University of California Davis  
Combinatorics MAT 145  
Sample Midterm Examination  
February 8 2019  
Time Limit: 50 Minutes

This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

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Do not write in the table to the right.
1. (20 points) Prove the following two statements.

(a) (10 points) Prove that for every \( n \in \mathbb{N} \)
\[ \binom{2n}{2} = 2 \binom{n}{2} + n^2. \]

**Solution.** The left hand side counts the number of 2-element subsets of a \((2n)\)-element set \(X\). Let us count the element of \(X\) differently. Consider \(X\) as a union of two copies of \(\{1, 2, \ldots, n\}\), so that
\[ X = \{1, 2, \ldots, n\} \cup \{1, 2, \ldots, n\}. \]
Then a pair of elements can come from the first copy, contributing \(\binom{n}{2}\), from the second copy, contributing \(\binom{n}{2}\), or each element from a different copy, giving \(n^2\). These three cases contribute a total of
\[ 2 \binom{n}{2} + n^2, \]
which is the right hand side, as requested. \(\square\)

(b) (10 points) Show that for every \( n \in \mathbb{N} \)
\[ \binom{n}{0} + 2^1 \binom{n}{1} + 2^2 \binom{n}{2} + \ldots + 2^k \binom{n}{k} + \ldots + 2^n \binom{n}{n} = 3^n. \]

**Solution.** The binomial theorem states that
\[ x^0 y^n \binom{n}{0} + xy^{n-1} \binom{n}{1} + x^2 y^{n-2} \binom{n}{2} + \ldots + x^k y^{n-k} \binom{n}{k} + \ldots + x^n y^0 \binom{n}{n} = (x+y)^n, \]
evaluating at \(x = 2\) and \(y = 1\) we obtain the above identity. \(\square\)
2. (20 points) Consider a 5×6 rectangular grid as depicted in Figure 1. A staircase walk is a path in the grid which moves only right or up.

![Figure 1: The 5×6 grid and a valid staircase walk from A to B avoiding P.](image)

(a) (10 points) Find the number of staircase walks from A to B.

Solution. There are \( \binom{5+6}{6} = \binom{11}{6} = 462 \) staircase walks from A to B, since we have a 11 unit walk and we decide in which of the 11 units we take our 6 rights.

(b) (10 points) How many staircase walks from A to B avoid the point P?

There are \( \binom{6}{3} \cdot \binom{5}{2} = 20 \cdot 10 = 200 \) staircase walks that pass through P, since there are \( \binom{6}{3} \) paths from A to P, and \( \binom{5}{2} \) paths from P to B. Thence, there are

\[
\binom{11}{6} - \binom{6}{3} \cdot \binom{5}{2} = 462 - 200 = 262,
\]

staircase walks from A to B which avoid P.
3. (20 points) Solve the following two questions.

(a) (10 points) Compute the number of positive integers \( n \in \mathbb{N} \) such that \( 1 \leq n \leq 10000 \), and which are \textbf{not} divisible by 6, 7, 9.

\textbf{Solution.} Let \( A_6, A_7 \) and \( A_9 \) be the sets of such integers divisible by 6, 7 and 9. There are \( \left\lfloor \frac{10000}{6} \right\rfloor = 1666 \) such integers which are divisible by 6, \( \left\lfloor \frac{10000}{7} \right\rfloor = 1428 \) such numbers which are divisible by 7 and \( \left\lfloor \frac{10000}{9} \right\rfloor = 1111 \) such integers which are divisible by 9, and thus

\[
|A_6| = 1666, \quad |A_7| = 1428, \quad |A_9| = 1111.
\]

The desired answer is \( 10000 - |A_6 \cup A_7 \cup A_9| \). We compute \( |A_6 \cup A_7 \cup A_9| \) via the Inclusion-Exclusion Principle. Since we have

\[
|A_6 \cap A_7| = \left\lfloor \frac{10000}{42} \right\rfloor = 238, \quad |A_6 \cap A_9| = \left\lfloor \frac{10000}{lcm(6, 9)} \right\rfloor = \left\lfloor \frac{10000}{18} \right\rfloor = 555,
\]

\[
|A_7 \cap A_9| = \left\lfloor \frac{10000}{63} \right\rfloor = 158,
\]

and also \( \left\lfloor \frac{10000}{lcm(6, 7, 9)} \right\rfloor = \left\lfloor \frac{10000}{126} \right\rfloor = 79 \), we conclude that

\[
|A_6 \cup A_7 \cup A_9| = 1666 + 1428 + 1111 - 238 - 555 - 158 + 79 = 3333.
\]

Thus, the requested answer is \( 10000 - 3262 = 6738 \). \( \square \)

(b) (10 points) Prove that in any group of six people there are either three people who all know each other, or three people who are mutual strangers.

\textbf{Solution.} Consider a person \( A \), by the pigeonhole principle that person either \textit{knows} at least three people \( P_1, P_2 \) and \( P_3 \) or does \textit{not know} at least three people, which we also call \( P_1, P_2 \) and \( P_3 \).

In the former case, either these three people do not know each other (and so we are done since \( P_1, P_2 \) and \( P_3 \) satisfies the requirement), or two of them, say \( P_1 \) and \( P_2 \), know each other, thus yielding three people, \( P_1, P_2 \) and \( A \), who know each other.

In the latter case, one proceed analogously. \( \square \)
4. (20 points) Consider a French deck of 52 cards, containing 4 suits with 13 values

\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, K, Q\}

in each of the four suits. We are dealt five cards from the deck. These five cards constitute our hand.

(a) (10 points) A hand of five cards is said to be a **trio** if it contains exactly three cards with the same value, and the remaining two cards have distinct values amongst them. Show that the probability of having a trio is approximately 0.021128.

**Solution.** The total number of hands of five cards is \(\binom{52}{5} = 2598960\). Let us count how many of such hands can be trios. For that we select a value for the trio, which gives \(\binom{13}{1} = 13\), a suit for the trio, which gives a 4. Then we select the remaining two cards, which have two possible distinct values, giving \(\binom{12}{2} = 66\), since they cannot be of the same value as the trio, and their possible suits contribute with \(4^2\). The probability of a trio is thus

\[
\frac{13 \cdot 4 \cdot 66 \cdot 4^2}{\binom{52}{5}} = \frac{54912}{2598960} = 0.021128451.
\]

(b) (10 points) A hand of five cards is said to be a **full house** if it contains exactly three cards with the same value, and the remaining two cards have the same value. Compute the probability of having a full house.

**Solution.** The same reasoning as above yields a total of \(13 \cdot 4 \cdot 12 \cdot \binom{4}{2}\) hands with a full house. Thus the probability of having a full house is

\[
\frac{13 \cdot 4 \cdot 12 \cdot \binom{4}{2}}{\binom{52}{5}} = \frac{3744}{2598960} = 0.00144057623.
\]
5. (20 points) Let us consider a die with six faces, each face with a value in \( \{1, 2, 3, 4, 5, 6\} \). Roll the die four consecutive times. Compute the probability that we will see a 3 in at least one of these four rolls.

**Solution.** The set of all possible outcomes of rolling a die four times has size \( 6^4 = 1296 \), if we consider that order matters. Let \( X_1 \) be the subset consisting of those four rolls such that the first roll is a 3. Similarly, let \( X_2 \) (and \( X_3 \) and \( X_4 \)) be the subsets consisting of those four rolls such that the second (the third or the fourth) roll is a 3.

We are interested in the cardinality of \( |X_1 \cup X_2 \cup X_3 \cup X_4| \), and we can use the Inclusion-Exclusion Principle to compute this cardinality. For that we require the following sizes:

\[
|X_1| = |X_2| = |X_3| = |X_4| = 6^3 = 216, \\
|X_i \cap X_j| = 6^2 = 36, \quad \forall i, j \in \{1, 2, 3, 4\}, \\
|X_i \cap X_j \cap X_k| = 6^1 = 6, \quad \forall i, j, k \in \{1, 2, 3, 4\}, \\
|X_1 \cap X_2 \cap X_3 \cap X_4| = 6^0 = 1.
\]

Thus the cardinality is

\[
|X_1 \cup X_2 \cup X_3 \cup X_4| = 4 \cdot 216 - \binom{4}{2} \cdot 36 + 4 \cdot 6 - 1 = 671,
\]

and the probability is \( \frac{671}{1296} = 0.51774691358 \).

**Solution II.** There is an alternative solution, which does not use the Inclusion-Exclusion Principle, in that the union involved consists of disjoint sets. The required probability can be computed as follows. First, the complement of the set \( X \) is

\[
X^c = (X_1 \cup X_2 \cup X_3 \cup X_4)^c = X_1^c \cap X_2^c \cap X_3^c \cap X_4^c,
\]

and since the sets \( X_i^c \), for \( i = 1, 2, 3 \) and 4, count independent outcomes, each with probability \( 1 - \frac{1}{6} \). Then we have that the probability of that intersection is

\[
\left(1 - \frac{1}{6}\right)^4 = 0.48225308642.
\]

Since we want the opposite event, the required probability is

\[
1 - \left(1 - \frac{1}{6}\right)^4 = 1 - 0.48225308642 = 0.51774691358,
\]

yielding the same result as above. \( \square \)