MAT 215B: PROBLEM SET 5

DUE TO FRIDAY FEB 28 AT 3:00PM

Abstract. This is the fifth problem set for the graduate course Algebraic Topology II in the Winter Quarter 2020. It was posted online on Saturday February 22 and is due Friday February 28 at 3:00pm to be handed it in class.

Purpose: The goal of this assignment is to practice properties and computations for **cohomology** and the Ext functor, as used and taught in Algebraic Topology II (MAT215B). In particular, we would like to become familiar with computations of *Ext groups*, *cohomology with coefficients* and the *Universal Coefficients Theorem*.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Task and Grade: Solve Problems 1 through 6 below. The second and last problems will not be graded but I trust that you will work on them. Problem 1 and Problems 3 to 5 will be graded. Each graded Problem is worth 25 points. The maximum possible grade is 100 points.

Textbook: We will use "Algebraic Topology" by A. Hatcher. Please contact me *immediately* if you have not been able to get a copy of any edition.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Main Definition: Let H be an Abelian group and $F_* \longrightarrow H$ a free resolution, $* \in \mathbb{N}$. We define the ith Ext group $Ext^i(H,G)$ to be the ith cohomology group of the dual complex $Hom(F_*, G)$. That is, if we write

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow H \longrightarrow 0,$$

for a free resolution, and consider the dual complex

$$\cdots \longleftarrow Hom(F_1, G) \longleftarrow Hom(F_0, G) \longleftarrow 0,$$

then $Ext^i(H,G)$ is the kernel of the $(i+1)$th arrow modulo the image of the ith arrow, where the arrows are indexed from the right, the rightmost arrow being the 0th arrow. All homomorphism in this Problem Set are considered to be homomorphisms of Abelian groups.

Problem 1. (**Properties of Ext**). Let G, H be finitely generated Abelian groups.

(a) Show that $Hom(-, G)$ is right exact, i.e. if $A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact, then $Hom(A, G) \longleftarrow Hom(B, G) \longleftarrow Hom(C, G) \longleftarrow 0$ is exact.

(b) Give an example which shows that $Hom(-, G)$ is *not* left exact.

(c) Show that $Ext^0(H, G) = Hom(H, G)$.

(d) Show that $Ext^i(H, G) = 0$ for $i \geq 2$.

(e) Give two examples of distinct Abelian groups H_1, H_2 such that

$$Ext^0(H_1, \mathbb{Z}) = Ext^0(H_2, \mathbb{Z}).$$

Thus, $Ext^0(H, \mathbb{Z})$ does *not* in general recover the Abelian group H.

Problem 2. (**Homological Algebra of Ext**). Let $0 \longrightarrow A \longrightarrow B \longrightarrow C$ be a short exact sequence of Abelian groups, and G an Abelian group. Show that there exists a long exact sequence

$$0 \longrightarrow Hom(C, G) \longrightarrow Hom(B, G) \longrightarrow Hom(A, G) \xrightarrow{\delta} Ext^1(C, G) \longrightarrow$$
$$\longrightarrow Ext^1(B, G) \longrightarrow Ext^1(A, G) \longrightarrow 0.$$

That is, similar to the homology functor H_*, applying Ext^* to a short exact sequence yields a long exact sequence of Ext groups. (Recall that $Ext^0(H, G) = Hom(H, G)$, as you proved in Problem 1.(c), and, $Ext^i(H, G) = 0$ for $i \geq 2$, after Problem 1.(d). Thus this long exact sequence is a sequence of *all* Ext groups.)

Problem 3. (**Computations of Ext I**). All groups H_1, H_2, G_1, G_2, H, G in this problem are finitely generated Abelian groups.

(a) Show that $Ext^*(H_1 \oplus H_2, G) = Ext^*(H_1, G) \times Ext^*(H_2, G)$.

(b) Prove that $H^*(X \vee Y, G) \cong H^*(X, G) \times H^*(Y, G)$.

(b) Is $Ext^*(H, G_1 \times G_2) = Ext^*(H, G_1) \times Ext^*(H, G_2)$ true ?

(c) Show that $Ext^1(H, G) = 0$ if H is a free group.

(d) Find the cohomology groups $H^*(S^n, G)$ with coefficients in a finitely generated group Abelian G.

(e) Find the cohomology groups $H^*(\mathbb{CP}^n, G)$.

Problem 4. (**Computations of Ext II**) Solve the following parts:

(a) Compute $Ext^*(H, \mathbb{Z})$ for all finitely generated Abelian groups H. In particular, conclude that the graded groups $Ext^*(H, \mathbb{Z})$ uniquely determine H.

(b) Show that $Ext^0(\mathbb{Z}_n, \mathbb{Z}_m) = \mathbb{Z}_d$ and $Ext^1(\mathbb{Z}_n, \mathbb{Z}_m) = \mathbb{Z}_d$, where $d = gcd(n, m)$.

(c) Find the cohomology groups $H^*(\mathbb{RP}^n, \mathbb{Z})$ with coefficients in $\mathbb{Z}$.

(d) Compute the cohomology groups $H^*(\mathbb{RP}^n, \mathbb{Z}_2)$ with coefficients in $\mathbb{Z}_2$.

(e) Which groups provide more information $H^*(\mathbb{RP}^n, \mathbb{Z}_2)$ or $H^*(\mathbb{RP}^n, \mathbb{Z}_3)$?

(f) For each odd $n \in \mathbb{N}$, give an example of a topological space X such that the cohomology groups $H^*(X, \mathbb{Z}_3) \cong H^*(S^n, \mathbb{Z}_3)$ are isomorphic, but X is not homotopy equivalent to S^n.

Problem 5. (**Group Extensions**) Let A, B be two $\mathbb{Z}$-modules, an *extension* of B by A is the data of a group G **and** a short exact sequence

$$0 \longrightarrow A \longrightarrow G \longrightarrow B \longrightarrow 0.$$

The set of all extensions of B by A is denoted $\mathcal{E}(B, A)$.

(a) Show that $G \cong A \oplus B$ if B is free, i.e. the only extensions of the Abelian groups $\mathbb{Z}^r$ are by direct sums.

(b) Show that $\mathcal{E}(\mathbb{Z}_p, \mathbb{Z}_p)$ contains p elements, if p is a prime.

(c) Given an extension $0 \longrightarrow A \longrightarrow G \longrightarrow B \longrightarrow 0$, consider the induced map $Hom(A, A) \xrightarrow{\delta} Ext^1(B, A)$ constructed above. Show that

$$\chi : \mathcal{E}(B, A) \longrightarrow Ext^1(B, A),$$
$$\chi\,(0 \longrightarrow A \longrightarrow G \longrightarrow B \longrightarrow 0) = \delta(id_A),$$

is a bijection.

(d) Conclude that there are exactly $\mathbb{Z}_{gcd(n,m)}$ extensions of $\mathbb{Z}_n$ by $\mathbb{Z}_m$.

Problem 6. (**Geometric Meaning of Ext**) This problem is only optional, in case you are taking an introductory class in Algebraic Geometry.

(a) Let $X = \text{Spec}(A)$ be an affine variety, $x \in X$ a closed point given by a maximal ideal $\mathfrak{m} \subseteq A$, and $k(x)$ its residue field. Show that $Ext^1_X(k(x), k(x))$ is isomorphic to the tangent space T_xX of X at x.

Since $Ext^1_X(k(x), k(x))$ is isomorphic to $Ext^1_A(A/\mathfrak{m}, A/\mathfrak{m})$, and T_xX is, by definition, $Hom_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2, A/\mathfrak{m})$, the above statement is equivalent to

$$Ext^1_A(A/\mathfrak{m}, A/\mathfrak{m}) \cong Hom_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2, A/\mathfrak{m}).$$

(b) Let $E \longrightarrow X$ be a coherent sheaf (e.g. a vector bundle) over a smooth algebraic variety X, and $\mathcal{M}_X$ the moduli space of coherent sheaves. Give an heuristic explanation for the isomorphism $Ext^1(E, E) \cong T_E\mathcal{M}_X$.