University of California Davis
Euclidean Geometry MAT 141

Name (Print): ____________________
Student ID (Print): ____________________

Solutions to Sample Final Examination
March 17 2020
Time Limit: 120 Minutes

This examination document contains 7 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

Do not write in the table to the right.

1. (20 points) (**Euclidean Geometry**) Consider the three Euclidean lines in $\mathbb{R}^2$ given by $L = \{(x, y) \in \mathbb{R}^2 : x = 0\}$, $M = \{(x, y) \in \mathbb{R}^2 : y = 3\}$ and $N = \{(x, y) \in \mathbb{R}^2 : x + y = 1\}$.

   (a) (5 points) Find the image of $(0, 3) \in \mathbb{R}^2$ under the isometry $\overline{r}_N \circ \overline{r}_M \circ \overline{r}_L$.

   Since $(0, 3)$ is in the lines L and M, it is fixed by $\overline{r}_L$ and $\overline{r}_M$, so

   $$\overline{r}_N \circ \overline{r}_M \circ \overline{r}_L(0, 3) = \overline{r}_N(0, 3).$$

   The final reflection can be found by drawing the line N and creating a square with diagonal N and corner $(0, 3)$. The opposite corner is

   $$\overline{r}_N(0, 3) = (-2, 1),$$

   which is our final image.

   (b) (5 points) Let $R = \{(x, y) \in \mathbb{R}^2 : y = x + 3\} \subseteq \mathbb{R}^2$ be a line and $P \in R$ an arbitrary point in the line. Show that $(\overline{r}_N \circ \overline{r}_M \circ \overline{r}_L)(P) \in R$.

   This suggests that our isometry is a glide reflection along the line R. Rotate M and L about $(0, 3)$ to new lines M' and L', so that M' becomes parallel to N (the rotation is by $-\pi/4$). Therefore,

   $$\overline{r}_N \circ \overline{r}_M \circ \overline{r}_L = \overline{r}_N \circ \overline{r}_{M'} \circ \overline{r}_{L'}.$$

   Actually, the line L' is exactly R, so we have

   $$\overline{r}_N \circ \overline{r}_M \circ \overline{r}_L = \overline{r}_N \circ \overline{r}_{M'} \circ \overline{r}_R.$$

   Now we are reflecting in R, followed by two reflections in lines perpendicular to R, so our isometry is a glide reflection. Glide reflections preserve the line of reflection (in this case R), which is what we needed to show.

   (c) (10 points) Find $\alpha \in \mathbb{R}^2$ and $K \subseteq \mathbb{R}^2$ a line such that

   $$\overline{r}_N \circ \overline{r}_M \circ \overline{r}_L = t_\alpha \circ \overline{r}_K.$$

   From (b), we have

   $$\overline{r}_N \circ \overline{r}_M \circ \overline{r}_L = \overline{r}_N \circ \overline{r}_{M'} \circ \overline{r}_R,$$

   where N and M' are parallel lines perpendicular to R. Therefore, $\overline{r}_N \circ \overline{r}_{M'}$ is a translation along R. The displacement between N and M' is half of the square described in the solution to (a). Then the vector α is the full diagonal, which is $(0, 3) - (-2, 1) = (2, 2)$. Therefore,

   $$\alpha = (2, 2) \qquad \text{and} \qquad K = R.$$

2. (20 points) (**Spherical Geometry I**) Consider the 2-sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^2 : x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3,$$

and its points $P = \frac{1}{\sqrt{2}}(1, 1, 0), Q = \frac{1}{\sqrt{2}}(1, 0, 1)$ and $R = \frac{1}{\sqrt{2}}(0, 1, 1)$.

(a) (5 points) Compute the three distances $d_{S^2}(P, Q)$, $d_{S^2}(R, Q)$ and $d_{S^2}(P, R)$.

Using the formula $d_{S^2}(P_1, P_2) = 2 \arcsin\left(\frac{1}{2} d_{\mathbb{R}^3}(P_1, P_2)\right)$, we find

$$d_{S^2}(P, Q) = d_{S^2}(T, Q) = d_{S^2}(P, T) = \frac{\pi}{3}.$$

(b) (10 points) Let $E \subseteq S^2$ be the unique line containing $P, Q \in S^2$. Show that

$$\Pi_E = \{(x, y, z) \in \mathbb{R}^3 : x - y - z = 0\}$$

is the unique 2-plane $\Pi_E \subseteq \mathbb{R}^3$ such that $\Pi_E \cap S^2 = E$.

Lines in S^2 are cut out by planes in $\mathbb{R}^3$ that pass through the origin. Since our two points P and Q are not collinear with the origin, the necessary plane that cuts out E is uniquely determined (because a plane is uniquely specified by three non-collinear points). Finally, it is clear that P and Q are contained in the plane Π_E given, so this must be the plane that cuts out E.

(c) (5 points) Find the image $\overline{r}_E(R)$ of R under the reflection $\overline{r}_E : S^2 \longrightarrow S^2$.

The reflection $\overline{r}_E$ corresponds to reflection in the plane Π_E. To perform this reflection on a single point, we can find the perpendicular line L from R to Π_E. Then $\overline{r}_E(R)$ will simply be the other point where L intersects the sphere. Since Π_E has normal vector $(1, -1, -1)$, the line L is

$$L = \{R + t(1, -1, -1) : t \in \mathbb{R}\} = \left\{\left(t, -t + \tfrac{1}{\sqrt{2}}, -t + \tfrac{1}{\sqrt{2}}\right) : t \in \mathbb{R}\right\}.$$

The points in L that intersect the sphere S^2 must satisfy

$$(t)^2 + \left(-t + \tfrac{1}{\sqrt{2}}\right)^2 + \left(-t + \tfrac{1}{\sqrt{2}}\right)^2 = 1.$$

The first solution is $t = 0$, which gives the point R as expected. The other is $t = 2\sqrt{2}/3$, which corresponds on L to the point

$$\overline{r}_E(R) = \left(\frac{2\sqrt{2}}{3}, -\frac{\sqrt{2}}{6}, -\frac{\sqrt{2}}{6}\right).$$

3. (20 points) (**Spherical Geometry II**) Consider the unit 2-sphere S^2 and the stereographic projection $\pi_N : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$ from the North Pole $N = (0, 0, 1)$. Consider the two points $P_1 = (1, 0, 0), P_2 = \frac{1}{\sqrt{3}}(1, -1, 1)$ and let $L \subseteq S^2$ be the unique line which contains P_1, P_2.

(a) (5 points) Find the image $\pi_N(P_1)$ and $\pi_N(P_2)$ of the two points

$$P_1 = (1, 0, 0), P_2 = \frac{1}{\sqrt{3}}(1, -1, 1).$$

The point P_1 is on the equator $\mathcal{E} = \{z = 0\} \cap S^2$ of the sphere, so it is fixed by stereographic projection: $\pi_N(P_1) = P_1$. From the formula

$$\pi_N(x, y, z) = \frac{1}{1-z}(x, y),$$

we have

$$\pi_N(P_2) = \frac{1}{1 - 1/\sqrt{3}}\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}-1}(1, -1).$$

(b) (5 points) Find the two intersection points $L \cap \mathcal{E}$, where $\mathcal{E}$ is the equator defined as $\mathcal{E} = \{(x, y, z) \in S^2 : z = 0\} \subseteq S^2$.

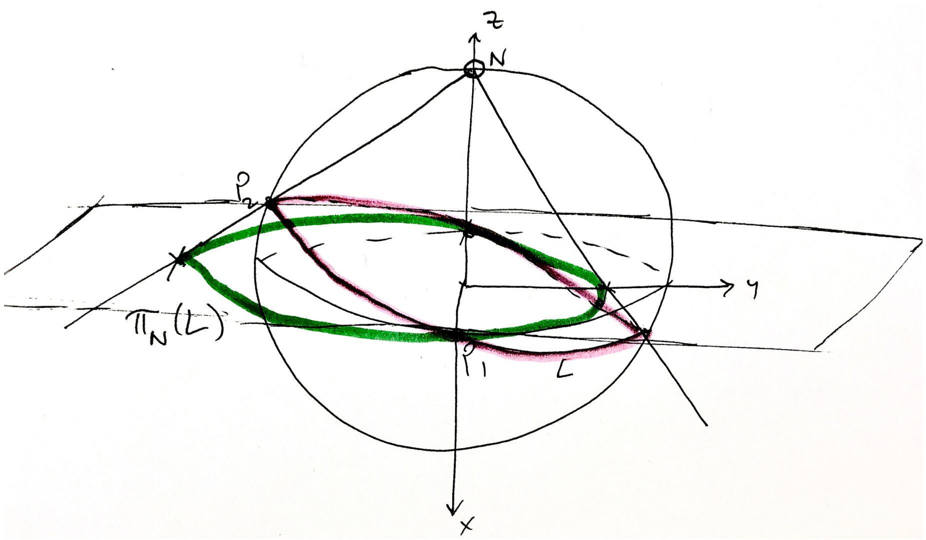
The point P_1 is already on the equator, so $P_1 \in L \cap \mathcal{E}$. Since lines in S^2 intersect in antipodal points, we must also have $-P_1 = (-1, 0, 0)$ as the second intersection point.

(c) (5 points) Find all the points of intersection of $\pi_N(L) \subseteq \mathbb{R}^2$ and the unit circle $\{(x, y) : x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$.

We know that stereographic projection is a bijection which fixes the equator $\mathcal{E}$. This equator is exactly the unit circle, so the intersection of $\pi_N(L)$ with the unit circle is precisely $\pi_N(L \cap \mathcal{E}) = L \cap \mathcal{E}$, which we found in (b) to be the two points $(1, 0, 0)$ and $(-1, 0, 0)$.

(d) (5 points) Qualitatively draw the image $\pi_N(L) \subseteq \mathbb{R}^2$ in the Euclidean plane.

The line L is in pink, and its stereographic projection is in green. The image is not to scale, but the important feature is that the circle has center on the negative y-axis and includes $(\pm 1, 0, 0)$.



4. (20 points) (**Hyperbolic Geometry in $\mathbb{H}^2$**) Let $k \in \mathbb{N}$ and consider the curves $\gamma_k \subseteq \mathbb{H}^2$ described as

$$\gamma_k = \{(x, y) \in \mathbb{H}^2 : 0 \leq x \leq 1, y = k\} \subseteq \mathbb{H}^2.$$

(a) (5 points) Compute the hyperbolic distance $d_{\mathbb{H}^2}(P, Q)$ between $P = (0, 1)$ and the point $Q = (1, 1)$.

One of the formulas for $d_{\mathbb{H}^2}(z_1, z_2)$ (where $z_1, z_2 \in \mathbb{C}^2$ are complex numbers) is

$$d_{\mathbb{H}^2}(z_1, z_2) = \log \frac{|z_1 - \overline{z}_2| + |z_1 - z_2|}{|z_1 - \overline{z}_2| - |z_1 - z_2|}.$$

In complex coordinates, $P = i$ and $Q = 1 + i$, so $|P - Q| = |-1| = 1$ and $\left|P - \overline{Q}\right| = |-1 + 2i| = \sqrt{5}$. We find

$$d_{\mathbb{H}^2}(P, Q) = \log \frac{\sqrt{5} + 1}{\sqrt{5} - 1} = 2 \log \varphi,$$

where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio.

(b) (5 points) Compute the length $l(\gamma_1)$ of γ_1, and use Part(a) to conclude that γ_1 is not part of a hyperbolic line.

Hint: Given a hyperbolic line L containing $P, Q \in \mathbb{H}^2$, the length of the bounded part in L between P and Q must be the distance $d_{\mathbb{H}^2}(P, Q)$.

We will use the parametrization for γ_k given by $(x(t), y(t)) = (t, k)$ in the interval $0 \leq t \leq 1$. We then find

$$l(\gamma_k) = \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\, dt = \int_0^1 \frac{1}{k}\, dt = \frac{1}{k}.$$

If γ_1 were part of a hyperbolic line L, then the length of γ_1 would be the bounded part of L between $(0, 1)$ and $(1, 1)$, which would tell us that $l(\gamma_1) = d_{\mathbb{H}^2}(P, Q)$. But $d_{\mathbb{H}^2}(P, Q) \neq 1$ (which you can see by noticing that $\varphi^2 \neq e$, because e is transcendental). We conclude that γ_1 is not part of any hyperbolic line.

(c) (10 points) Show that the length $l(\gamma_k)$ strictly decreases as $k \to \infty$ increases.

This follows from our calculation $l(\gamma_k) = 1/k$ in the solution to (b).

5. (20 points) For each of the ten sentences below, circle whether they are **true** or **false**. You do *not* need to justify your answer.

(a) (2 points) There exist no parallel hyperbolic lines in $(\mathbb{H}^2, d_{\mathbb{H}^2})$.

(1) True. (2) **False**.

(b) (2 points) Any two lines $L_1, L_2 \subseteq K$ in the Klein bottle must intersect once.

(1) True. (2) **False**.

(c) (2 points) A triangle $T \subseteq (\mathbb{H}^2, d_{\mathbb{H}^2})$ can have arbitrarily small interior angles.

(1) **True**. (2) False.

(d) (2 points) A triangle $T \subseteq (S^2, d_{S^2})$ can have three interior right angles.

(1) **True**. (2) False.

(e) (2 points) There exists a triangle $T \subseteq C$ in the Euclidean cylinder with two interior right angles.

(1) True. (2) **False**.

(f) (2 points) An isometry $f \in \mathrm{Iso}(S^2)$ which fixes the three points $(0,0,1), \frac{1}{\sqrt{2}}(0,1,1), (0,1,0)$ in the 2-sphere S^2, must be the identity.

(1) True. (2) **False**.

(g) (2 points) Let $P, Q \in S^2$ be two points in the 2-sphere S^2, then there exists a unique line $L \subseteq S^2$ containing P and Q.

(1) True. (2) **False**.

(h) (2 points) The product of two reflections $\overline{r}_1, \overline{r}_2 \in \mathrm{Iso}(S^2)$ is a rotation.

(1) **True**. (2) False.

(i) (2 points) The product of two reflections $\overline{r}_1, \overline{r}_2 \in \mathrm{Iso}(\mathbb{R}^2)$ is a rotation.

(1) True. (2) **False**.

(j) (2 points) The stereographic projection $\pi_N : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$ sends triangles in S^2 to triangles in $\mathbb{R}^2$.

(1) True. (2) **False**.