This examination document contains 7 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. Fill in all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may not use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.
1. (20 points) (Euclidean Geometry) Consider the three Euclidean lines in $\mathbb{R}^2$ given by $L = \{(x, y) \in \mathbb{R}^2 : x = 0\}$, $M = \{(x, y) \in \mathbb{R}^2 : y = 3\}$ and $N = \{(x, y) \in \mathbb{R}^2 : x + y = 1\}$.

(a) (5 points) Find the image of $(0, 3) \in \mathbb{R}^2$ under the isometry $r_N \circ r_M \circ r_L$.

Since $(0, 3)$ is in the lines $L$ and $M$, it is fixed by $r_L$ and $r_M$, so

$$r_N \circ r_M \circ r_L(0, 3) = r_N(0, 3).$$

The final reflection can be found by drawing the line $N$ and creating a square with diagonal $N$ and corner $(0, 3)$. The opposite corner is

$$r_N(0, 3) = (-2, 1),$$

which is our final image.

(b) (5 points) Let $R = \{(x, y) \in \mathbb{R}^2 : y = x + 3\} \subseteq \mathbb{R}^2$ be a line and $P \in R$ an arbitrary point in the line. Show that $(r_N \circ r_M \circ r_L)(P) \in R$.

This suggests that our isometry is a glide reflection along the line $R$. Rotate $M$ and $L$ about $(0, 3)$ to new lines $M'$ and $L'$, so that $M'$ becomes parallel to $N$ (the rotation is by $-\pi/4$). Therefore,

$$r_N \circ r_M \circ r_L = r_N \circ r_{M'} \circ r_{L'}.$$

Actually, the line $L'$ is exactly $R$, so we have

$$r_N \circ r_M \circ r_L = r_N \circ r_{M'} \circ r_R.$$

Now we are reflecting in $R$, followed by two reflections in lines perpendicular to $R$, so our isometry is a glide reflection. Glide reflections preserve the line of reflection (in this case $R$), which is what we needed to show.

(c) (10 points) Find $\alpha \in \mathbb{R}^2$ and $K \subseteq \mathbb{R}^2$ a line such that $r_N \circ r_M \circ r_L = t_\alpha \circ r_K$.

From (b), we have

$$r_N \circ r_M \circ r_L = r_N \circ r_{M'} \circ r_R,$$

where $N$ and $M'$ are parallel lines perpendicular to $R$. Therefore, $r_N \circ r_{M'}$ is a translation along $R$. The displacement between $N$ and $M'$ is half of the square described in the solution to (a). Then the vector $\alpha$ is the full diagonal, which is $(0, 3) - (-2, 1) = (2, 2)$. Therefore,

$$\alpha = (2, 2) \quad \text{and} \quad K = R.$$
2. (20 points) (Spherical Geometry I) Consider the 2-sphere

\[ S^2 = \{(x, y, z) \in \mathbb{R}^2 : x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3, \]

and its points \( P = \frac{1}{\sqrt{2}}(1, 1, 0), Q = \frac{1}{\sqrt{2}}(1, 0, 1) \) and \( R = \frac{1}{\sqrt{2}}(0, 1, 1) \).

(a) (5 points) Compute the three distances \( d_{S^2}(P, Q), d_{S^2}(R, Q) \) and \( d_{S^2}(P, R) \).

Using the formula \( d_{S^2}(P_1, P_2) = 2 \arcsin\left(\frac{1}{2}d_{\mathbb{R}^3}(P_1, P_2)\right) \), we find

\[ d_{S^2}(P, Q) = d_{S^2}(T, Q) = d_{S^2}(P, T) = \frac{\pi}{3}. \]

(b) (10 points) Let \( E \subseteq S^2 \) be the unique line containing \( P, Q \in S^2 \). Show that

\[ \Pi_E = \{(x, y, z) \in \mathbb{R}^3 : x - y - z = 0\} \]

is the unique 2-plane \( \Pi_E \subseteq \mathbb{R}^3 \) such that \( \Pi_E \cap S^2 = E \).

Lines in \( S^2 \) are cut out by planes in \( \mathbb{R}^3 \) that pass through the origin. Since our two points \( P \) and \( Q \) are not collinear with the origin, the necessary plane that cuts out \( E \) is uniquely determined (because a plane is uniquely specified by three non-collinear points). Finally, it is clear that \( P \) and \( Q \) are contained in the plane \( \Pi_E \) given, so this must be the plane that cuts out \( E \).

(c) (5 points) Find the image \( r_E(R) \) of \( R \) under the reflection \( r_E : S^2 \rightarrow S^2 \).

The reflection \( r_E \) corresponds to reflection in the plane \( \Pi_E \). To perform this reflection on a single point, we can find the perpendicular line \( L \) from \( R \) to \( \Pi_E \). Then \( r_E(R) \) will simply be the other point where \( L \) intersects the sphere. Since \( \Pi_E \) has normal vector \((1, -1, -1)\), the line \( L \) is

\[ L = \{R + t(1, -1, -1) : t \in \mathbb{R}\} = \left\{(t, -t + \frac{1}{\sqrt{2}}, -t + \frac{1}{\sqrt{2}}) : t \in \mathbb{R}\right\}. \]

The points in \( L \) that intersect the sphere \( S^2 \) must satisfy

\[ (t)^2 + \left(-t + \frac{1}{\sqrt{2}}\right)^2 + \left(-t + \frac{1}{\sqrt{2}}\right)^2 = 1. \]

The first solution is \( t = 0 \), which gives the point \( R \) as expected. The other is \( t = 2\sqrt{2}/3 \), which corresponds on \( L \) to the point

\[ r_E(R) = \left( \frac{2\sqrt{2}}{3}, -\frac{\sqrt{2}}{6}, -\frac{\sqrt{2}}{6} \right). \]
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3. (20 points) **(Spherical Geometry II)** Consider the unit 2-sphere $S^2$ and the stereographic projection $\pi_N : S^2 \setminus \{N\} \rightarrow \mathbb{R}^2$ from the North Pole $N = (0,0,1)$. Consider the two points $P_1 = (1,0,0), P_2 = \frac{1}{\sqrt{3}}(1,-1,1)$ and let $L \subseteq S^2$ be the unique line which contains $P_1, P_2$.

(a) (5 points) Find the image $\pi_N(P_1)$ and $\pi_N(P_2)$ of the two points

\[ P_1 = (1,0,0), P_2 = \frac{1}{\sqrt{3}}(1,-1,1). \]

The point $P_1$ is on the equator $E = \{z = 0\} \cap S^2$ of the sphere, so it is fixed by stereographic projection: $\pi_N(P_1) = P_1$. From the formula

\[ \pi_N(x,y,z) = \frac{1}{1-z}(x,y), \]

we have

\[ \pi_N(P_2) = \frac{1}{1-1/\sqrt{3}} \left( \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right) = \frac{1}{\sqrt{3}-1}(1,-1). \]

(b) (5 points) Find the two intersection points $L \cap E$, where $E$ is the equator defined as $E = \{(x,y,z) \in S^2 : z = 0\} \subseteq S^2$.

The point $P_1$ is already on the equator, so $P_1 \in L \cap E$. Since lines in $S^2$ intersect in antipodal points, we must also have $-P_1 = (-1,0,0)$ as the second intersection point.

(c) (5 points) Find all the points of intersection of $\pi_N(L) \subseteq \mathbb{R}^2$ and the unit circle $\{(x,y) : x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$.

We know that stereographic projection is a bijection which fixes the equator $E$. This equator is exactly the unit circle, so the intersection of $\pi_N(L)$ with the unit circle is precisely $\pi_N(L \cap E) = L \cap E$, which we found in (b) to be the two points $(1,0,0)$ and $(-1,0,0)$. 

(d) (5 points) Qualitatively draw the image $\pi_N(L) \subseteq \mathbb{R}^2$ in the Euclidean plane.

The line $L$ is in pink, and its stereographic projection is in green. The image is not to scale, but the important feature is that the circle has center on the negative $y$-axis and includes $(\pm 1, 0, 0)$. 

![Diagram](image-url)
4. (20 points) (Hyperbolic Geometry in $\mathbb{H}^2$) Let $k \in \mathbb{N}$ and consider the curves $\gamma_k \subseteq \mathbb{H}^2$ described as

$$
\gamma_k = \{(x, y) \in \mathbb{H}^2 : 0 \leq x \leq 1, y = k \} \subseteq \mathbb{H}^2.
$$

(a) (5 points) Compute the hyperbolic distance $d_{\mathbb{H}^2}(P, Q)$ between $P = (0, 1)$ and the point $Q = (1, 1)$.

One of the formulas for $d_{\mathbb{H}^2}(z_1, z_2)$ (where $z_1, z_2 \in \mathbb{C}^2$ are complex numbers) is

$$
d_{\mathbb{H}^2}(z_1, z_2) = \log \left| \frac{|z_1 - z_2| + |\overline{z_1} - \overline{z_2}|}{|z_1 - z_2| - |\overline{z_1} - \overline{z_2}|} \right|.
$$

In complex coordinates, $P = i$ and $Q = 1 + i$, so $|P - Q| = |-1| = 1$ and $|P - \overline{Q}| = |-1 + 2i| = \sqrt{5}$. We find

$$
d_{\mathbb{H}^2}(P, Q) = \log \frac{\sqrt{5} + 1}{\sqrt{5} - 1} = 2 \log \varphi,
$$

where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio.

(b) (5 points) Compute the length $l(\gamma_1)$ of $\gamma_1$, and use Part (a) to conclude that $\gamma_1$ is not part of a hyperbolic line.

*Hint:* Given a hyperbolic line $L$ containing $P, Q \in \mathbb{H}^2$, the length of the bounded part in $L$ between $P$ and $Q$ must be the distance $d_{\mathbb{H}^2}(P, Q)$.

We will use the parametrization for $\gamma_k$ given by $(x(t), y(t)) = (t, k)$ in the interval $0 \leq t \leq 1$. We then find

$$
l(\gamma_k) = \int_0^1 \sqrt{x'(t)^2 + y'(t)^2} \, dt = \int_0^1 \frac{1}{k} \, dt = \frac{1}{k}.
$$

If $\gamma_1$ were part of a hyperbolic line $L$, then the length of $\gamma_1$ would be the bounded part of $L$ between $(0, 1)$ and $(1, 1)$, which would tell us that $l(\gamma_1) = d_{\mathbb{H}^2}(P, Q)$. But $d_{\mathbb{H}^2}(P, Q) \neq 1$ (which you can see by noticing that $\varphi^2 \neq e$, because $e$ is transcendental). We conclude that $\gamma_1$ is not part of any hyperbolic line.

(c) (10 points) Show that the length $l(\gamma_k)$ strictly decreases as $k \to \infty$ increases.

This follows from our calculation $l(\gamma_k) = 1/k$ in the solution to (b).
5. (20 points) For each of the ten sentences below, circle whether they are **true** or **false**. You do not need to justify your answer.

(a) (2 points) There exist no parallel hyperbolic lines in $(\mathbb{H}^2, d_{\mathbb{H}^2})$.

(1) True. (2) **False**.

(b) (2 points) Any two lines $L_1, L_2 \subseteq K$ in the Klein bottle must intersect once.

(1) True. (2) **False**.

(c) (2 points) A triangle $T \subseteq (\mathbb{H}^2, d_{\mathbb{H}^2})$ can have arbitrarily small interior angles.

(1) **True**. (2) False.

(d) (2 points) A triangle $T \subseteq (S^2, d_{S^2})$ can have three interior right angles.

(1) **True**. (2) False.

(e) (2 points) There exists a triangle $T \subseteq C$ in the Euclidean cylinder with two interior right angles.

(1) True. (2) **False**.

(f) (2 points) An isometry $f \in \text{Iso}(S^2)$ which fixes the three points $(0, 0, 1), \frac{1}{\sqrt{2}}(0, 1, 1), (0, 1, 0)$ in the 2-sphere $S^2$, must be the identity.

(1) True. (2) **False**.

(g) (2 points) Let $P, Q \in S^2$ be two points in the 2-sphere $S^2$, then there exists a unique line $L \subseteq S^2$ containing $P$ and $Q$.

(1) True. (2) **False**.

(h) (2 points) The product of two reflections $r_1, r_2 \in \text{Iso}(S^2)$ is a rotation.

(1) **True**. (2) False.

(i) (2 points) The product of two reflections $r_1, r_2 \in \text{Iso}(\mathbb{R}^2)$ is a rotation.

(1) True. (2) **False**.

(j) (2 points) The stereographic projection $\pi_N : S^2 \setminus \{N\} \rightarrow \mathbb{R}^2$ sends triangles in $S^2$ to triangles in $\mathbb{R}^2$.

(1) True. (2) **False**.
