MAT 215B: PROBLEM SET 6

FOR PRACTICE - NO DUE DATE

ABSTRACT. This is the sixth problem set for the graduate course Algebraic Topology II in the Winter Quarter 2020. It was posted online on Saturday March 7.

Purpose: The goal of this sixth assignment is to practice properties and computations for **cohomology**, as used and taught in Algebraic Topology II (MAT215B). In particular, we would like to become familiar with computations using the *cup product*, the *Künneth formula* and *Poincaré Duality*.

Problem 1. (**Cup Product**). Solve each of these parts, studying the ring structure of the cohomology of a topological space.

(a) Let Σ_g be an orientable genus-g surface. Show that there exists a basis for $H^1(\Sigma_g, \mathbb{Z})$ such that the cup product is given by the $(2g \times 2g)$-matrix with g diagonal blocks given by the (2×2)-matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(b) Consider the two spaces $X_1 = \mathbb{RP}^3$ and $X_2 = \mathbb{RP}^2 \vee S^3$. Show that the cohomology groups $H^*(X_1, \mathbb{Z})$ and $H^*(X_2, \mathbb{Z})$ are isomorphic as graded Abelian groups, but $H^*(X_1, \mathbb{Z}) \not\cong H^*(X_2, \mathbb{Z})$ as graded commutative rings.

(c) Let us consider the natural inclusion $\mathbb{CP}^1 \subseteq \mathbb{CP}^\infty$ of the complex projective line inside the infinite-dimensional complex projective space. Compute the cohomology ring of $H^*(\mathbb{CP}^\infty/\mathbb{CP}^1, \mathbb{Z})$.

(d) Find the cohomology ring of the space

$$X = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 : z_0 z_1 = z_2 z_3\} \subseteq \mathbb{CP}^3.$$

(e) For all $n \in \mathbb{N}$, Compute the cohomology ring of the spaces

$$X_n = \{([z_0 : z_1 : z_2], [w_0 : w_1]) \in \mathbb{CP}^2 \times \mathbb{CP}^1 : z_0 w_0^n = z_1 w_1^n\} \subseteq \mathbb{CP}^2 \times \mathbb{CP}^1.$$

(f) (Bonus) Compute the cohomology ring of the blow-up of $\mathbb{CP}^2$ at n general points, for all $n \in \mathbb{N}$. That is, compute the cohomology ring

$$H^*(\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \# \overline{\mathbb{CP}}^2 \# \overset{(n)}{\cdots} \# \overline{\mathbb{CP}}^2)$$

of $\mathbb{CP}^2$ connected sum with n copies of $\overline{\mathbb{CP}}^2$, which is the same space as $\mathbb{CP}^2$ but with its orientation reversed.

Problem 2. (**Cohomology of Projective Spaces**). Solve each of these parts about the cohomology of projective spaces.

(a) Compute the cohomology ring structures for rings $H^*(\mathbb{RP}^n, \mathbb{Z}_2)$, $H^*(\mathbb{CP}^n, \mathbb{Z})$, $H^*(\mathbb{HP}^n, \mathbb{Z})$ for all $n \in \mathbb{N}$.

(b) Describe *geometrically* a generator for each of the rings in Part (a), and interpret geometrically the product structure for its powers.

(c) Let us consider a map $f : \mathbb{RP}^n \longrightarrow \mathbb{RP}^m$ such that

$$H^1(f) : H^1(\mathbb{RP}^m, \mathbb{Z}_2) \longrightarrow H^1(\mathbb{RP}^n, \mathbb{Z}_2).$$

is non-zero. Show that $m \geq n$.

(d) Let $m \geq n$, give an example of a map $f : \mathbb{RP}^n \longrightarrow \mathbb{RP}^m$ such that

$$H^1(f) : H^1(\mathbb{RP}^m, \mathbb{Z}_2) \longrightarrow H^1(\mathbb{RP}^n, \mathbb{Z}_2).$$

is non-zero.

(e) Let us consider a map $f : \mathbb{CP}^n \longrightarrow \mathbb{CP}^m$ such that

$$H^2(f) : H^2(\mathbb{CP}^m, \mathbb{Z}) \longrightarrow H^2(\mathbb{CP}^n, \mathbb{Z}).$$

is non-zero. Show that $m \geq n$.

(f) Consider the quotient of the map

$$f_d : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{C}^{n+1} \setminus \{0\}, \qquad (z_0, z_1, \ldots, z_n) \longmapsto (z_0^d, z_1^d, \ldots, z_n^d),$$

to the complex projective space $\widetilde{f}_d : \mathbb{CP}^n \longrightarrow \mathbb{CP}^n$. Find the image of the generator of $H^2(\mathbb{CP}^n, \mathbb{Z})$ under $H^2(\widetilde{f}_d)$.

(g) Give an example of a map $f : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^2$ such that

$$H^2(f) : H^2(\mathbb{CP}^2, \mathbb{Z}) \longrightarrow H^2(\mathbb{CP}^1, \mathbb{Z}_2).$$

is multiplication by 2.

(h) Formulate and prove an analogue of Parts (c) and (d) for maps between quaternionic projective spaces $\mathbb{HP}^n$.

(i) (Bonus) Compute the cohomology ring $H^*(\mathbb{OP}^2, \mathbb{Z})$ of the octonionic projective plane. ($\mathbb{OP}^2$ is defined as the affine plane $\mathbb{O}^2$ with its $\mathbb{O}$-line at infinity.)

Problem 3. (**Künneth Formula I**). Solve the following parts.

(a) Find the cohomology ring of $H^*(S^n \times S^m, \mathbb{Z})$ for all $n, m \in \mathbb{N}$.

(b) Show that the cohomology ring of $H^*(T^n, \mathbb{Z})$ is isomorphic to the exterior algebra on n elements. In particular, show that

$$\mathrm{rank}_{\mathbb{Z}}(H^k(T^n, \mathbb{Z})) = \binom{n}{k}.$$

(c) Let X be a CW complex. Compute the cohomology of $H^*(X \times S^1, \mathbb{Z})$ in terms of $H^*(X)$. (*You did that in the midterm with MV, now use Künneth !*).

Side comment: This article from October 2019 proves Künneth Formulas in Persistent Homology. The recent date should indicate to you how many interesting algebraic topology result still remain to be proven in the persistent setting.

Problem 4. (**Künneth Formula II**). A real division algebra structure on $\mathbb{R}^n$ is a bilinear multiplication map

$$\cdot : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

such that $ax = b$ and $xa = b$ are uniquely solvable whenever $a \neq 0$.

(a) Show that $\mathbb{R}, \mathbb{R}^2$ and $\mathbb{R}^4$ admit an real division algebra structure.

(b) Suppose $(\mathbb{R}^n, \cdot)$ is a real division algebra. Show that the map

$$g : S^{n-1} \times S^{n-1} \longrightarrow S^{n-1}, \quad g(x, y) = \frac{x \cdot y}{|x \cdot y|}$$

defined by restricting the division algebra product to the unit sphere (and normalize), descends to a map $\widetilde{g} : \mathbb{RP}^n \times \mathbb{RP}^n \longrightarrow \mathbb{RP}^n$.

(c) Let $\alpha \in H^1(\mathbb{RP}^{n-1}, \mathbb{Z}_2)$ be the generator. Show that the pull-back $\widetilde{g} * (\alpha^n)$ of the map $\widetilde{g}$ in Part (b) vanishes, i.e. $\widetilde{g} * (\alpha^n) = 0$.

(d) Show that if $(\mathbb{R}^n, \cdot)$ is a real division algebra, then $n = 2^k$ for some $k \in \mathbb{N}$.

Problem 5. (**Poincaré Duality I**) Let M be a closed orientable topological manifold of dimension $n \in \mathbb{N}$. The *Euler characteristic* is defined as the alternating sum

$$\chi(M) = \sum_{i \geq 0} \mathrm{rank}_{\mathbb{Z}}(H^i(M, \mathbb{Z})).$$

(a) Show that the Euler characteristic $\chi(M) = 0$ vanishes if $n \in \mathbb{N}$ is odd.

(b) Prove that the Euler characteristic $\chi(M)$ is even if $n \equiv 2 \mod 4$.

(c) By direct computation, show that the Euler characteristic $\chi(T^n) = 0$ of the n-torus vanishes.

(d) Let X be a CW-complex, prove that the Euler characteristic $\chi(X \times S^1) = 0$ of $X \times S^1$ necessarily vanishes.

(e) For each $n \in \mathbb{N}$, compute the Euler characteristic $\chi(\mathbb{CP}^n)$ of $\mathbb{CP}^n$.

Problem 6. (**Intersection Pairing**) Let M be a closed orientable topological manifold of dimension $n \in \mathbb{N}$ and $\mu \in H_n(M)$ a fundamental class. Consider the pairing:

$$\langle , \rangle : H^k(M, R) \times H^{n-k}(M, R) \longrightarrow \mathbb{Z},$$

given by $\langle \alpha, \beta \rangle = (a \cup b)(\mu)$. Let $H^*_{tors}(M, \mathbb{Z})$ be the torsion part of $H * (M, \mathbb{Z})$.

(a) Show that
$$\langle , \rangle : H^k(M, \mathbb{Z})/H^k_{tors}(M, \mathbb{Z}) \times H^{n-k}(M, \mathbb{Z})/H^{n-k}_{tors}(M) \longrightarrow \mathbb{Z},$$
is non-singular.

(b) Let k be a field. Show that
$$\langle , \rangle : H^k(M, k) \times H^{n-k}(M, k) \longrightarrow k,$$
is non-singular.

(c) Suppose $n \in \mathbb{N}$ is even, with $n = 2m$, $m \in \mathbb{N}$. Show that the pairing
$$\langle , \rangle : H^m(M, \mathbb{Z})/H^m_{tors}(M, \mathbb{Z}) \times H^m(M, \mathbb{Z})/H^m_{tors}(M) \longrightarrow \mathbb{Z},$$
is unimodular.

(d) In the hypothesis of Part(c), show that the pairing
$$\langle , \rangle : H^m(M, \mathbb{Z})/H^m_{tors}(M, \mathbb{Z}) \times H^m(M, \mathbb{Z})/H^m_{tors}(M) \longrightarrow \mathbb{Z},$$
is symmetric if m is even, and anti-symmetric if m is odd.

Problem 7. (**Bonus Computation**) Let X be defined by the zero set
$$X = \{[z_0 : z_1 : \ldots : z_n] : z_0^2 + z_1^2 + \ldots + z_n^2 = 0\} \subseteq \mathbb{CP}^n.$$
Show that $H^{2n-3}(X) = 0$.

Historical Note: This is a particular case of a remarkable and general result of the geometer S. Lefschetz (1924), known as the Lefschetz Hyperplane Theorem. His article was titled "L'Analysis situs et la géométrie algébrique". Remember that H. Poincaré founded topology in his 1895 book "Analysis Situs", and "*Analysis Situs*" is how they referred to "*topology*" in the creation of the field. S. Lefschetz's article was one of the first works connecting *topology* and *algebraic geometry*.