This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there are any pages missing, in which case you should let the instructor know. Fill in all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may not use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this and explain why the theorem may be applied.

(B) Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) Mysterious or unsupported answers will not receive full credit. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.
1. (20 points) (Isometries in $\mathbb{R}^2$) Consider the three points $P = (0, 0), Q = (1, 0), R = (0, 1) \in \mathbb{R}^2$ in the Euclidean plane. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an isometry such that $f(P) = (2, 2), f(Q) = (2, 3)$ and $f(R) = (3, 2)$.

(a) (5 points) Find the images $f(-1,0)$ and $f(8,2)$ of the points $(-1,0)$ and $(8,2)$ under the isometry $f$.

(b) (5 points) Prove that the isometry $f$ is not a translation, i.e. there exists no vector $(\alpha, \beta) \in \mathbb{R}^2$ such that $f = t(\alpha, \beta)$.

(c) (5 points) Show that there exists no point $S \in \mathbb{R}^2$ such that $f(S) = S$.

(d) (5 points) Find a set of at most three reflections $\{R_{L_1}, R_{L_2}, R_{L_3}\} \in \text{Iso}(\mathbb{R}^2)$ such that $f$ is a composition of these reflections.
2. (20 points) (Properties of $\Gamma \subseteq \text{Iso}(\mathbb{R}^2)$) Solve the following parts:

(a) (5 points) Show that $\Gamma = \langle t_{(2,3)}, \alpha \circ t_{(1,0)} \rangle$ is fixed point free.

Since \((t \circ t_{(1,0)})^n(t_{(2,3)})^m = t_{(2m, 3m)}(t \circ t_{(1,0)})^n\),
we can write all group elements as \((t_{(2,3)})^n(t_{(2,-3)})^m(t \circ t_{(1,0)})^n\). All fix no points if \(d \neq 0\). If \(d = 0\), then \(w = 0\), and so fixes no fixed pt.

(b) (5 points) Let $L = \{(x, y) \in \mathbb{R}^2 : x = y\}$ and $M = \{(x, y) \in \mathbb{R}^2 : x = 6\}$. Find an element $g \in \Gamma = \langle \overline{F}, \overline{F} \rangle$ which has a unique fixed point.

An orbit of \((x, y) \in \mathbb{R}^2\): \(\Gamma(x, y) = \{(x + 2n - 3k, y + 3n - 9k) : n, k \in \mathbb{Z}\}\).

Two such orbit points \((2n - 3m, 3n - 9m)\) for \(n' = n, m = k' - k\). This vector has length at least 1 (because $\gcd(2,3) = 1$), so no points in the orbit are within distance 1 of each other.

(d) (5 points) Find two elements $g_1, g_2$ in the group $\langle 1, 1 \rangle$ which generate $\Gamma$, i.e. such that $\Gamma = \langle g_1, g_2 \rangle$.

- $t_{(5, -15)} = (t_{(1, -3)})^5$ so it is redundant.
- $t_{(2, -3)} = (t_{(1, -15)})^2$.
- $t_{(-3, 9)} = (t_{(1, 3)})^3$.
- $t_{(-4, 6)} = (t_{(1, 15)})^4$.

So \[ \Gamma = \langle t_{(-1, 1)}, t_{(1, -15)} \rangle \]
3. (20 points) (\(\Gamma\)-Geometry for the Klein Bottle) Let \(K = \mathbb{R}^2/\Gamma\) be the Euclidean Klein Bottle, where \(\Gamma = \langle t_{(0,1)}, t_{(1,0)} \rangle \subseteq \text{Iso}(\mathbb{R}^2)\).

(a) (5 points) Draw the \(\Gamma\)-orbits of the following points:

\[ P = (0, 0), \ F = (0.5, 2), \ R = (1, -5), \ S = (3, -2.3) \in \mathbb{R}^2. \]

(b) (5 points) Find a fundamental domain \(D_\Gamma \subseteq \mathbb{R}^2\) which is not a square.

Chop the square in 2, and move one piece to another copy of itself, s.t. new propagations don’t intersect.

\[ D_\Gamma = \{ (x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1, \ 0 \leq y \leq x^2 \leq 1 \} \text{ works.} \]

(c) (5 points) Consider the lines

\[ L = \{ (x, y) \in K : x = 2y \}, \quad M = \{ (x, y) \in K : x = 0 \}. \]

Find all the intersection points \(L \cap M\).

\[ \pi(L) = \left\{ y = \frac{1}{2} x + m \right\} \cup \left\{ y = -\frac{1}{2} x + n \right\} \]

\[ \pi(M) = \{ x = k \} \]

\[ \pi(M) \cap \pi(L) = \{ (k, \frac{1}{2} k), (k, -\frac{1}{2} k + \frac{1}{2}) : k \in \mathbb{Z} \} \]

\[ = \{ (0, 0), (0, \frac{1}{2}) \} \]

(d) (5 points) Consider the line \(N = \{ (x, y) \in K : x = \pi \cdot y \}. \) Is the number of intersection points \(M \cap N\) finite or infinite?

Infinite. Crossings include at least

\[ \{ x = n^2 \} \cap \{ x = \pi y^2 \} = \{ (n, \frac{n}{\pi}) : n \in \mathbb{Z} \}. \]

None of which are related by group elements.
4. (20 points) **(The Cylinder)** In this problem, all points and lines are considered in the cylinder \( C = \mathbb{R}^2 / \Gamma \), where \( \Gamma = \langle (t(1,0)) \rangle \subseteq \text{Iso}(\mathbb{R}^2) \). Solve the following parts:

(a) (5 points) Consider the points \( P = (0.5,0), Q = (0.3,0.2), R = (5.9,-0.2) \in \mathcal{M} \).

Find an isometry \( f : C \rightarrow C \) such that

\[
f(P) = (0.7,0), \quad f(Q) = (0.5,-0.2), \quad f(R) = (6.1,0.2).
\]

(b) (5 points) Find infinitely many distinct lines \( \{ L_i \} \subseteq C, i \in \mathbb{N} \), which contain \( P, Q \), i.e. \( P, Q \in L_i \), for all \( i \in \mathbb{N} \).

\[
\text{Line through } (0.5+i,0) \text{ and (0.3,0.2)}
\]

**Slope is** \[
\frac{0.2}{-0.2 - i} \quad \text{so} \quad L_i = \left\{ \left( y - 0.2 \right) = \frac{0.2}{-0.2 - i} (x - 0.3) \right\}
\]

(c) (5 points) Let \( t_{(0,\pi)} : C \rightarrow C \) be a vertical translation, and \( H = \langle t_{(0,\pi)} \rangle \) the group of isometries of \( C \) it generates. Does the \( H \)-orbit of the point \( R \in C \) have limit points in the cylinder \( C \)? (Justify your answer.)

\[
\text{No. } H \cdot R = f (5.9, -0.2 + \pi n) : n \in \mathbb{Z}, \quad \text{and no two points in this set in } C \text{ are within distance 1.}
\]

(Note, we would have limit points from \( t_{(\pi,0)} \).

(d) (5 points) Consider the group \( A = \langle t_{(0,\sqrt{2})}, t_{(0,1)} \rangle \) as a subgroup of the group of isometries of \( C \). Prove that the \( A \)-orbit of \( P \) inside the cylinder \( C \) has limit points.

See the proof of the Theorem on pp. 33-34 in Stillwell.
5. (20 points) For each of the five sentences below, circle the unique correct answer. You do not need to justify your answer.

(a) (2 points) Let \((0,0),(0.5,0)\) be two points in the cylinder. The set of points equidistant to \((0,0)\) and \((0.5,0)\) consists of exactly:

(1) A line, (2) Empty (3) Two lines (4) Infinite Lines

(b) (2 points) Two lines \(L,M \subseteq T^2\) in the two torus must have:

(1) Finitely Many Intersection Points (2) Infinitely Many Intersection Points (3) No Intersection Points (4) None of the other answers. All are possible

(c) (2 points) A non-trivial subgroup \(\Gamma \subseteq \text{Iso}(\mathbb{R}^2)\) must:

(1) Contain a translation, (2) Be generated by at most two elements, \(\langle t(0,0) \rangle \) works.
(3) Be fixed point free, \(\langle R \rangle \) isn't (4) Contain a product of reflections. All isometries are a product \((1,2,3)\) and reflection. (Note, (1) and (4) are actually true because \(\text{Iso}(\mathbb{R}^2)\))

(d) (2 points) There exists a unique isometry which fixes

(1) Three collinear points (2) Three non-collinear points Theorem in Stillwell
(3) Four collinear points (4) The origin. \((1,3,4)\) all fixed by \(1\) and reflection.

(e) (2 points) Let \(\Gamma \subseteq \text{Iso}(\mathbb{R}^2)\) be an arbitrary discontinuous and fixed point free subgroup. Then the group \(\Gamma\)

(1) Cannot contain more than two translations, (2) contains a glide reflection,
(3) Cannot contain a rotation, \(\text{except identity}\) (4) Necessarily has a limit point. Non-trivial rotation would fix a point.