University of California Davis
Euclidean Geometry MAT 141

Name (Print): Solutions
Student ID (Print):

Sample Midterm Examination
Time Limit: 50 Minutes

February 7 2020

This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- (A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.
- (B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.
- (C) **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- (D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	0	
4	20	
5	20	
Total:	80	

1. (20 points) (**Rotations in $\mathbb{R}^2$**) Consider the two points $P = (0,0), Q = (1,0) \in \mathbb{R}^2$ in the Euclidean plane. Solve the following parts:

   (a) (5 points) Let $R_{P,\pi/2}$ be a rotation of angle $\pi/2$ centered at P. Compute the image $R_{P,\pi/2}(3,3)$ of the point $(3,3) \in \mathbb{R}^2$ under the isometry $R_{P,\pi/2}$.

$$R_{P,\frac{\pi}{2}} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\longrightarrow R_{P,\frac{\pi}{2}}(3,3) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \boxed{(-3, 3)}$$

   (b) (5 points) Let $R_{Q,-\pi/2}$ be a rotation of angle $-\pi/2$ centered at Q. Compute the image $R_{Q,-\pi/2}(4,5)$ of the point $(4,5) \in \mathbb{R}^2$ under the isometry $R_{Q,-\pi/2}$.

$$R_{Q,-\frac{\pi}{2}} = t_{(1,0)} R_{-\frac{\pi}{2}} \left(t_{(1,0)}\right)^{-1} = t_{(1,0)} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} t_{(-1,0)}$$

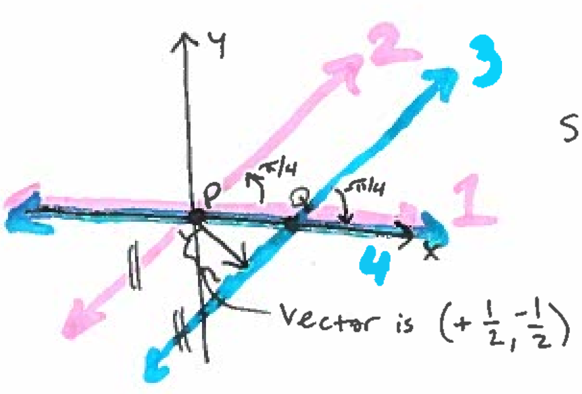
$$(4,5) \xmapsto{t_{(-1,0)}} (3,5) \xmapsto{R_{-\pi/2}} (5,-3) \xmapsto{t_{(1,0)}} \boxed{(6,-3)}$$

   (c) (5 points) Let $(x,y) \in \mathbb{R}^2$ be any point. Where does $(x,y) \in \mathbb{R}^2$ get send under the composition $R_{Q,-\pi/2} \circ R_{P,\pi/2}$?

By part (d), $R_{Q,-\frac{\pi}{2}} \circ R_{P,\frac{\pi}{2}}(x,y) = t_{(1,1)}(x,y)$

$$= \boxed{(x+1, y+1)}$$

   (d) (5 points) Show that $\boxed{R_{Q,-\pi/2}} \circ \boxed{R_{P,\pi/2}} = t_{(+1,1)}$.



So $R_{Q,-\frac{\pi}{2}} \circ R_{P,\frac{\pi}{2}} = \bar{r}\, t_{(1,-1)}\, \bar{r}$

and $\bar{r}\, t_{(1,-1)}\, \bar{r}\,(x,y) = \bar{r}\, t_{(1,-1)}(x,-y)$

$= \bar{r}(x+1, -y-1)$

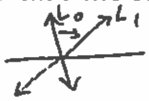
$= (x+1, y+1)$

$= t_{(1,1)}(x,y)$

So $R_{Q,-\pi} \circ R_{P,\pi}$ ... t

2. (20 points) (**Reflections in $\mathbb{R}^2$**) Consider the two lines $L_0 = \{y = 0\}, L_1 = \{x = y\} \subseteq \mathbb{R}^2$ and the two lines $M_0 = \{x = y + 1\}, M_1 = \{x = -y + 1\} \subseteq \mathbb{R}^2$.

(a) (5 points) Show that the only fixed point of the isometry $\overline{r}_{L_1} \circ \overline{r}_{L_0}$ is $(0, 0)$.

Draw lines: 

, so $\overline{r}_{L_1} \circ \overline{r}_{L_0} = R_{-\frac{\pi}{2}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

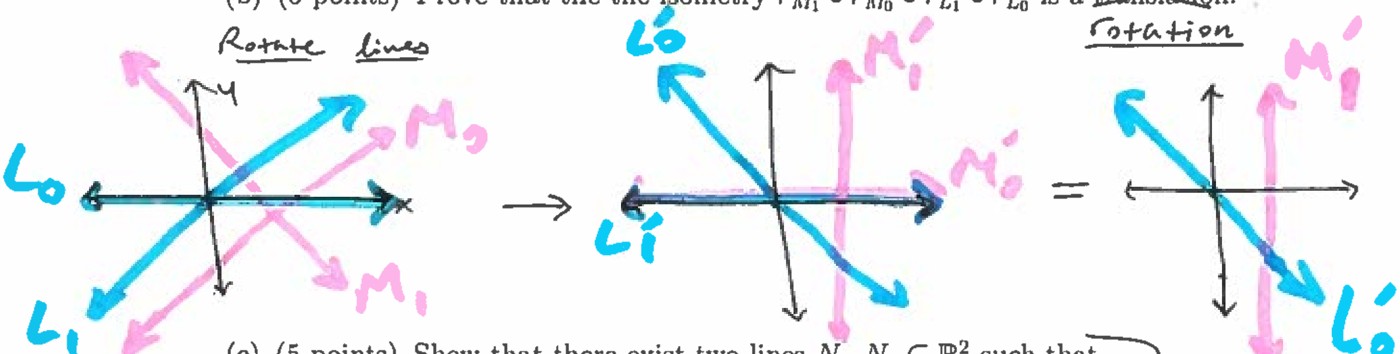
If $(x,y) \in \mathbb{R}^2$ is a fixed pt, then $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$,

So $x = y = -x$, so $x = 0$, $y = 0$,

So $(0,0)$ is the only fixed pt.

(b) (5 points) Prove that the the isometry $\overline{r}_{M_1} \circ \overline{r}_{M_0} \circ \overline{r}_{L_1} \circ \overline{r}_{L_0}$ is a ~~translation~~ rotation.

Rotate lines



$= \overline{r}_{M_1'} \circ \overline{r}_{L_0'}$

$= \boxed{\text{Rotation}}$

(c) (5 points) Show that there exist two lines $N_0, N_1 \subseteq \mathbb{R}^2$ such that

$$\overline{r}_{M_1} \circ \overline{r}_{M_0} \circ \overline{r}_{L_1} \circ \overline{r}_{L_0} = \overline{r}_{N_1} \circ \overline{r}_{N_0}.$$

$N_0 = L_0'$ above

$N_1 = M_1'$ above

(d) (5 points) Find the image of a point $(x, y) \in \mathbb{R}^2$ under the isometry $\overline{r}_{M_0} \circ \overline{r}_{L_1}$.

[redacted], [redacted]

[redacted]

L_1 and M_0 are parallel, with vector $(+\frac{1}{2}, -\frac{1}{2})$ from L_1 to M_0, so $\overline{r}_{M_0} \circ \overline{r}_{L_1}(x,y)$

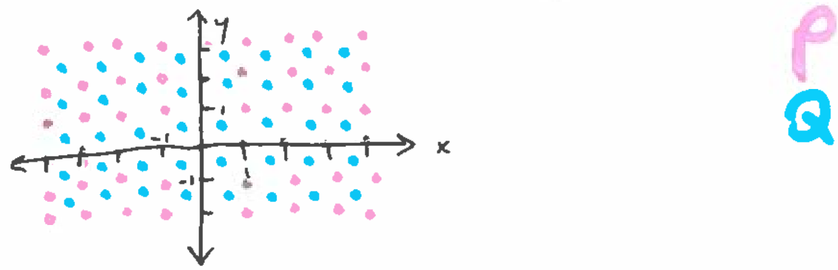
$= t_{(1,-1)}(x,y)$

$= \boxed{(x+1, y-1)}$

3. (20 points) (**Γ-Geometry for the 2-Torus**) Let $T^2 = \mathbb{R}^2/\Gamma$ be the Euclidean Torus, where $\Gamma = \langle t_{(0,1)}, t_{(1,0)} \rangle \subseteq \mathrm{Iso}(\mathbb{R}^2)$ is the group generated by the two translations

$$t_{(0,1)}, t_{(1,0)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$

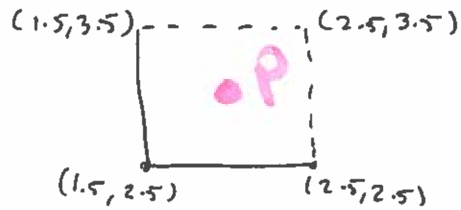
(a) (5 points) Draw the Γ-orbits of the two points $P = (2,3), Q = (0.5, -7.5) \in \mathbb{R}^2$.



(b) (5 points) Find a fundamental domain $D_\Gamma \subseteq \mathbb{R}^2$ which contains $P \in \mathbb{R}^2$.

Take $D_\Gamma = \{(x,y) \in \mathbb{R}^2 : 1.5 \leq x < 2.5,\ 2.5 \leq y < 3.5\} \subseteq \mathbb{R}^2$

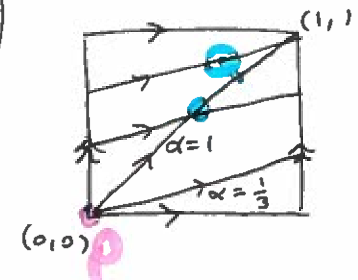
(many choices work)



(c) (5 points) Consider $P = (2,3), Q = (0.5, -7.5) \in \mathbb{R}^2/\Gamma$ as points in the 2-torus. Show that the line $\{(x,y) \in T^2 : x = y\} \subseteq T^2$ contains both P and Q.

In T^2, $P = (2,3) = (0,0) \in \{x = y\}$

In T^2, $Q = (0.5, -7.5) = (0.5, 0.5) \in \{x = y\}$



(d) (5 points) Find *all* lines $L \subseteq T^2$ such that $P, Q \in L$.

Any line connecting blue to pink in (a).

Use algebraic techniques or work in a fundamental domain

- Line must go through origin, so must be $\{y = \alpha x\} \subseteq T^2$.
- In $\mathbb{R}^2$, line should hit some $(0.5+n, 0.5+m)$ for $n, m \in \mathbb{Z}$

So $0.5 + m = \alpha(0.5+n)$, so $\alpha = \dfrac{2m+1}{2n+1}$, (all different slopes).

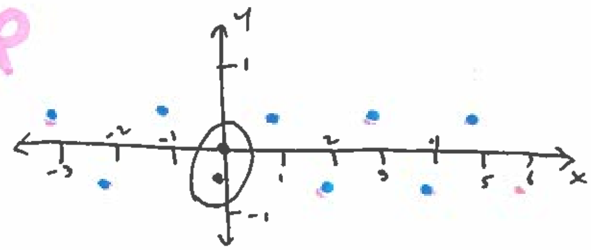
Lines are of the form $\{py = qx : p, q \in \mathbb{Z}$ odd$\}$

4. (20 points) (**Geometry in the Twisted Cylinder**) In this problem, *all* points and lines are considered in the twisted cylinder $M = \mathbb{R}^2/\Gamma$, where $\Gamma = \langle t_{(1,0)} \circ \overline{r} \rangle \subseteq \mathrm{Iso}(\mathbb{R}^2)$. Solve the following parts:

Q

P

(a) (5 points) Consider the points $P = (0,0), Q = (0.9, 0.2), R = (5.9, -0.2) \in M$. Find the three distances $d(P,Q), d(P,R), d(Q,R) \in M$.



Note $Q = R$, so $d(Q,R) = \boxed{0.}$

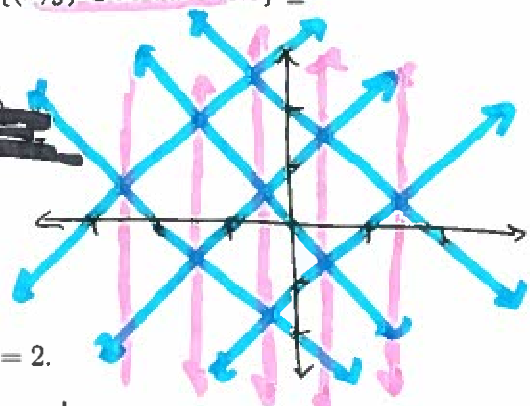
$d(P,Q) = d(P,R) = \boxed{\sqrt{(0.1)^2 + (0.2)^2}}$

L_1

(b) (5 points) Find the intersection points between the line $\{(x,y) \in M : x = 0.5\} \subseteq M$ and the line $\{(x,y) \in M : x = -y\} \subseteq M$.

L_2.

[redacted]

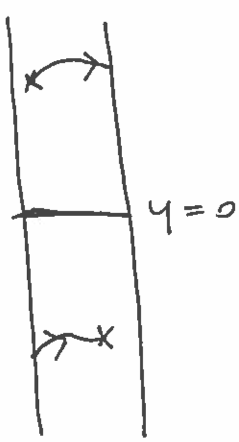


All crossings in $\mathbb{R}^2$ are glides of $\boxed{\{(0.5, -0.5 + 2n) : n \in \mathbb{Z}\}}$ These are all disstinct in M.

(c) (5 points) Find two lines $K, L \subseteq M$ such that $|L \cap K| = 2$.

$K = \{y = 1\}$ and $L = \{x = 0\}$ work

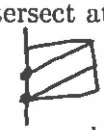
(d) (5 points) Show that given two points $S, T \in M$ in the complement of the line $H = \{(x,y) \in M : y = 0\} \subseteq M$, there exists a continuous path $\gamma \subseteq M$ from S to T such that $|H \cap \gamma| = 0$.



Since both halves can be connected, clearly any 2 points with $y \neq 0$ can be connected.

5. (20 points) For each of the ten sentences below, circle whether they are **true** or **false**. You do *not* need to justify your answer.

(a) (2 points) Two lines $K, L \subseteq T^2$ cannot intersect at more than one point.



(1) True. (2) False.

(b) (2 points) Let $\Gamma \subseteq \mathrm{Iso}(\mathbb{R}^2)$ be a discontinuous subgroup. Then an isometry $g \in \Gamma$ cannot have fixed points.

(1) True. (2) False.

The identity is in Γ. It fixes every point.

(c) (2 points) The composition of an even number of reflections cannot be a reflection.

(1) True. (2) False.

$\mathrm{Iso}^+(\mathbb{R}^2)$ is a closed group. (as are all groups).

(d) (2 points) A glide reflection admits infinitely many fixed points.

(1) True. (2) False.

Any non pure reflection glide fixes no points.

(e) (2 points) For any glide reflection $\overline{r}_1 \in \mathrm{Iso}(\mathbb{R}^2)$, there exists a glide reflection $\overline{r}_2 \in \mathrm{Iso}(\mathbb{R}^2)$ such that $\overline{r}_2 \circ \overline{r}_1 = \mathrm{Id}$.

(1) True. (2) False.

Just reflect in the same line but translate backwards.

(f) (2 points) An isometry $f \in \mathrm{Iso}(\mathbb{R}^2)$ which fixes $(0,0), (3,4), (-6,6) \in \mathbb{R}^2$ must send the point $(-5, 9.8)$ to the point $(-5, 9.8)$, i.e. it will fix the point $(-5, 9.8)$.

(1) True. (2) False.

These points are not collinear, so f is uniquely determined. $f = \mathrm{Id}$.

(g) (2 points) For any pair of points $P, Q \in K$ in the Klein bottle, there are infinitely many distinct lines $L \subseteq K$ containing $P, Q \in K$.

(1) True. (2) False.

Draw the lattice of points in $\mathbb{R}^2$. ∞-many slopes are possible.

(h) (2 points) There exist rotations $R_{P,\theta}, R_{Q,\phi} \in \mathrm{Iso}(\mathbb{R}^2)$ such that the composition $R_{P,\theta} \circ R_{Q,\phi}$ is *not* a rotation.

(1) True. (2) False.

See HW2, Problem 4, or 6.

Rot $\circ$ Rot = Rotation OR Translation.

(i) (2 points) Two lines $L, K \subseteq M$ in the twisted cylinder either intersect 0,1 or infinitely many times.

(1) True. (2) False.

See 4(c) above.

(j) (2 points) Let $\Gamma \subseteq \mathrm{Iso}(\mathbb{R}^2)$ be generated by a finite number of translations. Then there exists a fundamental domain $D_\Gamma \subseteq \mathbb{R}^2$ of finite area.

(1) True. (2) False.

$\Gamma = \langle t_{(1,0)}, t_{(\sqrt{2},0)} \rangle$

has no such D_Γ