MAT 215B: PROBLEM SET 3

DUE TO FRIDAY JAN 24 AT 3:00PM

ABSTRACT. This is the third problem set for the graduate course Algebraic Topology II in the Winter Quarter 2020. It was posted online on Saturday Jan 18 and is due Friday Jan 24 at 3:00pm to be handed it in class.

Purpose: The goal of this assignment is to practice the basic concepts and computations for **homological algebra** and **singular homology** as used and taught in the second week of Algebraic Topology II (MAT215B). In particular, we would like to become familiar with the algebraic notion of a *chain complex*, a *chain homotopy*, an the induced *long exact sequences*.

Assumption: In this Problem Set we shall *assume* that the simplicial homology groups $H_*^\Delta(X)$ and the singular homology groups $H_*(X)$ of a topological space X are isomorphic, and $H_*^\Delta(X)$ is independent of the choice of Δ-complex structure.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Task and Grade: Solve Problems 1 through 7 below. The first 2 problems will not be graded but I trust that you will work on them. Problems 3 to 6 will be graded. Problems 1 and 2 are review exercises. Problems 3 and 4 are geometric in nature, and Problems 5 and 6 belong to homological algebra. Each graded Problem is worth 25 points. The maximum possible grade is 100 points.

Textbook: We will use "Algebraic Topology" by A. Hatcher. Please contact me *immediately* if you have not been able to get a copy of any edition.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. Let (A_*, ∂^A) and (B_*, ∂^B) be two chain complexes of Abelian groups, indexed by $* \in \mathbb{N}$. Let $i_* : (A_*, \partial^A) \longrightarrow (B_*, \partial^B)$ be a map of chain complexes, i.e. $i_{*-1} \circ \partial^A = \partial^B \circ i_*$. Show that $i_* : (A_*, \partial^A) \longrightarrow (B_*, \partial^B)$ descends to a map

$$H_*(i) : (A_*, \partial^A) \longrightarrow (B_*, \partial^B),$$

between the homologies of these chain complexes.

Problem 2. Let $f, g : X \longrightarrow Y$ be two continuous maps between topological spaces X, Y, which we assume to be CW complexes. Let $(C_\bullet(X), \partial^X)$ be the singular chain complex for X and $(C_\bullet(Y), \partial^Y)$ be the singular chain complex for Y.

(a) Show that $f_\# : (C_\bullet(X), \partial^X) \longrightarrow (C_\bullet(Y), \partial^Y)$ satifies

$$f_\# \circ \partial^X = \partial^Y \circ f_\#.$$

In particular, $f_\#$ descends to a map in homology.

(b) Suppose that $f \simeq g$ and $F : X \times [0,1] \longrightarrow Y$ is a homotopy between f and g. Show that there exists a *chain homotopy*, constructed using F, between the two maps $f_\#$ and $g_\#$ of chain complexes.

(c) Show that $X \simeq Y$ implies that their singular homology groups $H_*(X) \cong H_*(Y)$ are all isomorphic.

Parts 2.(a),2(c) and almost all of 2.(b) were discussed in detail in class. The purpose of this exercise is that you review the material and perform all the computations needed for the chain homotopy in Part 2.(b).

Problem 3. Compute the singular homology $H_*(X)$ for each of the following topological spaces. In choosing the spaces X for this exercise, I am trying to *make a point.* Explore each of this spaces and see if you can find it.

(a) $X = \{(z_1, z_2, z_3, z_4) : z_1 z_2 z_3 z_4 = 0\} \subseteq \mathbb{C}^4$.

(b) $X = \{(z_1, z_2, z_3, z_4, z_5) : z_1^2 + z_2^3 + z_3^7 + z_4 z_5^3 = 0\} \subseteq \mathbb{C}^5$.

(c) $X = \{(z_1, z_2, z_3, z_4, z_5) : |z_1|^2 + z_2^3 + |z_3|^9 + |z_4| z_5^3 = 0\} \subseteq \mathbb{C}^5$.

(d) $X = \{(x, y, z, w) \in \mathbb{R}^4 : 5x^3y^3 + xy^9 - 2y^{12} + x^2z^6 + 7y^8w^2 = 0\} \subseteq \mathbb{R}^4$.

(e) $X = \{A \in M_n(\mathbb{R}) : a_{ij} = 0 \text{ if } i < j\}$ where $M_n(\mathbb{R}) = \mathrm{End}(\mathbb{R}^n)$ denotes the space of $(n \times n)$-matrices $A = (a_{ij})$, $1 \leq i, j \leq n$, with real entries $a_{ij} \in \mathbb{R}$.

(f) Let V be a finite-dimensional $\mathbb{R}$-vector space. Let X be the space of all inner products $\langle \cdot, \cdot \rangle : V \otimes V \longrightarrow \mathbb{R}$.

(g) Let $V = \mathbb{R}^\infty$ be the infinite-dimensional $\mathbb{R}$-vector space. Let $X = \mathbb{R}^\infty \setminus \{0\}$ be the set of its non-zero vectors.

The next two bonus items will not be graded. They might be of interest if you are also taking the Analysis Sequence, or as a matter of general interest.

(h) (Bonus I) Let H be a Hilbert space. For instance, let us choose

$$H = \{(z_1, z_2, \ldots) : \sum_{i=1}^{\infty} |z_n|^2 \text{ convergent}\} \subseteq \mathbb{C}^\infty,$$

the set of all infinite sequences of complex numbers, or $H = \mathbb{C}^\infty$. Choose $X = H \setminus \{0\}$.

(i) (Bonus II) Let $X = \{f \in L^2(\mathbb{C}, \mathbb{R}) : \left(\int_{\mathbb{C}} |f|^2\right)^{1/2} = 1\}$, where the space $L^2(\mathbb{C}, \mathbb{R})$ of L^2-integrable functions $f : \mathbb{C} \longrightarrow \mathbb{R}$ is given the topology induced by the inner product

$$\langle f, g\rangle = \int_{\mathbb{C}} f(z)\overline{g(z)}dz.$$

Problem 4. Solve the following problems.

(a) Let $X_2 = \{l \subseteq \mathbb{R}^3 : l \text{ is an oriented line}\}$ be the space of oriented lines in $\mathbb{R}^3$. Show that $H_0(X_2) = H_2(X_2) \cong \mathbb{Z}$ and $H_i(X_2) = 0$ if $i \in \mathbb{N}$, $i \neq 0, 2$.

(b) Let $X_3 = \{l \subseteq \mathbb{R}^4 : l \text{ is an oriented line}\}$ be the space of oriented lines in $\mathbb{R}^4$. Show that $H_0(X_3) = H_3(X_3) \cong \mathbb{Z}$ and $H_i(X_3) = 0$ if $i \in \mathbb{N}$, $i \neq 0, 3$.

(c) Let $\mathrm{SL}(2, \mathbb{C}) = \{A \in M_2(\mathbb{C}) : \det(A) = 1\} \subseteq M_2(\mathbb{C})$ be the space of (2×2)-matrices with complex entries and determinant one. Show that we have $H_0(\mathrm{SL}(2, \mathbb{C})) \cong \mathbb{Z}$, $H_3(\mathrm{SL}(2, \mathbb{C})) \cong \mathbb{Z}$ and $H_i(\mathrm{SL}(2, \mathbb{C})) = 0$ if $i \in \mathbb{N}$, $i \neq 0, 3$.

Problem 5. Let $(A_*, \partial^A), (B_*, \partial^B)$ and (C_*, ∂^C) be three chain complexes of Abelian groups, indexed by $* \in \mathbb{N}$. Let $i_* : (A_*, \partial^A) \longrightarrow (B_*, \partial^B)$ and $j_* : (B_*, \partial^B) \longrightarrow (C_*, \partial^C)$ be two maps of chain complexes, i.e. $i_{*-1} \circ \partial^A = \partial^B \circ i_*$ and $j_{*-1} \circ \partial^B = \partial^C \circ j_*$.

Suppose that i is an injective map, i.e. i_* is injective for all $* \in \mathbb{N}$, j is a surjective map, and $im(i_*) = \ker(j_*)$.

(a) Show that the image of $H_*(i) : H_*(A_*, \partial^A) \longrightarrow H_*(B_*, \partial^B)$ equals the kernel of $H_*(j) : H_*(B_*, \partial^B) \longrightarrow H_*(C_*, \partial^C)$.

(b) For each $n \in \mathbb{N}$, let $c_n \in C_n$ be an n-cycle, and consider *any* element $b_n \in B_n$ such that $j(b_n) = c_n$. Show that there exists an element $a_{n-1} \in A_{n-1}$ such that $i_{n-1}(a_{n-1}) = \partial^B b_n$.

(c) Let us now define the map

$$\delta_* : H_*(C_*, \partial^C) \longrightarrow H_{*-1}(A_*, \partial^A), \quad \delta([c_n]) = [a_{n-1}],$$

where a_{n-1} is *any* element obtained as in Part 5.(b). Here $[c] \in H_n(X)$ denotes the homology class of a cycle $c \in C_n(X)$. Show that $\delta_n([c_n]) = [a_{n-1}]$ is independent of the choice of b_n and a_{n-1} in Part 5.(b), and thus it is a well-defined map.

(d) Show that $im(j_*) = \ker(\delta_*)$ as subgroups of $H_*(C_*, \partial^C)$.

(e) Show that $im(\delta_*) = \ker(i_{*-1})$ as subgroups of $H_{*-1}(A_*, \partial^A)$.

Problem 6. Let (A_*, ∂^A) and (B_*, ∂^B) be two chain complexes of Abelian groups, indexed by $* \in \mathbb{N}$. A map of chain complexes $f_* : (A_*, \partial^A) \longrightarrow (B_*, \partial^B)$ is said to be a quasi-isomorphism if $H_*(f)$ are isomorphisms.

A map of chain complexes $f_* : (A_*, \partial^A) \longrightarrow (B_*, \partial^B)$ is said to be a chain homotopy equivalence if there exists $g_* : (B_*, \partial^B) \longrightarrow (A_*, \partial^A)$ such that $f_* \circ g_* \simeq id$ and $g_* \circ f_* \simeq id$ are chain homotopic to the identity.

(a) Show that a chain homotopy equivalence is a quasi-isomorphism.

(b) Give an example of a chain map $f_* : (A_*, \partial^A) \longrightarrow (B_*, \partial^B)$ which is a quasi-isomorphism but *not* a chain homotopy equivalence.

(c) Given an example of a chain complex (A_*, ∂^A) whose homology groups are zero and such that $id_A : (A_*, \partial^A) \longrightarrow (A_*, \partial^A)$ is not homotopic to the zero map.