This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there are any pages missing, in which case you should let the instructor know. Fill in all the requested information on the top of this page, and put your initials at the top of every page, in case the pages become separated.

You may not use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this and explain why the theorem may be applied.

(B) Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) Mysterious or unsupported answers will not receive full credit. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.
1. (20 points) (Isometries in \( \mathbb{R}^2 \)) Consider the three points \( P = (0, 0), Q = (1, 0), R = (0, 1) \in \mathbb{R}^2 \) in the Euclidean plane. Let \( f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \) be an isometry such that \( f(P) = (2, 2), f(Q) = (2, 3) \) and \( f(R) = (3, 2) \).

(a) (5 points) Find the images \( f(-1, 0) \) and \( f(8, 2) \) of the points \((-1, 0)\) and \((8, 2)\) under the isometry \( f \).

(b) (5 points) Prove that the isometry \( f \) is not a translation, i.e. there exists no vector \((\alpha, \beta) \in \mathbb{R}^2\) such that \( f = t_{(\alpha, \beta)} \).

(c) (5 points) Show that there exists no point \( S \in \mathbb{R}^2 \) such that \( f(S) = S \).

(d) (5 points) Find a set of at most three reflection \( \{r_{L_1}, r_{L_2}, r_{L_3}\} \in \text{Iso}(\mathbb{R}^2) \) such that \( f \) is a composition of these reflections.
2. (20 points) (Properties of $\Gamma \subseteq \text{Iso}(\mathbb{R}^2)$) Solve the following parts:

(a) (5 points) Show that $\Gamma = \langle t_{(2,3)}, r \circ t_{(1,0)} \rangle$ is fixed point free.

(b) (5 points) Let $L = \{(x, y) \in \mathbb{R}^2 : x = y\}$ and $M = \{(x, y) \in \mathbb{R}^2 : x = 6\}$. Find an element $g \in \Gamma = \langle \overline{r}_L, \overline{r}_M \rangle$ which has a unique fixed point.

(c) (5 points) Show that $\Gamma = \langle t_{(2,3)}, t_{(-3,9)} \rangle$ is discontinuous.

(d) (5 points) Find two elements $g_1, g_2$ in the group

$$\Gamma := \langle t_{(-4,6)}, t_{(-3,9)}, t_{(5,-15)}, t_{(2,-3)}, t_{(-1,3)}, t_{(1,-1.5)} \rangle$$

which generate $\Gamma$, i.e. such that $\Gamma = \langle g_1, g_2 \rangle$. 

3. (20 points) **(Γ-Geometry for the Klein Bottle)** Let \( K = \mathbb{R}^2 / \Gamma \) be the Euclidean Klein Bottle, where \( \Gamma = \langle t_{(0,1)}, \rho \circ t_{(1,0)} \rangle \subseteq \text{Iso}(\mathbb{R}^2) \).

(a) (5 points) Draw the \( \Gamma \)-orbits of the following points:

\[ P = (0, 0), Q = (0.5, 2), R = (1, -5), S = (3, -232) \in \mathbb{R}^2. \]

(b) (5 points) Find a fundamental domain \( D_\Gamma \subseteq \mathbb{R}^2 \) which is not a square.

(c) (5 points) Consider the lines

\[ L = \{(x, y) \in K : x = 2y\}, \quad M = \{(x, y) \in K : x = 0\}. \]

Find all the intersection points \( L \cap M \).

(d) (5 points) Consider the line \( N = \{(x, y) \in K : x = \pi \cdot y\} \). Is the number of intersection points \( M \cap N \) finite or infinite?
4. (20 points) (The Cylinder) In this problem, all points and lines are considered in the cylinder $C = \mathbb{R}^2 / \Gamma$, where $\Gamma = \langle t_{(1,0)} \rangle \subseteq \text{Iso}(\mathbb{R}^2)$. Solve the following parts:

(a) (5 points) Consider the points $P = (0.5, 0), Q = (0.3, 0.2), R = (5.9, -0.2) \in M$. Find an isometry $f : C \longrightarrow C$ such that

$$f(P) = (0.7, 0), \quad f(Q) = (0.5, -0.2), \quad f(R) = (6.1, 0.2).$$

(b) (5 points) Find infinitely many distinct lines $\{L_i\} \subseteq C$, $i \in \mathbb{N}$, which contain $P, Q$, i.e. $P, Q \in L_i$, for all $i \in \mathbb{N}$.

(c) (5 points) Let $t_{(0,\pi)} : C \longrightarrow C$ be a vertical translation, and $H = \langle t_{(0,\pi)} \rangle$ the group of isometries of $C$ it generates. Does the $H$-orbit of the point $R \in C$ have limit points in the cylinder $C$? (Justify your answer.)

(d) (5 points) Consider the group $A = \langle t_{(0,\sqrt{2})}, t_{(0,1)} \rangle$ as a subgroup of the group of isometries of $C$. Prove that the $A$-orbit of $P$ inside the cylinder $C$ has limit points.
5. (20 points) For each of the five sentences below, circle the unique correct answer. You do not need to justify your answer.

(a) (2 points) Let \((0, 0), (0.5, 0) \in C\) be two points in the cylinder. The set of points equidistant to \((0, 0)\) and \((0.5, 0)\) consists of exactly:

(1) A line, (2) Empty (3) Two lines (4) Infinite Lines

(b) (2 points) Two lines \(L, M \subseteq T^2\) in the two torus must have:

(1) Finitely Many Intersection Points (2) Infinitely Many Intersection Points

(3) No Intersection Points. (4) None of the other answers.

(c) (2 points) A non-trivial subgroup \(\Gamma \subseteq \text{Iso}(\mathbb{R}^2)\) must:

(1) Contain a non-trivial translation, (2) Be generated by at most two elements,

(3) Be fixed point free, (4) Contain a product of reflections.

(d) (2 points) There exists a unique isometry which fixes

(1) Three collinear points (2) Three non-collinear points

(3) Four collinear points (4) The origin.

(e) (2 points) Let \(\Gamma \subseteq \text{Iso}(\mathbb{R}^2)\) be an arbitrary discontinuous and fixed point free subgroup. Then the group \(\Gamma\)

(1) Cannot contain more than 2 non-trivial translations,

(2) Cannot contain a non-trivial rotation,

(3) Contains a non-trivial glide reflection,

(4) Necessarily has a limit point.