MAT 215B: PROBLEM SET 2

DUE TO FRIDAY JAN 17 AT 3:00PM

Abstract. This is the second problem set for the graduate course Algebraic Topology II in the Winter Quarter 2020. It was posted online on Saturday Jan 11 and is due Friday Jan 17 at 3:00pm via online submission.

Purpose: The goal of this assignment is to practice the basic concepts and computations for **simplicial homology** taught in the first week of Algebraic Topology II (MAT215B). In particular, we would like to become familiar with many examples and develop an intuition the geometry captured by the homology groups.

Task: Solve Problems 1 through 7 below.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 20 points, five problems will be graded. The maximum possible grade is 100 points.

Textbook: We will use "Algebraic Topology" by A. Hatcher. Please contact me *immediately* if you have not been able to get a copy of any edition.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. Let $\Delta^3 = [v_0, v_1, v_2, v_3]$ be the 3-simplex. Consider the topological space X obtained by first identifying the face $[v_0, v_1, v_2]$ with the face $[v_0, v_1, v_3]$, and then identifying the face $[v_1, v_2, v_3]$ with the face $[v_0, v_2, v_3]$. Show that X is homeomorphic to the 3-sphere S^3.

Problem 2. Let X be a topological space endowed with a Δ-complex structure and let $r = |\pi_0(X)|$ be the number of path-connected components.

(a) Show that $H_0^\Delta(X) \cong \mathbb{Z}^r$.

(b) Describe geometrically r generators for $H_0^\Delta(X)$.

Problem 3. Let X be a topological space endowed with a Δ-complex structure C with finitely many chains. Suppose that each group $H_*^{\Delta}(X)$ is independent of the choice of Δ-complex structure.

Consider the rank $c_i(X,C) := \mathrm{rk}(\Delta_i(X,C))$ of the $\mathbb{Z}$-module of i-chains, $1 \leq i \leq n$, and the rank

$$b_i(X) := \mathrm{rk}(H_i^{\Delta}(X))$$

of the i-th simplicial homology group, $1 \leq i \leq n$. (The $b_i(X)$ is called the ith Betti number of X.)

(a) Give an example of a space X with two distinct Δ-complex structres C_1, C_2 such that $c_i(X,C_1) \neq c_i(X,C_2)$ for all $0 \leq i \leq 2$.

(b) Show that

$$\sum_{i\geq 0}(-1)^i c_i(X,C) = \sum_{i \geq 0}(-1)^i b_i(X).$$

This quantity is known as the Euler characteristic $\chi(X)$ of X.

(c) Conclude that $\chi(X)$ is independent of the choice of the choice Δ-complex structure and compute $\chi(S^1)$.

Problem 4. (Orientable Surfaces) Let Σ_g be the orientable surface of genus g, $g \geq 0$.

(a) Describe a Δ-complex structure on Σ_g, for any $g \geq 0$.

(b) Compute the simplicial homology groups $H_*^{\Delta}(\Sigma_g)$, $* \geq 0$.

(c) Describe geometrically the generators of $H_1^{\Delta}(\Sigma_g)$.

(d) Show that $H_*^{\Delta}(\Sigma_g) \cong H_*^{\Delta}(\Sigma_h)$ implies $g = h$.

(Thus, simplicial homology characterizes *all* orientable surfaces.)

(e) Let $\pi_1(\Sigma_g)$ be the fundamental group and $[\pi_1(\Sigma_g), \pi_1(\Sigma_g)] \subseteq \pi_1(\Sigma_g)$ the commutator subgroup. Show that $\pi_1(\Sigma_g)/[\pi_1(\Sigma_g), \pi_1(\Sigma_g)] \cong \mathbb{Z}^{2g}$.

(Hence, in a sense, the difference between $H_1(\Sigma_g)$ and $\pi_1(\Sigma_g)$ is that in H_1 elements *commute*. Intuitively, going around a curve $\alpha \subseteq \Sigma_g$ and then around another curve $\beta \subseteq \Sigma_g$ is $\alpha + \beta$ in $H_1(\Sigma_g)$, which equals $\beta + \alpha$.)

Problem 5. Let $\mathbb{RP}^2$ be the real projective plane and K the Klein bottle.

(a) Describe a Δ-complex structure for $\mathbb{RP}^2$ and K.

(b) Compute the homology groups $H_*^{\Delta}(\mathbb{RP}^2)$ and $H_*^{\Delta}(K)$.
(You should obtain $H_1^{\Delta}(\mathbb{RP}^2) \cong \mathbb{Z}_2$.)

(c) Describe geometrically the generator γ of $H_1^{\Delta}(\mathbb{RP}^2)$ and explain, geometrically, why it has order 2 in homology, i.e. $2\gamma = 0$.

Problem 6. Compute the simplicial homology of the following spaces:

(a) The spheres S^1, S^2 and S^3.

(b) The real projective line $\mathbb{RP}^1$, $\mathbb{RP}^2$ and $\mathbb{RP}^3$.

(c) The 3-torus T^3. This is the space obtained by identifying the six faces of a solid cube $Q = \{(x, y, z) \in \mathbb{R}^3 : -1 \leq x, y, z \leq 1\}$ as follows: each face is uniquely identified with the *only* face parallel to it, and the three map identifications, one for each of the three pairs of identified faces, are all the identity.

(d) Let X, Y be topological spaces endowed with Δ-complex structures. Compute the simplicial homology $H_*^\Delta(X \vee Y)$ of the wedge $X \vee Y$.

(e) Exercise 8 in Section 2.1 in Hatcher (Page 131).

Problem 7. Suppose that simplicial homology $H_*^\Delta(X)$ is a homotopy invariant of X. Let $k \geq 1$, and consider the following topological spaces:

$$S_k = \{(z_1, z_2) \in \mathbb{C}^2 : z_1^2 = z_2^k - 1\}, \quad M_k = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 + z_2^2 = z_3^k - 1\}.$$

(a) Compute $H_*^\Delta(S_k)$ for each $k \geq 0$.

(b) Compute $H_*^\Delta(M_k)$ for each $k \geq 0$.

(c) Consider the topological spaces

$$L_k = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : z_1^2 + z_2^2 + z_3^2 = z_4^k - 1\}.$$

Find the simplicial homology groups $H_*^\Delta(L_k)$, for each $k \geq 0$.

(d) Find the simplicial homology groups of

$$X = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : z_1 z_2 + z_3 z_4 = z_5^7 - 1\}.$$