University of California Davis **Name (Print):** ____________________

Euclidean Geometry MAT 141 **Student ID (Print):** ____________________

Sample Midterm Examination **February 7 2020**

Time Limit: 50 Minutes

This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	0	
4	20	
5	20	
Total:	80	

1. (20 points) (**Rotations in $\mathbb{R}^2$**) Consider the two points $P = (0,0), Q = (1,0) \in \mathbb{R}^2$ in the Euclidean plane. Solve the following parts:

   (a) (5 points) Let $R_{P,\pi/2}$ be a rotation of angle $\pi/2$ centered at P. Compute the image $R_{P,\pi/2}(3,3)$ of the point $(3,3) \in \mathbb{R}^2$ under the isometry $R_{P,\pi/2}$.

   (b) (5 points) Let $R_{Q,-\pi/2}$ be a rotation of angle $-\pi/2$ centered at Q. Compute the image $R_{Q,-\pi/2}(4,5)$ of the point $(4,5) \in \mathbb{R}^2$ under the isometry $R_{Q,-\pi/2}$.

   (c) (5 points) Let $(x,y) \in \mathbb{R}^2$ be any point. Where does $(x,y) \in \mathbb{R}^2$ get send under the composition $R_{Q,-\pi/2} \circ R_{P,\pi/2}$?

   (d) (5 points) Show that $R_{Q,-\pi/2} \circ R_{P,\pi/2} = t_{(1,1)}$.

2. (20 points) (**Reflections in** $\mathbb{R}^2$) Consider the two lines $L_0 = \{y = 0\}, L_1 = \{x = y\} \subseteq \mathbb{R}^2$ and the two lines $M_0 = \{x = y + 1\}, M_1 = \{x = -y + 1\} \subseteq \mathbb{R}^2$.

   (a) (5 points) Show that the only fixed point of the isometry $\overline{r}_{L_1} \circ \overline{r}_{L_0}$ is $(0, 0)$.

   (b) (5 points) Prove that the isometry $\overline{r}_{M_1} \circ \overline{r}_{M_0} \circ \overline{r}_{L_1} \circ \overline{r}_{L_0}$ is a rotation.

   (c) (5 points) Show that there exist two lines $N_0, N_1 \subseteq \mathbb{R}^2$ such that

$$\overline{r}_{M_1} \circ \overline{r}_{M_0} \circ \overline{r}_{L_1} \circ \overline{r}_{L_0} = \overline{r}_{N_1} \circ \overline{r}_{N_0}.$$

   (d) (5 points) Find the image of a point $(x, y) \in \mathbb{R}^2$ under the isometry $\overline{r}_{M_0} \circ \overline{r}_{L_1}$.

3. (20 points) (Γ**-Geometry for the 2-Torus**) Let $T^2 = \mathbb{R}^2/\Gamma$ be the Euclidean Torus, where $\Gamma = \langle t_{(0,1)}, t_{(1,0)} \rangle \subseteq \mathrm{Iso}(\mathbb{R}^2)$ is the group generated by the two translations

$$t_{(0,1)}, t_{(1,0)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$

(a) (5 points) Draw the Γ-orbits of the two points $P = (2,3), Q = (0.5, -7.5) \in \mathbb{R}^2$.

(b) (5 points) Find a fundamental domain $D_\Gamma \subseteq \mathbb{R}^2$ which contains $P \in \mathbb{R}^2$.

(c) (5 points) Consider $P = (2,3), Q = (0.5, -7.5) \in \mathbb{R}^2/\Gamma$ as points in the 2-torus. Show that the line $\{(x,y) \in T^2 : x = y\} \subseteq T^2$ contains both P and Q.

(d) (5 points) Find *all* lines $L \subseteq T^2$ such that $P, Q \in L$.

4. (20 points) (**Geometry in the Twisted Cylinder**) In this problem, *all* points and lines are considered in the twisted cylinder $M = \mathbb{R}^2/\Gamma$, where $\Gamma = \langle t_{(1,0)} \circ \overline{r} \rangle \subseteq \text{Iso}(\mathbb{R}^2)$. Solve the following parts:

   (a) (5 points) Consider the points $P = (0,0), Q = (0.9, 0.2), R = (5.9, -0.2) \in M$. Find the three distances $d(P,Q), d(P,R), d(Q,R) \in M$.

   (b) (5 points) Find the intersection points between the line $\{(x,y) \in M : x = 0.5\} \subseteq M$ and the line $\{(x,y) \in M : x = -y\} \subseteq M$.

   (c) (5 points) Find two lines $K, L \subseteq M$ such that $|L \cap K| = 2$.

   (d) (5 points) Show that given two points $S, T \in M$ in the complement of the line $H = \{(x,y) \in M : y = 0\} \subseteq M$, there exists a continuous path $\gamma \subseteq M$ from S to T such that $|H \cap \gamma| = 0$.

5. (20 points) For each of the ten sentences below, circle whether they are **true** or **false**. You do *not* need to justify your answer.

(a) (2 points) Two lines $K, L \subseteq T^2$ cannot intersect at more than one point.

(1) True. (2) False.

(b) (2 points) Let $\Gamma \subseteq \text{Iso}(\mathbb{R}^2)$ be a discontinuous subgroup. Then an isometry $g \in \Gamma$ cannot have fixed points.

(1) True. (2) False.

(c) (2 points) The composition of an even number of reflections cannot be a reflection.

(1) True. (2) False.

(d) (2 points) A glide reflection admits infinitely many fixed points.

(1) True. (2) False.

(e) (2 points) For any glide reflection $\overline{r}_1 \in \text{Iso}(\mathbb{R}^2)$, there exists a glide reflection $\overline{r}_2 \in \text{Iso}(\mathbb{R}^2)$ such that $\overline{r}_2 \circ \overline{r}_1 = \text{Id}$.

(1) True. (2) False.

(f) (2 points) An isometry $f \in \text{Iso}(\mathbb{R}^2)$ which fixes $(0,0), (3,4), (-6,6) \in \mathbb{R}^2$ must send the point $(-5, 9.8)$ to the point $(-5, 9.8)$, i.e. it will fix the point $(-5, 9.8)$.

(1) True. (2) False.

(g) (2 points) For any pair of points $P, Q \in K$ in the Klein bottle, there are infinitely many distinct lines $L \subseteq K$ containing $P, Q \in K$.

(1) True. (2) False.

(h) (2 points) There exist rotations $R_{P,\theta}, R_{Q,\phi} \in \text{Iso}(\mathbb{R}^2)$ such that the composition $R_{P,\theta} \circ R_{Q,\phi}$ is *not* a rotation.

(1) True. (2) False.

(i) (2 points) Two lines $L, K \subseteq M$ in the twisted cylinder either intersect 0,1 or infinitely many times.

(1) True. (2) False.

(j) (2 points) Let $\Gamma \subseteq \text{Iso}(\mathbb{R}^2)$ be generated by a finite number of translations. Then there exists a fundamental domain $D_\Gamma \subseteq \mathbb{R}^2$ of finite area.

(1) True. (2) False.