University of California Davis
Euclidean Geometry MAT 141

Name (Print): Solutions
Student ID (Print):

Sample Midterm Examination II
Time Limit: 50 Minutes

February 7 2020

This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	0	
4	20	
5	20	
Total:	80	

1. (20 points) (**Isometries in $\mathbb{R}^2$**) Consider the three points $P = (0,0), Q = (1,0), R = (0,1) \in \mathbb{R}^2$ in the Euclidean plane. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an isometry such that $f(P) = (2,2)$, $f(Q) = (2,3)$ and $f(R) = (3,2)$.

   (a) (5 points) Find the images $f(-1,0)$ and $f(8,2)$ of the points $(-1,0)$ and $(8,2)$ under the isometry f.

P, Q, R, are not collinear, so f is uniquely determined. Note that, for $L_1 = \{x=y\} \subseteq \mathbb{R}^2$, $\overline{r}_L t_{(2,2)}$ achieves all values given, so $f = \overline{r}_{L_1} t_{(2,2)}$.

$f(-1,0) = \overline{r}_{L_1} t_{(2,2)}(-1,0) = \overline{r}_{L_1}(1,2) = \boxed{(2,1)}$

$f(8,2) = \overline{r}_{L_1} t_{(2,2)}(8,2) = \overline{r}_{L_1}(10,4) = \boxed{(4,10)}$

   (b) (5 points) Prove that the isometry f is not a translation, i.e. there exists no vector $(\alpha, \beta) \in \mathbb{R}^2$ such that $f = t_{(\alpha,\beta)}$.

f is orientation reversing, but all translations preserve orientation.

   (c) (5 points) Show that there exists no point $S \in \mathbb{R}^2$ such that $f(S) = S$.

$f(x,y) = \overline{r}_{L_1} t_{(2,2)}(x,y) = \overline{r}_L(x+2, y+2) = (y+2, x+2)$,

so if (x,y) is fixed by f, then $x = y+2 = (x+2)+2 = x+4$,

impossible,

   (d) (5 points) Find a set of *at most* three reflection $\{\overline{r}_{L_1}, \overline{r}_{L_2}, \overline{r}_{L_3}\} \in \mathrm{Iso}(\mathbb{R}^2)$ such that f is a composition of these reflections.

$t_{(2,2)} = \overline{r}_{L_2} \circ \overline{r}_{L_3}$, where $L_3 = \{y = -x\}$

$L_2 = \{y - 1 = -(x-1)\}$

So $\boxed{f = \overline{r}_{L_1} \circ \overline{r}_{L_2} \circ \overline{r}_{L_3}}$

2. (20 points) (**Properties of $\Gamma \subseteq \mathrm{Iso}(\mathbb{R}^2)$**) Solve the following parts:

(a) (5 points) Show that $\Gamma = \langle t_{(2,3)}, \overline{r} \circ t_{(1,0)} \rangle$ is fixed point free.

Since $(\overline{r} \circ t_{(1,0)})^n (t_{(2,3)})^m = t_{(2m, (-1)^n 3m)} (\overline{r} \circ t_{(1,0)})^n$,

we can write all group elements as $(t_{(2,3)})^{n_1} (t_{(2,-3)})^{n_2} (\overline{r} \circ t_{(1,0)})^{n_3}$

$=$ $t_{(k,3\ell)}\, \overline{r}$ or $t_{(k,3\ell)}$, for some $k, \ell \in \mathbb{Z}$.

All fix no points if $\ell \neq 0$. If $\ell = 0$, then $m = 0$, and so just a glide $\Rightarrow$ no fixed pts.

(b) (5 points) Let $L = \{(x,y) \in \mathbb{R}^2 : x = y\}$ and $M = \{(x,y) \in \mathbb{R}^2 : x = 6\}$. Find an element $g \in \Gamma = \langle \overline{r}_L, \overline{r}_M \rangle$ which has a *unique* fixed point.

$\overline{r}_M \circ \overline{r}_L$ is a rotation, because L and M cross,

so it fixes only the intersection of L and M.

(c) (5 points) Show that $\Gamma = \langle t_{(2,3)}, t_{(-3,9)} \rangle$ is discontinuous.

An orbit of (x,y) is $\Gamma \cdot (x,y) = \{(x + 2n - 3k, y + 3n - 9k) : n, k \in \mathbb{Z}\}$.

Two such orbit points for $n, k \in \mathbb{Z}$ and $n', k' \in \mathbb{Z}$ have difference vector $(2\ell - 3m, 3\ell - 9m)$ for $\ell = n' - n$, $m = k' - k$. This vector has length at least 1 (because $\gcd(2,3) = 1$), so no points in the orbit are within distance 1 from each other. (no limit pts)

(d) (5 points) Find two elements g_1, g_2 in the group

$$\Gamma := \langle t_{(-4,6)}, t_{(-3,9)}, t_{(5,-15)}, t_{(2,-3)}, t_{(-1,3)}, t_{(1,-1.5)} \rangle$$

which generate Γ, i.e. such that $\Gamma = \langle g_1, g_2 \rangle$.

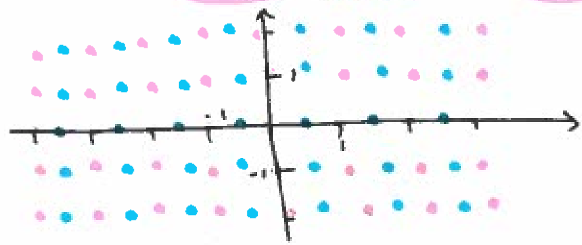
- $t_{(5,-15)} = (t_{(-1,3)})^{-5}$ so it is redundant
- $t_{(2,-3)} = (t_{(1,-1.5)})^2$ " " " "
- $t_{(-3,9)} = (t_{(-1,3)})^3$ " " " "
- $t_{(-4,6)} = (t_{(1,-1.5)})^{-4}$ " " " "

So $\boxed{\Gamma = \langle t_{(-1,3)}, t_{(1,-1.5)} \rangle}$

3. (20 points) (**Γ-Geometry for the Klein Bottle**) Let $K = \mathbb{R}^2/\Gamma$ be the Euclidean Klein Bottle, where $\Gamma = \langle t_{(0,1)}, \overline{r} \circ t_{(1,0)} \rangle \subseteq \mathrm{Iso}(\mathbb{R}^2)$.

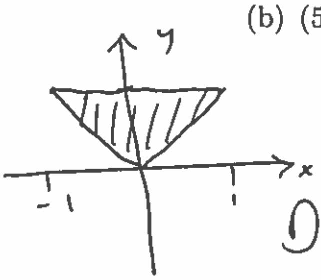
(a) (5 points) Draw the Γ-orbits of the following points:

$$P = (0,0), Q = (0.5, 2), R = (1, -5), S = (3, -232) \in \mathbb{R}^2.$$



Missing pink dots on axes
(didn't show on scan)

(b) (5 points) Find a fundamental domain $D_\Gamma \subseteq \mathbb{R}^2$ which is *not* a square.



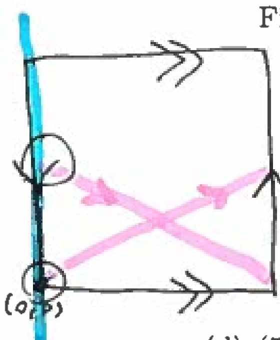
Chop the square in 2, and move one piece to another copy of itself, s.t. new propagations don't intersect.

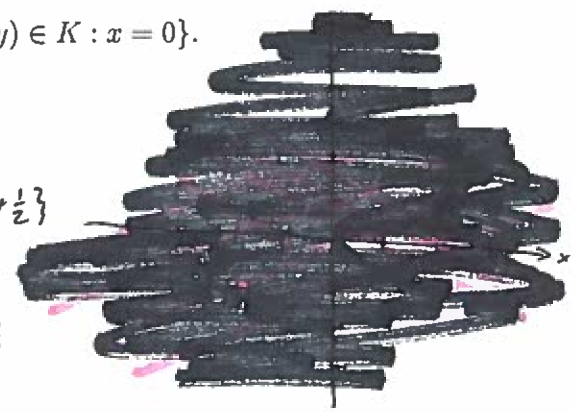
$D_\Gamma = \{(x,y) \in \mathbb{R}^2 \mid -1 \le x \le 1,\ 0 \le y \le x\} \subseteq \mathbb{R}$ works.

(c) (5 points) Consider the lines

$$L = \{(x,y) \in K : x = 2y\}, \quad M = \{(x,y) \in K : x = 0\}.$$

Find *all* the intersection points $L \cap M$.





(1,1) $\boxed{\text{Just } (0,0),\ (0, 0.5)}$

$\pi(L) = \{y = \frac{1}{2}x + n\} \cup \{y = -\frac{1}{2}x + n + \frac{1}{2}\}$

$\pi(M) = \{x = k\}$

$\pi(M) \cap \pi(L) = \{(k, \frac{k}{2}), (k, -\frac{k}{2} + \frac{1}{2}) : k \in \mathbb{Z}\}$

$= \{(0,0), (0, \frac{1}{2})\}$

(d) (5 points) Consider the line $N = \{(x,y) \in K : x = \pi \cdot y\}$. Is the number of intersection points $M \cap N$ finite or infinite ?

Infinite. Crossings include at least

$\{x = n\} \cap \{x = \pi y\} = \{(n, \frac{n}{\pi}) : n \in \mathbb{Z}\}$

None of which are related by group elements.

4. (20 points) (**The Cylinder**) In this problem, *all* points and lines are considered in the cylinder $C = \mathbb{R}^2/\Gamma$, where $\Gamma = \langle t_{(1,0)} \rangle \subseteq \mathrm{Iso}(\mathbb{R}^2)$. Solve the following parts:

(a) (5 points) Consider the points $P = (0.5, 0), Q = (0.3, 0.2), R = (5.9, -0.2) \in M$. Find an isometry $f : C \longrightarrow C$ such that

$$f(P) = (0.7, 0), \quad f(Q) = (0.5, -0.2), \quad f(R) = (6.1, 0.2).$$

$f = t_{(0.2,0)}\overline{r}$

(b) (5 points) Find infinitely many distinct lines $\{L_i\} \subseteq C$, $i \in \mathbb{N}$, which contain P, Q, i.e. $P, Q \in L_i$, for all $i \in \mathbb{N}$.

L_i = line through $(0.5+i, 0)$ and $(0.3, 0.2)$

slope is $\frac{0.2}{-0.2-i}$. So $\boxed{L_i = \left\{(y-0.2) = \frac{0.2}{-0.2-i}(x-0.3)\right\}}$

(c) (5 points) Let $t_{(0,\pi)} : C \longrightarrow C$ be a vertical translation, and $H = \langle t_{(0,\pi)} \rangle$ the group of isometries of C it generates. Does the H-orbit of the point $R \in C$ have limit points in the cylinder C ? (Justify your answer.)

No. $H \cdot R = \{(5.9, -0.2 + \pi n) : n \in \mathbb{Z}\}$, and no two points in this set in C are within distance 1.
(Note, we would have limit points from $t_{(\pi,0)}$).

(d) (5 points) Consider the group $A = \langle t_{(0,\sqrt{2})}, t_{(0,1)} \rangle$ as a subgroup of the group of isometries of C. Prove that the A-orbit of P inside the cylinder C has limit points.

See the proof of the Theorem on pp. 33-34 in Stillwell.

5. (20 points) For each of the five sentences below, circle the **unique** correct answer. You do *not* need to justify your answer.

Two lines

(a) (2 points) Let $(0,0), (0.5,0) \in C$ be two points in the cylinder. The set of points equidistant to $(0,0)$ and $(0.5,0)$ consists of exactly:

(1) A line, (2) Empty (3) Two lines (4) Infinite Lines

(b) (2 points) Two lines $L, M \subseteq T^2$ in the two torus must have:

rational ratio of slopes — irrational ratio of slopes

(1) Finitely Many Intersection Points (2) Infinitely Many Intersection Points

(3) No Intersection Points. (4) None of the other answers. All are possible

~~equal~~ parallel.

(c) (2 points) A non-trivial subgroup $\Gamma \subseteq \mathrm{Iso}(\mathbb{R}^2)$ must:

(implies (4))

(1) Contain a translation, (2) Be generated by at most two elements, $\langle t_{(1,0)} \rangle$ works.

(3) Be fixed point free, (4) Contain a product of reflections. all isometries are a product (1, 2, or 3) of reflections

$\langle R_\theta \rangle$ isn't

(Note, (1) and (4) are actually true since $Id \in \Gamma$)

(d) (2 points) There exists a unique isometry which fixes

(1) Three collinear points (2) Three non-collinear points Theorem in Stillwell

(3) Four collinear points (4) The origin.

(1, 3, 4) all fixed by Id and reflection.

(e) (2 points) Let $\Gamma \subseteq \mathrm{Iso}(\mathbb{R}^2)$ be an arbitrary discontinuous and fixed point free subgroup. Then the group Γ

(1) Cannot contain more than two translations, (2) contains a glide reflection,

(3) Cannot contain a rotation, (4) Necessarily has a limit point.

(except identity)

Non-trivial rotation would fix a point.