MAT 215B: PROBLEM SET 4

DUE TO FRIDAY JAN 31 AT 3:00PM

Abstract. This is the fourth problem set for the graduate course Algebraic Topology II in the Winter Quarter 2020. It was posted online on Friday Jan 24 and is due Friday Jan 31 at 3:00pm to be handed it in class.

Purpose: The goal of this assignment is to practice properties and computations for **singular homology** as used and taught in the third of Algebraic Topology II (MAT215B). In particular, we would like to become familiar with the *Relative Homology Groups*, the homology of a *quotient space*, the *Excision Theorem* and the *Mayer-Vietoris* sequence.

Assumption: In this Problem Set we shall *assume* that the simplicial homology groups $H_*^\Delta(X)$ and the singular homology groups $H_*(X)$ of a topological space X are isomorphic, and $H_*^\Delta(X)$ is independent of the choice of Δ-complex structure.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Task and Grade: Solve Problems 1 through 7 below. The first 2 problems will not be graded but I trust that you will work on them. Problems 3 to 6 will be graded. Problems 1 and 2 are review exercises. Problems 3 and 4 are geometric in nature, and Problems 5 and 6 belong to homological algebra. Each graded Problem is worth 25 points. The maximum possible grade is 100 points.

Textbook: We will use "Algebraic Topology" by A. Hatcher. Please contact me *immediately* if you have not been able to get a copy of any edition.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. Let $X_1, X_2 \subseteq X$ be subspaces such that $X = \mathring{X}_1 \cup \mathring{X}_2$.

(a) Show that the inclusion $i : (X_1, X_1 \cap X_2) \longrightarrow (X_1 \cup X_2, X_2) = (X, X_2)$ induces isomorphisms

$$H_*(i) : H_*(X_1, X_1 \cap X_2) \longrightarrow H_*(X, X_2) \quad \forall * \in \mathbb{N}.$$

(b) Given an example of a space X with two subspaces $X_1, X_2 \subseteq X$ such that $X = X_1 \cup X_2$ and the inclusion $i : (X_1, X_1 \cap X_2) \longrightarrow (X_1 \cup X_2, X_2) = (X, X_2)$ is not an isomorphism.

Problem 2. Let (X, x_0) be a based topological space. This exercise relates the **fundamental group** $\pi_1(X, x_0)$ with the **first homology group** $H_1(X)$.

Consider the group homomorphism $a : \pi_1(X, x_0) \longrightarrow H_1(X)$, $a([f]) = [\sigma_f]$, where $f : (S^1, 0) \longrightarrow (X, x_0)$ is a based map, $[f]$ its homotopy class, and $[\sigma_f]$ denotes the homology class of the singular 1-chain $\sigma_f : \Delta^1 \longrightarrow X$ given by $\sigma_f(p) = f(p)$, where $\Delta^1 = [0, 1]$ and $S^1 = [0, 1]/(0 \sim 1)$.

(a) Show that $\sigma_f : \Delta^1 \longrightarrow X$ is a 1-cycle, and thus a indeed descends to $H_1(X)$.

(b) Show that a is well-defined, i.e. it only depends on the homotopy class $[f]$.

(c) Show that a is surjective.

(d) Prove that the commutator subgroup $[\pi_1(X, x_0), \pi_1(X, x_0)] \subseteq \pi_1(X, x_0)$ lies in the kernel of a.

(e) Prove that every element in $\ker(a)$ belongs to the commutator $[\pi_1(X, x_0), \pi_1(X, x_0)]$.

(f) Conclude the group isomorphism

$$H_1(X) \cong \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)].$$

Notation: Given a group G, the quotient $G/[G, G]$ is referred to as the **abelianization** *of G, as it consists of the elements in G where we have forced that all elements commute.*

(g) Let $K \subseteq S^3$ be a knot and $G = \pi_1(S^3 \setminus K)$ its knot group. Let us conclude that the group G is "*highly non-Abelian*", i.e. the inclusion

$$[\pi_1(S^3 \setminus K), \pi_1(S^3 \setminus K)] \subseteq \pi_1(S^3 \setminus K)$$

is almost the entire group. In precise terms, show that

$$\pi_1(S^3 \setminus K)/[\pi_1(S^3 \setminus K), \pi_1(S^3 \setminus K)] \cong \mathbb{Z}.$$

(See also Problem 6.(a), where you will be asked to do this computation.)

Problem 3. Solve the following exercises:

(a) Compute the relative homology groups for $(D^n, \partial D^n)$ for all $n \in \mathbb{N}$.

(b) Compare the groups in Part 2.(a) with the homology groups for the n-sphere S^n for all $n \in \mathbb{N}$. Explain this relation geometrically.

(c) Compute the relative homology groups (S^n, A) where $A \subseteq S^n$ is an equator $A \cong S^{n-1}$ of the n-sphere S^n, for all $n \in \mathbb{N}$.

(d) Compare the groups in Part 2.(c) with the homology groups for the wedge $S^n \vee S^n$ of two n-sphere S^n. Explain this relation geometrically.

(e) Let $B \subseteq T^2$ be an embedded curve in the 2-torus which is not null-homologous. Compute the relative homology groups $H_*(T^2, B)$.

(f) Let $p, q \in S^2$ be two distinct points. Compare the groups in Part 2.(e) with the relative homology groups $H_*(S^2, \{p, q\})$, explaining their relation geometrically.

Problem 4. This exercise discusses **suspensions**.

(a) Let SX be the suspension $SX := (X \times [0,1])/\sim$, with $(x_1, 0) \sim (x_2, 0)$, for any $x_1, x_2 \in X$, and $(x_1, 1) \sim (x_2, 1)$ for any $x_1, x_2 \in X$. Compute the homology groups $H_*(SX)$ in terms of $H_*(X)$.

(b) Consider the subspace

$$Z = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : z_1 z_2 z_3 + z_4^2 = 1\} \subseteq \mathbb{C}^4.$$

Compute the homology groups $H_*(Z)$.

(c) Consider the subspace

$$W = \{(z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbb{C}^5 : z_1 z_2 z_3 z_4 + z_5^2 + z_6^2 = 1\} \subseteq \mathbb{C}^6.$$

Compute the homology groups $H_*(W)$.

Problem 5. This exercise discusses an important class of 3-manifolds, called **lens spaces**. Let $A \in \mathrm{SL}(2, \mathbb{Z})$ be the matrix

$$A = \begin{pmatrix} m & p \\ n & q \end{pmatrix}, \quad mq - np = 1.$$

Consider the 2-torus $T^2 = \mathbb{R}^2/\sim$ with $(x, y) \sim (x + 1, y)$ and $(x, y) \sim (x, y + 1)$.

(a) Show that the linear map A descends to a diffeomorphism $\varphi_A : T^2 \longrightarrow T^2$, induced by $A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

(b) Consider the space $X_A = ((S^1 \times D^2) \cup (D^2 \times S^1))/\sim_{\varphi_A}$, where the gluing along the boundaries $T^2 = \partial(S^1 \times D^2) = \partial(D^2 \times S^1)$ is given φ_A.

(c) Show that X_{Id} is homeomorphic to the 3-sphere S^3.

(d) Compute the singular homology groups $H_*(X_A)$ in terms of $A \in \mathrm{SL}(2, \mathbb{Z})$.

(e) Find a matrix A such that X_A is homeomorphic to real projective 3-space $\mathbb{RP}^3$.

(f) Show that there exists a matrix A such that X_A is homeomorphic to $S^1 \times S^2$.

Problem 6. In this problem we shall study the singular homology of complements of embedded spheres. In particular, the complements of knots in all dimensions.

(a) Let $K \subseteq S^3$ be a knot, i.e. the image of an embedding $i : S^1 \longrightarrow S^3$. Compute the singular homology groups $H_*(S^3 \setminus K)$.

(b) Give an example of two knots $K_1, K_2 \subseteq S^3$ such that the complements $S^3 \setminus K_1$ and $S^3 \setminus K_2$ are *not* homotopy equivalent.

(c) Let $X = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_3(z_1^2 + z_2^3) = 1, |z_1|^2 + |z_2|^2 = 1\} \subseteq \mathbb{C}^3$. Compute the homology groups $H_*(X)$.

(d) Let $k, n \in \mathbb{N}$ with $k \leq n-1$. Show that for *any* embedding $i : D^k \longrightarrow S^n$ we have $\widetilde{H}_*(S^n \setminus i(D^k)) = 0$, for all $* \in \mathbb{N}$.

(e) Let $k, n \in \mathbb{N}$ with $k \leq n-1$. Show that for *any* embedding $i : S^k \longrightarrow S^n$ we have $\widetilde{H}_{n-k-1}(S^n \setminus i(S^k)) = \mathbb{Z}$ and $\widetilde{H}_*(S^n \setminus i(S^k)) = 0$, for all $* \in \mathbb{N} \setminus \{n-k-1\}$.

(f) Let $Y = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : z_4(z_1^2 + z_2^2 + z_3) = 1, |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\} \subseteq \mathbb{C}^4$. Compute the homology groups $H_*(Y)$.

Problem 7. This exercise discusses the first historically known space with **the same** homology as S^3, which is not homotopically S^3.

Let $SO(3)$ be the group of rotations in $\mathbb{R}^3$ centered at the origin. Let $I \subseteq \mathbb{R}^3$ be a regular icosahedron centered at the origin and X_I the space of configurations of this icosahedron, i.e. all the possible images of $I \subseteq \mathbb{R}^3$ under the action of $SO(3)$ in $\mathbb{R}^3$.

(a) Let $G_I \subseteq SO(3)$ be the subgroup of isometries such that $G_I(I) = I$, i.e. the stabilizer of I under the $SO(3)$-action on $\mathbb{R}^3$.

(b) Show that X_I is homeomorphic to the quotient of $SO(3)/G_I$.

(c) Show that X_I is *not* homotopy equivalent to S^3.

(d) Show that $H_1(X_I) \cong \{0\}$.

(e) (Bonus) Show that $H_*(X_I) \cong H_*(S^3)$. Thus, X_I has the **same homology as the 3-sphere** but it is not homotopy equivalent to it.

Piece of History: This is the space that made H. Poincaré introduce the fundamental group and homology groups (Betti numbers). The article ends with a question, known nowadays as the Poincaré Conjecture.