University of California Davis **Name (Print):** ____________
Algebraic Topology II MAT 215B **Student ID (Print):** ____________

Final Examination **March 14 2020**
Time Limit: Due March 17 at 9:00pm (81 Hours)

This examination document contains 5 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the Professor Casals know. Put your initials on the top of every page, as well as the Problem number, in case the pages become separated or are scanned incorrectly.

You are to submit this *take home* Final Examination for MAT 215B Algebraic Topology II via **Gradescope by Tuesday March 17 at 9:00pm**. Please make sure to write clearly and that the quality of the scanning allows for clear readability.

You are required to show your work on each problem on this exam. You are *not* allowed to discuss the problems of this examination with *anyone* until March 17 at 9:00pm. Answers should be complete and self-contained, never referring to any external resource, nor using mysterious knowledge coming from the Internet. You are welcome to use the material discussed in class and in the Problem Sets, just refer to it in a precise manner.

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) Let us consider the topological space

$$X = \mathbb{R}^3 \setminus (C_1 \cup C_2),$$

where C_1, C_2 are the 1-spheres defined by

$$C_1 = \{(x, y, z) \in \mathbb{R}^3 : x^2+y^2 = 1, z = 0\}, \quad C_2 = \{(x, y, z) \in \mathbb{R}^3 : (x-1)^2+z^2 = 1, y = 0\}.$$

(a) (15 points) Show that $X \simeq S^2 \vee T^2$.

(b) (10 points) Compute the homology $H_*(X)$.

2. (25 points) Let $X_d = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 : z_0^d + z_1^d + z_2^d = 0\} \subseteq \mathbb{CP}^2$, $d \in \mathbb{N}$, $d \geq 1$.

   (a) (6 points) Find the homology groups $H_*(X_d)$.

   Hint: Consider the decomposition $X = (X \setminus X \cap B) \cup (X \cap B)$ *where* B *is a projective line* $B = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 : z_0 = 0\} \subseteq \mathbb{CP}^2$.

   (b) (6 points) Describe the cohomology ring $H^*(X_n \times X_m, \mathbb{Z})$ for any $n, m \in \mathbb{N}^*$.

   (c) (6 points) Let $Y_2 = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 : z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0\} \subseteq \mathbb{CP}^3$. Show that $H^3(Y_2, \mathbb{Z}) = 0$.

   (d) (7 points) Show that the cohomology rings $H^*(X_2 \times Y_2, \mathbb{Z}) \cong H^*(S^2 \times S^2 \times S^2, \mathbb{Z})$ are isomorphic as commutative graded rings.

3. (25 points) Let us consider the $(2n+1)$-dimensional lens space $L_m = S^{2n+1}/\mathbb{Z}_m$, $n \geq 0$, where $\mathbb{Z}_m$ acts on $S^{2n+1} = \{(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} : |z_0|^2 + \ldots + |z_n|^2 = 1\}$ via

$$\mathbb{Z}_m \times S^{2n+1} \longrightarrow S^{2n+1}, \quad (k, z) \longmapsto e^{2\pi i k/m} \cdot z.$$

(a) (9 points) Compute the cohomology groups $H^*(L_m, \mathbb{Z}_q)$ for all $q \in \mathbb{N}$, $q \geq 2$.

Hint: Find a CW-structure on L_m with a unique cell $e : D^k \longrightarrow L_m$ for each $0 \leq k \leq 2n+1$. The case $L_2 = \mathbb{RP}^{2n+1}$ can be a useful example to keep in mind.

(b) (10 points) Describe the cohomology ring $H^*(L_m, \mathbb{Z}_m)$ under cup product.

(c) (6 points) Find the cohomology ring $H^*(K(\mathbb{Z}_m, 1), \mathbb{Z})$, where $K(\mathbb{Z}_m, 1)$ is the Eilenberg-MacLane space for the Abelian group $\mathbb{Z}_m$ in degree 1.

4. (25 points) All topological spaces are path-connected CW-complexes. For each of the sentences below, circle **true** or **false**. You do *not* need to justify your answer.

(a) (2.5 points) There exists a closed manifold M such that $H^*(M,\mathbb{Z}) \cong H^*(\mathbb{CP}^2,\mathbb{Z})$ as Abelian groups, but $H^*(M,\mathbb{Z}) \ncong H^*(\mathbb{CP}^2,\mathbb{Z})$ as rings.

(1) True. (2) False.

(b) (2.5 points) Let $\pi : M^3 \longrightarrow S^2$ be a fiber bundle with fiber S^1. Then it must be that $H^*(M^3,\mathbb{Z}) \cong H^*(S^2 \times S^1,\mathbb{Z})$ as Abelian groups.

(1) True. (2) False.

(c) (2.5 points) Let X be a compact topological space such that $H_*(X) = 0$ for $* \geq n$, and $H_c^k(X,\mathbb{Z}) \cong H_{n-k}(X)$ for all $0 \leq k \leq n$. Then X is a topological n-manifold.

(1) True. (2) False.

(d) (2.5 points) $\widetilde{H}_*((S^n \times S^m) \setminus \{pt\}) \cong \widetilde{H}_*(S^n) \oplus \widetilde{H}_*(S^m)$ for all $n,m \in \mathbb{N}$, $n,m \geq 1$.

(1) True. (2) False.

(e) (2.5 points) Let G be a finitely generated Abelian group with non-zero torsion. Then $K(G,1)$ does not admit a CW-decomposition with finitely many cells.

(1) True. (2) False.

(f) (2.5 points) The Euler characteristic $\chi(X \times S^7) = 0$ for *any* finite CW-complex X.

(1) True. (2) False.

(g) (2.5 points) For $n \geq 1$, the suspension $\Sigma\mathbb{CP}^n$ of the complex projective space $\mathbb{CP}^n$ is a topological manifold if and only if $n = 1$.

(1) True. (2) False.

(h) (2.5 points) Let G, H be finitely generated Abelian groups. If $Ext^0_{\mathbb{Z}}(G,H) = 0$ vanishes, then it must be that $Ext^1_{\mathbb{Z}}(G,H) \neq 0$ does not vanish.

(1) True. (2) False.

(i) (2.5 points) Let $T^n = S^1 \times \overset{(n)}{\cdots} \times S^1$ be the n-torus, $n \geq 1$. The cup products of the cohomology ring $H^*(\Sigma T^n,\mathbb{Z})$ are all zero.

(1) True. (2) False.

(j) (2.5 points) Let X be a closed orientable topological 6-manifold whose homology groups satisfy $H_*(X) \cong H_*(S^3 \times S^3)$. Then X is homotopy equivalent to $S^3 \times S^3$.

(1) True. (2) False.