbb{R}^3 : y = 0\} \cap S^2$ and $E_3 = \{(x,y,z) \in \mathbb{R}^3 : z = 0\} \cap S^2$.

(a) (5 points) Find the images $(\overline{r}_{E_2} \circ \overline{r}_{E_1})(N)$ and $(\overline{r}_{E_2} \circ \overline{r}_{E_1})(S)$ of both the North pole $N = (0,0,1)$ and the South Pole $S = (0,0,-1)$ under the isometry

$$\overline{r}_{E_2} \circ \overline{r}_{E_1} : S^2 \longrightarrow S^2.$$

On a general point $(x,y,z) \in S^2$, this map acts by

$$\overline{r}_{E_2} \circ \overline{r}_{E_1}(x,y,z) = \overline{r}_{E_2}(-x,y,z) = (-x,-y,z).$$

Therefore, $\overline{r}_{E_2} \circ \overline{r}_{E_1}(0,0,1) = (0,0,1)$ and $\overline{r}_{E_2} \circ \overline{r}_{E_1}(0,0,-1) = (0,0,-1)$. Since isometry is a rotation about the z-axis, it fixes the poles.

—

(b) (10 points) Show that the composition of isometries $\overline{r}_{E_3} \circ \overline{r}_{E_2} \circ \overline{r}_{E_1} \in \mathrm{Iso}(S^2)$ is neither a rotation nor a reflection.

All reflections and rotations have fixed points, but this isometry has none. Indeed, suppose $(x,y,z) \in S^2$ is an isometry of $\overline{r}_{E_3} \circ \overline{r}_{E_2} \circ \overline{r}_{E_1}$. Then

$$\begin{aligned}(x,y,z) &= \overline{r}_{E_3} \circ \overline{r}_{E_2} \circ \overline{r}_{E_1}(x,y,z)\\ &= \overline{r}_{E_3} \circ \overline{r}_{E_2}(-x,y,z)\\ &= \overline{r}_{E_3}(-x,-y,z)\\ &= (-x,-y,-z).\end{aligned}$$

But this implies that $(x,y,z) = (0,0,0)$, which is not a point in S^2. We conclude that this isometry has no fixed points, so it is neither a rotation nor a reflection.

(c) (5 points) Is it true that $\overline{r}_{E_3} \circ \overline{r}_{E_2} \circ \overline{r}_{E_1} = \overline{r}_{E_1} \circ \overline{r}_{E_2} \circ \overline{r}_{E_3}$?

Yes. With a calculation like that shown in the solution to (b), both of these are the antipodal map $(x,y,z) \mapsto (-x,-y,-z)$. In general, reflections in orthogonal planes commute.

3. (20 points) (**Stereographic Projection in the 2-sphere** S^2) Consider the 2-sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^2 : x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3,$$

the stereographic projection $\pi_N : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$ and the two points $P = (3/5, 4/5, 0)$ and $S = (0, 0, -1)$.

(a) (5 points) Find the images $\pi_N(P)$ and $\pi_N(S)$.

For any $(x, y, z) \in S^2$, we have the formula

$$\pi_N(x, y, z) = \frac{1}{1 - z}(x, y).$$

Therefore,

$$\pi_N(3/5, 4/5, 0) = (3/5, 4/5) \qquad \text{and} \qquad \pi_N(0, 0, -1) = (0, 0).$$

(b) (10 points) Let $L_{P,S} \subseteq S^2$ be the unique line containing $P, S \in S^2$. Show that the image $\pi_N(L) \subseteq \mathbb{R}^2$ is a line in the Euclidean 2-plane $\mathbb{R}^2$.

Since $L_{P,S}$ contains the South pole S, it must also contain the North pole. We know that all lines containing the North pole map under π_N to Euclidean lines in $\mathbb{R}^2$ (see the first half of Problem 4(a) in Problem Set 6 for a proof of this fact). So we conclude that $\pi_N(L)$ is a Euclidean line.

(c) (5 points) Give an example of a line $M \subseteq S^2$ such that its image $\pi_N(M)$ is *not* a line in $\mathbb{R}^2$.

Take the equator $M = \{(x, y, z) \in S^2 : z = 0\}$. Then $\pi_N(M) = M$ is the unit circle in $\mathbb{R}^2$, which is not a Euclidean line.

4. (20 points) (**Lines in the Hyperbolic Upper-Half Plane** $\mathbb{H}^2$) Let $P, Q \in \mathbb{H}^2$ be the points $P = (0,1) = i, Q = (0,2) = 2i$, and consider the line

$$L = \{z \in \mathbb{H}^2 : |z| = 1\} = \{(x,y) \in \mathbb{H}^2 : x^2 + y^2 = 1\}.$$

(a) (5 points) Show that $M = \{z \in \mathbb{H}^2 : |z + 3/2| = 5/2\}$ is a hyperbolic line which contains $Q = 2i$.

Geometrically, M is the set of points in $\mathbb{H}^2$ at distance $5/2$ from the complex number $3/2$. This is a circle with center $-3/2$, which is on the x-axis, so M is a hyperbolic line. Algebraically,

$$\begin{aligned} M &= \{z \in \mathbb{H}^2 : |z + \tfrac{3}{2}| = \tfrac{5}{2}\} \\ &= \{(x,y) \in \mathbb{H}^2 : |(x,y) + (\tfrac{3}{2}, 0)| = \tfrac{5}{2}\} \\ &= \{(x,y) \in \mathbb{H}^2 : (x + \tfrac{3}{2})^2 + y^2 = \tfrac{25}{4}\}, \end{aligned}$$

which confirms the same. Plugging in $(x,y) = Q = (0,2)$, we have

$$(0 + \tfrac{3}{2})^2 + (2)^2 = \tfrac{9}{4} + 4 = \tfrac{25}{4},$$

so $Q \in M$.

(b) (5 points) Show that M is parallel to L.

Suppose $z \in \mathbb{H}^2$ is a point on L, so $|z| = 1$. From the triangle inequality,

$$\left|z + \tfrac{3}{2}\right| \leq |z| + \left|\tfrac{3}{2}\right| = 1 + \tfrac{3}{2} = \tfrac{5}{2},$$

with equality if and only if z and $\frac{3}{2}$ are collinear with the origin. This would imply that z is on the x-axis, which is impossible because the hyperbolic plane does not include the x-axis. We conclude that the inequality is strict, so $\left|z + 3/2\right| \neq 5/2$. This means that z is not a point on the line L, so $M \cap L = \varnothing$.

We could also compare the equations for M and L. If $(x,y) \in L$ then $x^2 + y^2 = 1$. If this point is also in M then

$$\tfrac{25}{4} \;=\; (x + \tfrac{3}{2})^2 + y^2 \;=\; x^2 + y^2 + 3x + \tfrac{9}{4} \;=\; 1 + 3x + \tfrac{9}{4},$$

which implies $x = 1$. This implies $y = 0$, so (x,y) is not in $\mathbb{H}^2$.

(c) (10 points) Find a hyperbolic line $N \subseteq \mathbb{H}^2$ which is distinct from M, parallel to L and contains Q, i.e. $N \neq M$, $L \cap N = \emptyset$ and $Q \in N$.

Take the half-circle $N = \{z \in \mathbb{H}^2 : |z| = 2\}$ of radius 2 and center $(0,0)$. The line N is concentric with L and with different radius, so $L \cap N = \varnothing$. Since $|Q| = 2$, we have $Q \in N$. Finally, Since N and M have different centers (and radii), $N \neq M$.

5. (20 points) For each of the five sentences below, circle the **unique** correct answer. You do *not* need to justify your answer.

   (a) (2 points) Euclid's Fifth Postulate "Given a line and a point not on it, at most one line parallel to the given line can be drawn through the point." does not hold in the:

   (1) **Hyperbolic Plane $\mathbb{H}^2$**, (2) The 2-sphere S^2,

   (3) The complement $S^2 \setminus \{(0,0,1)\}$, (4) None of these three.

   (b) (2 points) Two hyperbolic lines in $(\mathbb{H}^2, d_{\mathbb{H}^2})$ cannot:

   (1) Be parallel, (2) **Intersect in more than one point**,

   (3) Be Euclidean lines, (4) None of the other answers.

   (c) (2 points) Every isometry $\varphi \in \text{Iso}(S^2)$ must:

   (1) Be a reflection or a rotation, (2) Have a fixed point,

   (3) Be a product of one or two reflections, (4) **None of the other answers**.

   (d) (2 points) The stereographic projection $\pi_N : S^2 \setminus \{(0,0,1)\} \longrightarrow \mathbb{R}^2$:

   (1) Sends lines to lines, (2) Is an isometry,

   (3) Maps lines to circles, (4) **None of the other answers**.

   (e) (2 points) Let $\Gamma \subseteq \text{Iso}(\mathbb{R}^2)$ be a discontinuous fixed-point free subgroup which contains a glide reflection. Then $\mathbb{R}^2/\Gamma$ must be

   (1) The Euclidean Klein bottle,

   (2) The Euclidean Möbius band,

   (3) The Hyperbolic Plane $\mathbb{H}^2$,

   (4) **None of the three answers above is correct**.