MAT 215B: PROBLEM SET 1

DUE TO FRIDAY JAN 10 AT 3:00PM

Abstract. This is the first problem set for the graduate course Algebraic Topology II in the Winter Quarter 2020. It was posted online on Saturday Jan 4 and is due Friday Jan 10 at 3:00pm via online submission.

Purpose: The goal of this assignment is to review and practice the basic concepts from Algebraic Topology I (MAT215B). In particular, we would like to become familiar with many examples of topological spaces, including fiber bundles and covering spaces, as well as with the algebra appearing from homotopy groups..

Task: Solve Problems 1 through 7 below.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 20 points, five problems will be graded. The maximum possible grade is 100 points.

Textbook: We will use "Algebraic Topology" by A. Hatcher. Please contact me *immediately* if you have not been able to get a copy of any edition.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. Let X_1 be the thrice-punctured 2-sphere S^2 and X_2 the once-punctured 2-torus $T^2 = S^1 \times S^1$.

(a) Show that X_1 is *not* homeomorphic to X_2.

(b) Show that X_1 *is* homotopy equivalent to X_2,

(c) Compute the homotopy groups $\pi_*(X_1)$ and $\pi_*(X_2)$.

(d) Is $X_1 \times \mathbb{R}$ homeomorphic to $X_2 \times \mathbb{R}$?

Problem 2. Let $\Sigma_{g,n}$ an n-punctured genus-g surface.

(a) Find a finitely generated presentation for the group $\pi_1(\Sigma_{g,0})$.

(b) Find a presentation for the group $\pi_1(\Sigma_{g,n})$ if $n \geq 1$.

(c) Compute the higher-homotopy groups $\pi_*(\Sigma_{g,n})$ for all $g, n \in \mathbb{N}$ and $* \geq 2$.

Problem 3. Let $\mathbb{RP}^n$ be the real projective n-space.

(a) Show that the fundamental group is $\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2$.

(b) Give an explicit map $i : S^1 \longrightarrow \mathbb{RP}^n$ such that the induced morphism
$$i_* : \pi_1(S^1) \longrightarrow \pi_1(\mathbb{RP}^n)$$
maps the generator in $\pi_1(S^1) \cong \mathbb{Z}$ to the generator of $\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2$.

(c) Let $\mathbb{CP}^n$ be the *complex* projective n-space. Prove that $\pi_1(\mathbb{CP}^n) = 0$ and compute the second homotopy group $\pi_2(\mathbb{CP}^n)$.

(d) Given Part (c), the statements for Part (a) and Part (b) do not *exactly* translate from $\mathbb{RP}^n$ to $\mathbb{CP}^n$ in an interesting manner (since the fundamental groups are zero). State and solve the "correct" analogue of Parts (a) and (b) for $\mathbb{CP}^n$.

Problem 4. Consider the Hopf fibration $H : S^3 \longrightarrow S^2$.

(a) Write down the long exact sequence in homotopy associated to the fibration H.

(b) Show that the connecting morphism $\delta(H) : \pi_2(S^2) \longrightarrow \pi_1(S^1)$ in the long exact sequence in Part (a) is an isomorphism.

(c) Deduce from Part (a) that the homotopy class of the map H generates the third homotopy group $\pi_3(S^2) \cong \mathbb{Z}$.

(d) State the generalization of Part (c) for the general Hopf fibration
$$H : S^{2n+1} \longrightarrow \mathbb{CP}^n$$
given by quotienting the unit sphere $S^{2n+1} \subseteq (\mathbb{C}^{n+1} \setminus \{0\})$ by the diagonal circle action (which reparametrizes complex lines).

(e) Conclude that $\pi_*(\mathbb{CP}^n) \cong \pi_*(S^{2n+1})$ for $* \geq 3$.

Problem 5. Let $X_1 = S^2 \times \mathbb{RP}^3$ and $X_2 = S^3 \times \mathbb{RP}^2$.

(a) Find the universal covers $\widetilde{X}_1$ and $\widetilde{X}_2$ of X_1 and X_2.

(b) Show that $\pi_*(X_1) \cong \pi_*(X_2)$ in all degrees $* \in \mathbb{N}$.

(c) (Guess) Do you think X_1 and X_2 are homeomorphic ?

Problem 6. Let $SO(3) \subseteq M_3(\mathbb{R})$ be the set of orthogonal (3×3)-matrices with positive determinant, i.e.

$$SO(3) = \{A \in M_3(\mathbb{R}) : AA^t = \mathrm{Id}, \det(A) = 1\}.$$

This is a topological space when endowed with the induced topology from the Euclidean ambient space $M_3(\mathbb{R}) \cong \mathbb{R}^9$.

(a) Show that $SO(3)$ is homeomorphic to $\mathbb{RP}^3$.

(b) Let $SU(2) \subseteq M_2(\mathbb{C})$ be the set of unitary (2×2)-matrices with complex coefficients and determinant one, i.e.

$$SU(2) = \{A \in M_2(\mathbb{C}) : AA^\dagger = \mathrm{Id}, \det(A) = 1\}.$$

This is a topological space when endowed with the induced topology from the Euclidean ambient space $M_2(\mathbb{C}) \cong \mathbb{C}^4 \cong \mathbb{R}^8$. The goal is to understand the topology of $SU(2)$ as a topological space.

Let $(z,w) \in \mathbb{C}^2$. Show that

$$SU(2) = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \in M_2(\mathbb{C}), \text{ such that} |z|^2 + |w|^2 = 1 \right\}.$$

(c) Show that $SU(2)$ is homeomorphic to S^3.

(d) Let $\Re(z)$ and $\Im(z)$ be the real and imaginary parts of $z \in \mathbb{C}$. Consider the map $\pi : SU(2) \longrightarrow SO(3)$ given by

$$\pi\left(\begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \right) = \begin{pmatrix} \Re(z^2 - w^2) & \Im(z^2 + w^2) & -2\Re(zw) \\ -\Im(z^2 - w^2) & \Re(z^2 + w^2) & 2\Im(zw) \\ 2\Re(z\overline{w}) & 2\Im(z\overline{w}) & |z|^2 - |w|^2 \end{pmatrix}.$$

Compute the preimage of the identity Id$\in SO(3)$. In the homeomorphism identifications $SU(2) \cong S^3$ in Part (c) and $SO(3) \cong \mathbb{RP}^3$ in Part (a), which map the above $\pi : S^3 \longrightarrow \mathbb{RP}^3$?

Problem 7. Let $k \geq 1$, and consider the following topological spaces:

$$S_k = \{(z_1, z_2) \in \mathbb{C}^2 : z_1^2 = z_2^k - 1\}, \quad S = \{(z_1, z_2) \in \mathbb{C}^2 : z_1^2 = \sin(z_2)\}.$$

(a) Show that these spaces are pairwise not homotopic, i.e. $S_k \not\simeq S_l$ if $k \neq l$.

(b) Compute the fundamental groups $\pi_1(S)$.