This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. Fill in all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may not use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this and explain why the theorem may be applied.

(B) Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) Mysterious or unsupported answers will not receive full credit. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.
1. (20 points) (Graph Profiling) Consider the graph $G = (V, E)$ in Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{graph.png}
\caption{The graph for Problem 1.}
\end{figure}

(a) (5 points) Show that $G$ does not admit an Euler cycle.

(b) (5 points) Prove that there exists a Hamiltonian cycle.

(c) (5 points) Prove that $\chi(G) \geq 3$, i.e. $G$ is not bipartite.

(d) (5 points) Let $T$ be a spanning graph for $G$. How many edges does $T$ have?
2. (20 points) **(Trees and Cayley’s Theorem)** Let $B_n$ be the $(n,n)$-dumbbell graph, obtained by joining two disjoint complete $K_n$ graphs with one edge, as depicted in Figure 2 for the cases $n = 4, 5, 6, 7$.

(a) (10 points) Show that the number of spanning trees of $B_n$ is $n^{2n-4}$.

(b) (10 points) Let $T_n$ be the number of unlabeled trees in $n$ vertices. Show that this number satisfies the inequality

$$n^{n-2} \leq n!T_n.$$
3. (20 points) (Perfect Matchings) Solve the following two parts.
   
   (a) (5 points) Show that a tree $T$ has at most one perfect matching.

   (b) (10 points) Prove that a bipartite graph $G$ such that all vertices have the same degree admits a perfect matching.

   (c) (5 points) Construct a graph $G = (V, E)$ in which all vertices have the same degree but $G$ does not admit a perfect matching.
4. (20 points) (Planarity And Colorings) Consider the graph $G = (V, E)$ in Figure 3.

(a) (10 points) Show that $G$ is not planar.

![The Petersen Graph](image)

Figure 3: The Petersen Graph for Problem 4.

(b) (10 points) Find the chromatic number $\chi(G)$ of $G$. 


5. (20 points) **(Chromatic Properties)**

(a) (10 points) Show that the chromatic polynomial for the complete bipartite graph $K_{2,3}$ is given by the polynomial

$$
\pi_{K_{2,3}} = x(x - 1)^3 + x(x - 1)(x - 2)^3.
$$

These graphs $C_n$ are depicted in Figure 4 for $n = 3, 4$ and 5.

![Figure 4: The complete bipartite graph $K_{2,3}$.](image)

(b) (10 points) Let $G$ be a graph with chromatic polynomial

$$
\pi_G(x) = x^6 - 15x^5 + 85x^4 - 225x^3 + 274x^2 - 120x.
$$

Show that $G$ is non-planar.