09/27/2013: The Fluid Continuum

0.1 The fluid continuum

- Blob of fluid: made of molecules but we’ll treat it as a continuum (the material is infinitely divisible).
  - Properties such as density, temperature, etc. can be described as a continuous on the blob: $T(\bar{x})$.
  - Describe the motion by a velocity field, regardless of whether there is a molecule at that point.

- Is this a valid approximation?
  - **Short answer:** On the appropriate scale, YES.
  - **Long answer:** For water the spacing between molecules is what?

$$\rho = 1.00 \text{ g/cm}^3(@ \text{ room temperature, @ atm pressure})$$

$$\text{molecular weight} \approx 18 \text{ g/mol}$$

$$\Rightarrow 18 \text{ cm}^3/\text{mol} = \frac{18}{6 \times 10^{23}} \text{ cm}^3/\text{molecules} = 3 \times 10^{-23} \text{ cm}^3/\text{molecules}$$

  - Then, approximately $3 \times 10^{-8} \text{ cm}$ between molecules or 3 angstroms (about the same size as molecules themselves). Size of bacterium: 1 $\mu$m (while most cells are around 10 $\mu$m) is 10,000 times bigger. The size of a virus is 100 nm which is 1000 times bigger. The diameter of DNA is around 2-3 nm (10 times larger). Therefore, the fluid continuum approximation is valid for anything larger than a few nanometers.

- **What about gases?** Also a fluid but density changes much more substantially with pressure and temperature (think of Ideal Gas Law). Characterized by much more space between molecules.
  - If $\ell$ is the mean free path of the molecules (avg. distance a molecule travels between collisions) and $L$ is the lengthscale of the problem, then we define the **Knudsen number**:

$$\text{Kn} = \ell / L$$

  - Note that $\text{Kn} \ll 1$ for air at room temperature and atm pressure with $\ell \approx 50 \text{ nm}$ (gas continuum).
09/30/2013: Eulerian and Lagrangian Coordinates

0.2 Eulerian and Lagrangian Coordinate Systems

• Let $S_0$ be a region containing fluid at $t = 0$. Under flow at time $t = T$ the region becomes $S_T$. Suppose there is some map $S_0 \rightarrow S_T$.

  – Let $a \in S_0$ and $x = X(a, t) \in S_T$. Let $X$ be the position of a fluid particle at time $T$ that was at position $a$ at time zero (i.e. $X(a, 0) = a$).

  – $a$ is called the **Lagrangian coordinate** of the fluid particles. It moves with the material.

  – Typically we describe fluids in a fixed coordinate system, **Eulerian coordinate system**. This coordinate system is fixed in space.

• What is the velocity of “particle $a$”?

\[
\frac{\partial X}{\partial t} = \tilde{u}(a, t)
\]

  is the velocity in the Lagrangian coordinate system.

  – Given $X(a, t)$ it is difficult to find $u(x, t)$ (the Eulerian velocity) because you need to invert to find $a = X^{-1}(x, t)$.

  – Given the flow field $u(x, t)$ can we find $X$? We can construct $X(a, t)$ by solving:

\[
\frac{dX}{dt} = u,
\]

$X(a, 0) = a$

• **Material derivative**: is the Lagrangian time derivative expressed in Eulerian coordinates.

  – Let $P$ be a property of the fluid (temperature, density, color, etc.). Assume that $P$ follows the flow.

  – $\tilde{P}(a, t)$ be the Lagrangian representation of $P$.

  – $P(x, t)$ be the Eulerian representation of $P$.

  – Now, let’s suppose $P$ is invariant with the flow, i.e. $\frac{\partial \tilde{P}}{\partial t} = 0$

  – Then,

\[
\tilde{P}(a, t) = P(X(a, t), t) = P(x, t)
\]

\[
\Rightarrow \frac{\partial \tilde{P}}{\partial t} = \frac{d}{dt}P(X(a, t), t)\text{ where }\frac{d}{dt}\text{ is the total time derivative}
\]

\[
= \frac{D}{Dt}P(x, t)\text{ where }\frac{D}{Dt}\text{ denotes the material derivative}
\]

\[
= \frac{d}{dt}P(X(a, t), Y(a, t), Z(a, t), t)
\]

\[
= \frac{\partial P}{\partial X} \frac{\partial X}{\partial t} + \frac{\partial P}{\partial Y} \frac{\partial Y}{\partial t} + \frac{\partial P}{\partial Z} \frac{\partial Z}{\partial t}
\]

\[
= \frac{\partial P}{\partial t} + u \frac{\partial P}{\partial X} + v \frac{\partial P}{\partial Y} + w \frac{\partial P}{\partial Z}
\]

\[
= \frac{\partial P}{\partial t} + \mathbf{u} \cdot \nabla P.
\]
– The material derivative is:
\[
\frac{DP}{Dt} = \frac{\partial P}{\partial t} + u \cdot \nabla P
\]

– For \( P \) to be invariant with the flow we require: \( DP/Dt = 0 \).

**Balance or Conservation Laws:**
– e.g. mass, momentum, energy, etc.
– Will be of the form: \( \frac{DP}{Dt} = F \),
\[
\frac{\partial P}{\partial t} + u \cdot \nabla P = F
\]
\[
\frac{\partial P}{\partial t} = -u \cdot \nabla P + F
\]
where \( \cdot \) term is movement past a fixed location and \( F \) represents a source or sink.

**How to compute acceleration?**
– In Lagrangian coordinates,
\[
a = \frac{\partial^2}{\partial t^2} X(a,t)
\]

– In Eulerian coordinates, given flow velocity \( u(x,t) \) the acceleration is: \( Du/Dt \),
\[
a = \frac{Du}{Dt} = u_t + u \cdot \nabla u
\]
\[
= u_t + (u \cdot \nabla) u
\]
\[
= u_t + (u_\partial_x + u_\partial_y + u_\partial_z) u
\]

**Streamlines, particle paths, streaklines**
– For all the above definitions we are given \( u(x,t) \).
– **Particle paths:**
\[
\frac{dX}{dt} = u
\]
\[
X(0) = X_0
\]
– The trajectories that individual fluid particles follow.
– **Instantenous streamlines:** the integral curves of \( u(x,t) \) frozen in time.
\[
\frac{dX}{ds} = u(x,t) \text{ note it is independent of } s
\]
\[
X(0) = X_0
\]
– If flow is steady \( u_t = 0 \), then streamlines and particle paths are the same.
– It is instantaneously tangent to the velocity vector of the flow. These show the direction a fluid element will travel in at any point in time.
Figure 1: Streaklines.

- **Streaklines**: Inject dye at fixed point in space for time \(0 \leq \tau \leq t\)

\[
\frac{dX}{dt} = u \\
X(t = \tau) = X_0
\]

- Get curves with \(\tau\) as a parameter.

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10/02/2013: Eulerian and Lagrangian Coordinates (cont)

0.3 Example (Streamlines, Particles Paths, and Streaklines):

**\(u = (1 \ t)^T\)**

- **Streamlines**:

\[
\frac{dx}{ds} = 1 \quad \frac{dy}{ds} = t \\
x(s = 0) = x_0 \\
x = s + x_0 \quad y = ts + y_0 \\
y = t(x - x_0) + y_0 \text{ straight lines}
\]

- **Particle Paths**:

\[
\frac{dx}{dt} = 1 \quad \frac{dy}{dt} = t \\
x(t = 0) = x_0 \\
x = t + x_0 \quad y = t^2/2 + y_0 \\
y = \frac{1}{2}(x - x_0)^2 + y_0 \text{ parabolas}
\]
• Streaklines:

\[
\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = t
\]

\[
x(t = \tau) = x_0
\]

\[
x = (t - \tau) + x_0, \quad y = \left(t^2 - \tau^2\right)/2 + y_0
\]

\[
y = \frac{1}{2}(x - x_0)^2 + y_0 \text{ parabolas}
\]

• Let’s pick \(x_0 = (0, 0)\) and \(0 \leq \tau \leq 1 = t\).
  
  – Streamlines: \(y = x\);
  
  – Particle paths: \(y = x^2/2\);
  
  – Streaklines: \(x = 1 - \tau, \quad y = \frac{1}{2}(1 - \tau^2)\) or \(y = \frac{1}{2} - \frac{1}{2}(1 - x)^2\)

0.4 Changes in Local Volume

• Let \(\Omega(0)\) be a set of points in the fluid with flow field \(u\) taken to \(\Omega(t) = \{X(\alpha, t) \mid \alpha \in \Omega(0)\}\) at a later time \(t\). Then the volume changes can be tracked:

\[
V(t) = \int_{\Omega(t)} dx
\]

\[
\frac{dV}{dt} = \frac{d}{dt} \int_{\Omega(t)} dx
\]

\[
\frac{dV}{dt} = \frac{d}{dt} \int_{\Omega(0)} |J| da \text{ where } J_{ij} = \frac{\partial X_i}{\partial a_j}
\]

\[
\frac{dV}{dt} = \int_{\Omega(0)} \frac{\partial |J|}{\partial t} da
\]

• Taking derivatives of determinants:

\[
|J| = \begin{vmatrix}
\frac{\partial X_1}{\partial a_1} & \frac{\partial X_1}{\partial a_2} \\
\frac{\partial X_2}{\partial a_1} & \frac{\partial X_2}{\partial a_2}
\end{vmatrix}
\]

\[
\frac{\partial |J|}{\partial t} = \frac{\partial}{\partial t} \frac{\partial X_i}{\partial a_j} = \frac{\partial u_i}{\partial a_j}
\]

\[
\frac{\partial |J|}{\partial a_j} = \frac{\partial u_1}{\partial a_1} \frac{\partial X_1}{\partial a_j} + \frac{\partial u_1}{\partial X_2} \frac{\partial X_2}{\partial a_j}
\]

\[
\frac{\partial |J|}{\partial a_2} = \frac{\partial u_1}{\partial X_1} \frac{\partial X_1}{\partial a_2} + \frac{\partial u_1}{\partial X_2} \frac{\partial X_2}{\partial a_2}
\]

\[
\frac{\partial |J|}{\partial t} = \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) |J|
\]

\[
= \nabla \cdot u |J|
\]
Then, going back to the change in volume expression,

\[
\frac{dV}{dt} = \int_{\Omega(0)} \nabla \cdot |J| da
\]

\[
= \int_{\Omega(t)} \nabla \cdot u \, dx
\]

\[
= \int_{\partial\Omega(t)} u \cdot n \, dS
\]

N.B. This says that the volume changes by fluid moving across the boundary.

- **Incompressible flow**: incompressible if volume of blobs of fluid do not change in time, i.e. \( \frac{dV}{dt} = \int_{\Omega(t)} \nabla \cdot u \, dx = 0 \) for any \( \Omega(t) \).

  Flow is incompressible iff \( \nabla \cdot u = 0 \).

### 10/04/2013: Tensor Notation and Calculus

**Streamfunctions**

- Let \( u = (u(x, y), v(x, y)) \) be a 2D planar flow. Then, \( \nabla \cdot u = u_x + v_y \). Suppose:

  \[
  u = \psi_y \quad v = -\psi_x
  \]

  for scalar function \( \psi(x, y) \). So, \( \nabla \cdot u = \psi_{yx} - \psi_{xy} = 0 \). Then for any \( \psi(x, y) \) this defines a 2D incompressible flow.

- \( \nabla \psi = (\psi_x, \psi_y) \) and \( u = (\psi_y, -\psi_x) \). So: \( u \cdot \nabla \psi = 0 \). Directional derivative of \( \psi \) in the flow direction is zero. Thus, \( \psi \) is constant on streamlines (or particle paths in this case since it’s a steady flow). We call \( \psi \) the **streamfunction**. The level sets of \( \psi \) are the streamlines.

- Given a 2D incompressible flow, is there a streamfunction? Yes, but it is unique up to an additive constant.

- There are other 2D nonplanar flows. If \( u = \nabla \times B \) defines an incompressible flow. In 2D planar flow, we say \( u = \nabla \times (\psi(x, y)e_z) = (\psi_y, -\psi_x, 0) \). If we have spherical axisymmetry (no dependence on \( \theta \) variable), \( u = \nabla \times (\psi(r, \phi)e_\theta) \). But is this \( \psi \) a stream function? No (see below for explanation).

- For cylindrical axis symmetry: \( u = \nabla \times (\psi(r, z)e_\theta) \)

  \[
  u_r = -\psi_z \quad , u_z = \frac{1}{r} \frac{\partial}{\partial r} (r\psi)
  \]

  \[
  \nabla \psi = \frac{\partial \psi}{\partial r} e_r + \frac{\partial \psi}{\partial z} e_z
  \]

  Generally, \( u \cdot \nabla \psi \neq 0 \).
Let’s try again, \( \mathbf{u} = \nabla \times \left( \frac{1}{r} \psi(r, z) \mathbf{e}_\theta \right) \). Then:

\[
\mathbf{u} = -\frac{1}{r} \frac{\partial \psi}{\partial z} \mathbf{e}_r + \frac{1}{r} \frac{\partial \psi}{\partial r} \mathbf{e}_z
\]

Now, \( \mathbf{u} \cdot \nabla \psi = 0 \).

\[\Psi = \frac{1}{r} \psi(r, z) \text{ is the Stokes streamfunction.}\]

**Gradients of velocity - Linearization of flow**

- Let’s say I have a scalar function, \( f(\mathbf{x}) : \mathbb{R}^3 \rightarrow \mathbb{R} \). Taylor expand \( f \) about \( \mathbf{x}_0 \):

\[
f(\mathbf{x}_0 + \mathbf{x}) = f(\mathbf{x}_0) + \frac{\partial f}{\partial x_1} x_1 + \frac{\partial f}{\partial x_2} x_2 + \frac{\partial f}{\partial x_3} x_3 + O(||\mathbf{x}||^2_2)
\]

\[
= f(\mathbf{x}_0) + \sum_{i=1}^{3} \frac{\partial f(\mathbf{x}_0)}{\partial x_i} x_i + O(||\mathbf{x}||^2_2)
\]

\[
= f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot \mathbf{x} + O(||\mathbf{x}||^2_2), \text{ this is a linear function of } 3 \text{ variables.}
\]

Think of the gradient as a linear operator on \( \mathbb{R}^3 \rightarrow \mathbb{R} \).

**Einstein Summation Convention**

\[
\sum_{i=1}^{3} \frac{\partial f(\mathbf{x}_0)}{\partial x_i} x_i = \frac{\partial f(\mathbf{x}_0)}{\partial x_i} x_i
\]

Sum is implicit with repeated indices in a product. Here are a couple examples:

\[
\mathbf{x} \cdot \mathbf{y} = x_i y_i
\]

\[
A \mathbf{x} = \mathbf{b} \Rightarrow b_i = A_{ij} x_j
\]

\[
\mathbf{c} = \mathbf{a} \times \mathbf{b} \Rightarrow c_i = \epsilon_{ijk} a_j b_k \text{ where } \epsilon \text{ is the alternating tensor}
\]

\[
\epsilon_{ijk} = \begin{cases} 
1 & \text{if } (i, j, k) \text{ is even permutation of } (1, 2, 3) \\
-1 & \text{if } (i, j, k) \text{ is odd permutation of } (1, 2, 3) \\
0 & \text{otherwise.}
\end{cases}
\]

\[
c_1 = \epsilon_{123} a_2 b_3 + \epsilon_{132} a_3 b_2 = a_2 b_3 - a_3 b_2
\]

**Linearization of flow**

- \( \mathbf{u}(\mathbf{x}) : \mathbb{R}^3 \rightarrow \mathbb{R} \). Let us linearize \( \mathbf{u}(\mathbf{x}_0 + \mathbf{x}) \) about \( \mathbf{x}_0 \):

\[
u_i(\mathbf{x}_0 + \mathbf{x}) = u_i(\mathbf{x}_0) + \frac{\partial u_i}{\partial x_j} x_j + O(||\mathbf{x}||_2)
\]

\[
(\nabla \mathbf{u})_{ij} = \frac{\partial u_i}{\partial x_j} \text{ the } (i, j) \text{ entry of a matrix or 2-tensor}
\]
10/07/2013: Tensor Calculus and Notation (cont’d)

Linearization of flow (cont’d)

Split \( \nabla \mathbf{u} \) into a symmetric part \( (D) \) + antisymmetric part \( (A) \):

\[
\nabla \mathbf{u} = D + A
\]

where \( D = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \) and \( A = \frac{1}{2}(\nabla \mathbf{u} - \nabla \mathbf{u}^T) \) (notice \( A^T = -A \)).

\[
\frac{dx}{dt} = \mathbf{u}(x_0 + x) - \mathbf{u}(x_0) = D\mathbf{x} + A\mathbf{x} + \text{h.o.t.}
\]

Look at each term separately:

- **Symmetric part:** \( \frac{dx}{dt} = D\mathbf{x} \)
  - Because \( D \) is symmetric, eigenvalues are real;
  - Eigenvectors are orthogonal (coordinate transformation is just a rotation);
  - Express in eigencoordinates: \( \frac{dy}{dt} = \lambda_i t \mathbf{y} \Rightarrow y_i(t) = y_i(0)e^{\lambda_i t} \) for \( i = 1, 2, 3 \);
  - \( D \) describes the rate (and directions) at which boxes of fluid are stretched;
  - Notice that \( D = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \) has units of \( 1/T \);
  - Called the rate of strain tensor and the deformation rate tensor;
  - Take a box aligned with the eigenvectors of \( D \), let’s write an expression for volume of box and how it changes over time:

\[
V(t) = y_1(t)y_2(t)y_3(t) = y_1(0)y_2(0)y_3(0)e^{(\lambda_1 + \lambda_2 + \lambda_3)t}
\]

\[
= V(0)e^{(\lambda_1 + \lambda_2 + \lambda_3)t}
\]

\[
D_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

\[
\text{Tr}(D) = \frac{\partial u_i}{\partial x_i} = \nabla \cdot \mathbf{u} = \text{Tr}(\nabla \mathbf{u}) \text{ because Tr}(A) = 0.
\]

Again we see that \( \nabla \cdot \mathbf{u} = 0 \) implies no changes in local volume.

- **Antisymmetric part:** \( \frac{dx}{dt} = A\mathbf{x} \) where \( A^T = -A \). Assume \( D = 0 \)
  - Pure imaginary eigenvalues;
  - Start in 2D:

\[
A = \begin{pmatrix}
0 & -\omega \\
\omega & 0
\end{pmatrix}
\]
Eigenvalues are $\pm i\omega$ with:

$$x(t) = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} x(0)$$

This is a rigid body rotation with angular velocity $\omega$.

- In 3D, special case:

$$A = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

One of the eigenvalues must be a zero eigenvalue. Describes a rotation about the z-axis.

$$Ax = \begin{pmatrix} -\omega y \\ \omega z \\ 0 \end{pmatrix} = (\omega e_z) \times x$$

We could have said instead: $dx/dt = \Omega \times x$ where $\Omega = (0 \ 0 \ \omega)^T$.

- In general 3D:

$$A = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

$$Ax = \Omega \times x \text{ where } \Omega = (\omega_1 \ \omega_2 \ \omega_3)$$

$$A\Omega = 0 \text{ because } \Omega \times \Omega = 0.$$ 

Describes a rotation about axis with the same direction as $\Omega$ with angular velocity $|\Omega|$. You can work out the relationship between $\Omega$ and $A$:

$$\Omega = \frac{1}{2} (\nabla \times u)$$

$$\omega = \nabla \times u$$

is the vorticity of the flow. Local change of a vector NOT a global rotation of the system. Vorticity measures how material vectors are rotating and not how points are rotating.

$$\frac{1}{2} (\nabla \times u)$$: angular velocity of rotation

Finally,

$$u(x_0 + x) = u(x_0) + \frac{1}{2} (\nabla u + \nabla u^T)x + \frac{1}{2} (\nabla u - \nabla u^T)x + ...$$

First term describes translation, second term describes stretching, and third term describes rotation.
Figure 2: Velocity profile for the example below, \( u = (\Omega r)e_\theta \).

10/09/13: Vorticity and Conservation/Balance Laws

Vorticity Examples:

- Recall vorticity describes how tiny boxes of fluid move with the flow not how particles move with the flow. The leaf going down the gutter: Tracking the leaf has nothing to do with vorticity however the spinning of the leaf is representative of vorticity.

- Let \( u = r\Omega e_\theta \). To compute vorticity:
  
  \[
  \omega = \nabla \times u = 0e_r + 0e_\theta + (2\Omega)e_z
  \]

  This is constant vorticity. This is just a rigid body rotation.

- However, rotation of the fluid does not always indicate vorticity. Let \( u = (\Omega/r)e_\theta \). To compute vorticity:
  
  \[
  \omega = \nabla \times u = 0e_r + 0e_\theta + 0e_z = (0 0 0)^T
  \]

  away from \( r = 0 \). Clearly all the flow is moving in the angular direction, yet no vorticity. Why?

Figure 3: Example of no vorticity but angular rotation is nonzero, \( u = (\Omega/r)e_\theta \).
Circulation

Let $C$ be a simple (= it's not self intersecting) closed smooth, oriented curve. The circulation around that curve (tendency for things to move around it) is:

$$\Gamma = \int_C \mathbf{u} \cdot d\mathbf{s}$$

Equivalent to integrating the tangential component of velocity (flow) around the curve.

Let $S$ denote the surface with boundary $C$, i.e. $\partial S = C$. Let $\mathbf{n}$ be the normal vector of the surface $S$. Using Stokes’ theorem, we can alternatively express the circulation as:

$$\Gamma = \int_S (\nabla \times \mathbf{u}) \cdot \mathbf{n} \, dA = \int_S \mathbf{\omega} \cdot \mathbf{n} \, dA$$

Therefore, indeed the circulation is related to looking at the total vorticity in the surface.

![Diagram of fluid flow with normal and velocity](image)

Figure 4: Blob of fluid with normal $\mathbf{n}$ and flow velocity $\mathbf{u}$.

**Example:** Recall that $\mathbf{u} = (\Omega/r)\mathbf{e}_{\theta}$ has zero vorticity. Let $C$ be the circle: $\mathbf{x}(\theta) = (r \cos(\theta), r \sin(\theta))$ for $0 \leq \theta \leq 2\pi$.

\[
\Gamma = \int_0^{2\pi} \mathbf{u} \cdot \mathbf{x}'(\theta) \, d\theta \\
= \int_0^{2\pi} \mathbf{u} \cdot r \mathbf{e}_\theta d\theta \\
= \int_0^{2\pi} \Omega d\theta \\
= 2\pi \Omega \\
\Gamma = \int_D (\nabla \times \mathbf{u}) \, d\mathbf{x} \text{ but not defined at } r = 0
\]

Yes, there is vorticity but a $\delta$-function vorticity at the origin. Called **point vortex**.
Conservation of Mass and Linear Momentum

Let $\Omega$ be a fixed region of space within the fluid. Let $\rho(x, t)$ (in Eulerian coordinates) be the mass density of the fluid with units of mass/vol.

$$\int_{\Omega} \rho(x, t)dx = \text{mass of the fluid in } \Omega$$

$$\frac{d}{dt} \int_{\Omega} \rho(x, t)dx = - \int_{\partial\Omega} \rho \mathbf{u} \cdot \mathbf{n} dS = - \int_{\partial\Omega} \mathbf{J} \cdot \mathbf{n} dS$$

where $\mathbf{J} = \rho \mathbf{u}$ is the mass flux density and $\mathbf{n}$ is an outward normal vector.

Because $\Omega$ is fixed:

$$\int_{\Omega} \rho_t dV = - \int_{\partial\Omega} \rho \mathbf{u} \cdot \mathbf{n} dS$$

$$\int_{\Omega} \rho_t + \nabla \cdot (\rho \mathbf{u}) dV = 0$$

Because $\Omega$ is arbitrary and $\rho, \mathbf{u}$ is smooth enough, this leads us to the differential form of conservation of mass:

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0$$

10/11/2013: Conservation Laws

The differential equation for the conservation of mass is,

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0$$

where $\rho$ is the mass density of the fluid. If there were sources or sinks,

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = Q$$

where $Q$ has units of mass/(vol-time).

Reynold’s Transport Theorem

Let $f(x, t)$ be density of something in the fluid.

$$\frac{d}{dt} \int_{\Omega(t)} f(x, t) dV = \int_{\Omega(t)} (f_t + \nabla \cdot (f \mathbf{u})) dV$$

Now imagine $f$ is in dimensions of eff/mass. Then:

$$\frac{d}{dt} \int_{\Omega(t)} \rho f(x, t) dV = \int_{\Omega(t)} \rho \frac{Df}{Dt} dV$$

12
where $D/Dt$ is a material derivative and we assume no sources/sinks of mass. We’ll next prove these equalities.

\[
\frac{d}{dt} \int_{\Omega(t)} f(x, t) dV = \frac{d}{dt} \int_{\Omega(0)} \tilde{f}(a, t)|J| da \\
= \int_{\Omega(0)} \frac{d\tilde{f}}{dt} |J| + \tilde{f} \nabla \cdot u |J| da \\
= \int_{\Omega(0)} \left[ \frac{d\tilde{f}}{dt} + \tilde{f} \nabla \cdot u \right] |J| da \\
= \int_{\Omega(t)} \left[ \frac{Df}{Dt} + f \nabla \cdot u \right] dV \\
= \int_{\Omega(t)} \left[ f_t + u \cdot \nabla f + f \nabla \cdot u \right] dV \\
= \int_{\Omega(t)} \left[ f_t + \nabla \cdot (fu) \right] dV.
\]

\[
\frac{d}{dt} \int_{\Omega(t)} \rho f(x, t) dV = \int_{\Omega(t)} (\rho f)_t + \nabla \cdot (\rho fu) dV \\
= \int_{\Omega(t)} \rho f_t + \rho f_t + f \nabla \cdot (\rho u) + \rho u \nabla f dV \\
= \int_{\Omega(t)} \left[ \rho_t + \nabla \cdot (\rho u) \right] f + [f_t + u \cdot \nabla f] \rho dV \\
= 0 + \int_{\Omega(t)} \rho \frac{Df}{Dt} dV \\
= \int_{\Omega(t)} \rho \frac{Df}{Dt} dV.
\]

**Balance of Linear Momentum**

It’s *Newton’s Second Law*, $F = ma$, in disguise. More generally,

\[
\frac{d}{dt}(mv) = \sum_j F_j
\]

time change of momentum is balanced by the sum of applied forces. In the continuum realm, $\rho u$ is the momentum density or momentum per unit volume. Then, Newton’s second law becomes:

\[
\frac{d}{dt} \int_{\Omega(t)} \rho u dV = \int_{\Omega(t)} f dV + \int_{\partial \Omega(t)} f_s dS
\]

where $f$ is a body force (units of force per unit volume) and $f_s$ is a surface force (units of force per unit area). To express this as a differential equation, we’ll use Reynold’s Transport Theorem:

\[
\int_{\Omega(t)} \rho (u_t + u \cdot \nabla u) dV = \int_{\Omega(t)} f dV + \int_{\partial \Omega(t)} f_s dS
\]
What happens as $\ell \to 0$? We can approximate this by, $\ell^3$ value = $\ell^3$ value + $\ell^2(\ldots) \Rightarrow$ value = value + $\ell^2/\ell^3$ ($\ldots$). As $\ell \to 0$, then:

$$\int_{\text{small vol}} f_s \, dS \approx \ell \to 0$$

as $\ell \to 0$. This implies that the net force from surface forces must be zero ($O(\ell)$) on very small volumes.

**How do you define surface force density “at a point”?**

![Figure 5: Considering infinitesimal tetrahedron to find the surface force at a point as a linear function of the normal to the surface.](image)

Using the fact that the tetrahedron is small, we approximate the integral as:

$$f_n \delta A_n = f_1 \delta A_1 + f_2 \delta A_2 + f_3 \delta A_3$$

Can show that $\delta A_j = \delta A_n \, (e_j \cdot n)$.

$$f_n \delta A_n = \delta A_n [f_1(e_1 \cdot n) + f_2(e_2 \cdot n) + f_3(e_3 \cdot n)]$$

$$f_n = f_1(e_1 \cdot n) + f_2(e_2 \cdot n) + f_3(e_3 \cdot n)$$

$$f_n = (f_1e_1^T + f_2e_2^T + f_3e_3^T) \, n$$

where $n$ is a column vector. From this we can conclude that the surface force is a linear function of the normal to the surface. $f_n = \sigma n$ where $\sigma$ is called the **stress tensor**.

**10/14/2013: Conservation Laws**

**Balance of Linear Momentum (cont’d):**

$$\int_{\Omega(t)} \rho (u_t + u \cdot \nabla u) \, dV = \int_{\partial\Omega(t)} f_s \, dS + \int_{\Omega(t)} f \, dV$$
Figure 6: Considering infinitesimal tetrahedron to find the surface force at a point as a linear function of the normal to the surface.

Given point and a normal to surface \( \mathbf{n} = (n_1, n_2, n_3) \) is sufficient information to determine the surface force density at a point:

\[
\mathbf{f}_n = \begin{pmatrix} f_1 & f_2 & f_3 \\ \end{pmatrix} \mathbf{n}
\]

where \( \mathbf{n} \) is a column vector and the matrix is \( 3 \times 3 \). From this we can conclude that the surface force is a linear function of the normal to the surface. \( \mathbf{f}_n = \sigma \mathbf{n} \) where \( \sigma \) is called the stress tensor. Then:

\[
\int_{\partial \Omega(t)} \mathbf{f}_s \, dS = \int_{\partial \Omega(t)} \sigma \mathbf{n} \, dS
\]

\[
= \int_{\Omega(t)} \nabla \cdot \sigma \, dV
\]

Familiar divergence theorem:

\[
\int_{\partial \Omega(t)} \mathbf{v} \cdot \mathbf{n} \, dS = \int_{\Omega(t)} \nabla \cdot \mathbf{v} \, dV
\]

\[
\int_{\partial \Omega(t)} v_j n_j \, dS = \int_{\Omega(t)} \frac{\partial}{\partial x_j} v_j \, dV
\]

Applying the same definition of divergence:

\[
\int_{\partial \Omega(t)} \sigma_{ij} n_j \, dS = \int_{\Omega(t)} \frac{\partial}{\partial x_j} \sigma_{ij} \, dV
\]
Balance of mom:

\[
\int_{\Omega(t)} \rho \left( u_t + u \cdot \nabla u \right) dV = \int_{\Omega(t)} \nabla \cdot \sigma + f \, dV \\
\rho \left( u_t + u \cdot \nabla u \right) = \nabla \cdot \sigma + f \\
\rho t + \nabla \cdot ( \rho u ) = 0
\]

The unknowns are \( \rho, u, \sigma \). At this point this holds for a fluid or solid. The choice of constitutive law for \( \sigma \) will determine if it’s a fluid or solid.

\[
\sigma = \begin{pmatrix}
\sigma_{11} & \sigma_{12} \\
\sigma_{21} & \sigma_{22}
\end{pmatrix}
\]

\( \sigma_{ij} \) is the \( i^{th} \) component of the surface force density on a surface with normal in the \( j \)–direction.

**Balance of Angular Momentum**

This is a homework problem (Set #2). For almost all materials, the stress tensor is symmetric. Follows from the balance of angular momentum.

Recall the local balance argument,

\[
\frac{1}{\ell} \int_{\Omega_\ell} f_s \, dS \to 0 \text{ as } \ell \to 0
\]

Consider balance of angular momentum on a small box:

\[
I \frac{d\theta}{dt} = \sum L
\]
where \( I = \int V r^2 \rho(r) dV \) is the moment of inertia and \( L = \int_S r \times \mathbf{f}_s dS \) is a surface torque. Assume no body torques. By scaling, \( I \approx \ell^5 \) and \( T \approx \ell^3 \). Based on this simple argument, \( \frac{d\theta}{dt} \approx \frac{1}{\ell^2} \) if there are no restrictions on surface forces. More carefully, \( \frac{d\theta}{dt} = C \ell^2 (\sigma_{21} - \sigma_{12}) \rightarrow \infty \) unless we require \( \sigma_{12} = \sigma_{21} \).

### 10/16/2013: Fluid stress

The equations of motion are:

\[
\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 \\
\rho (\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \mathbf{f}
\]

**Ideal Fluids (ignore internal friction)**

The only stress is an isotropic (mathematically the same in all coordinates) pressure: \( \sigma_{ij} = -p \delta_{ij} = -p I \), where \( \delta_{ij} \) is the Kronecker \( \delta \)-function.

\[
(\nabla \cdot \sigma)_i = \frac{\partial}{\partial x_j} (-p \delta_{ij}) = -\frac{\partial p}{\partial x_i} = (-\nabla p)_i
\]

Then the equations of motion for an ideal fluid are:

\[
\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 \\
\rho (\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \mathbf{f}
\]

The unknowns are: \( \rho, \mathbf{u}, p \). Total of 5 scalar unknowns and only 4 equations. Need more information or assumptions to close the system. One way, for example in compressible gas dynamics, there is an equation of state which relates the pressure and density:

\[
p = C \rho^\gamma
\]

for \( \gamma \geq 1 \). For example: Ideal gas law says: \( p = C \rho \). Another way to close the system is to assume incompressible flow: \( \nabla \cdot \mathbf{u} = 0 \).
Incompressible Euler Equations:

\[
\begin{align*}
\rho (u_t + u \cdot \nabla u) &= -\nabla p + f \\
\nabla \cdot u &= 0 \\
\rho_t + \nabla \cdot (\rho u) &= 0
\end{align*}
\]

If \( \rho \) is constant, implies that the \( \nabla \cdot u = 0 \). However, the converse is not true, i.e. \( \nabla \cdot u = 0 \not\Rightarrow \rho = \text{constant} \). Generally, we’ll always think of constant density fluids.

**What is \( f \)?**

Gravity is the most common. Assume the \( z^+ \) direction is up. Then: \( f = (0 \ 0 \ -\rho g)^T \) where \( g \) is the acceleration due to gravity. I can write this force as: \( f = -\rho \nabla \chi \) where \( \chi = gz \). Note that then we observe this is a **conservative force**. Then the incompressible Euler equations are:

\[
\begin{align*}
u_t + u \cdot \nabla u &= -\nabla (\rho^{-1} p + \chi) \\
\nabla \cdot u &= 0
\end{align*}
\]

**Why do some people not include gravity effects?**

Imagine fluid is enclosed in some domain with no free surfaces. What happens if we consider no flow, \( u = 0 \)? So \( p_0 = -\rho g z + c \). This is called **hydrostatic pressure**. For nonzero flow, let’s express \( p = p_0 + \tilde{p} \) where \( \tilde{p} \) is (maybe) called the **dynamic pressure**.

The hydrostatic pressure doesn’t matter to me when I look at small blobs of fluid in a domain with no free surfaces (incompressible flow in closed systems). **Pressure is whatever it needs to be to enforce incompressibility on the system.** Pressure is the Lagrange multiplier to the system.

Some vector identity,

\[
\begin{align*}
u \cdot \nabla u &= (\nabla \times u) \times u + \frac{1}{2} \nabla ||u||^2 \\
\Rightarrow u_t + (\nabla \times u) \times u &= -\nabla (\rho^{-1} p + \chi + \frac{1}{2} ||u||^2) \\
\Rightarrow u_t + (\nabla \times u) \times u &= -\nabla H
\end{align*}
\]

Assume steady flow, \( u_t = 0 \). Now, we can show \( u \cdot ((\nabla \times u) \times u) = 0 \). This gives: \( u \cdot \nabla H = 0 \). \( H \) is constant along streamlines (or particle paths since steady flow).

**Bernoulli Equation:** In a steady flow of an ideal fluid of constant density, we get that:

\[
H = \rho^{-1} p + \chi + \frac{1}{2} ||u||^2
\]

is constant streamlines. (Kind of looks like conservation of energy. Pressure goes up, flow goes down) If in addition the flow is irrotational, \( \nabla \times u = 0 \), then \( H \) is constant everywhere.
10/18/2013: Irrotational flows

Steady, constant density, ideal fluid: \( \rho^{-1} p + \chi + \frac{1}{2} |u|^2 \) = constant on streamlines (particle paths).

Steady, constant density, ideal fluid AND irrotational, \( \nabla \times u = 0 \), then \( \rho^{-1} p + \chi + \frac{1}{2} |u|^2 \) = constant.

Irrotational flows:

If \( \nabla \times u = 0 \) there is a potential \( \phi \) such that \( u = \nabla \phi \). If the domain is simply connected, then \( \phi \) is uniquely defined up to an additive constant by: \( \phi = \int_0^P u \cdot ds \). Simply connected means if you can take any curve and shrink it down to a point while remaining in the domain. If \( u = \nabla \phi \), then \( \nabla \times u = 0 \) since \( \epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \phi = 0 \).

- **Proof**: Suppose simply connected domain with \( \nabla \times u = 0 \). Define \( \phi(P) = \int_0^P u \cdot ds \). Show that \( \phi(P) \) is independent of the path from 0 to \( P \). This shows \( u \) is conservative, i.e. it is a gradient field.

\[ \int_{\partial S} u \cdot ds = \int_{C_1} u \cdot ds - \int_{C_2} u \cdot ds = \int_S (\nabla \times u) \cdot n dA = 0 \]

where we have employed Stokes’ theorem. Therefore, \( \phi(P) \) is the same for all paths. Why do we need simply connected. You would need to include the path integral around the hole as well.

- **Irrotational and incompressible**: \( \nabla \times u = 0 \) and \( \nabla \cdot u = 0 \)

\( \nabla \cdot u = \nabla \cdot \nabla \phi = 0 \Rightarrow \Delta \phi = 0 \) means \( \phi \) is a harmonic function. Harmonic functions define irrotational, and incompressible flows.

Let \( \Omega \) be a simply connected domain. Want to solve \( \Delta \phi = 0 \) on this domain. Neumann BC are reasonable: \( \frac{\partial \phi}{\partial n} = g \) on \( \partial \Omega \). This problem has a solution if \( \int_{\partial \Omega} g dS = 0 \). The gradient of the solution is unique. The velocity \( u = \nabla \phi \) is the solution to \( u \cdot \nabla u = -\nabla p \) and \( \nabla \cdot u = 0 \) with BC \( u \cdot n = g \) on \( \partial \Omega \).
**Unsteady potential flows**: Still irrotational.

\[
\mathbf{u}_t + \omega \times \mathbf{u} = -\nabla \left( \rho^{-1} p + \chi + \frac{1}{2} |\mathbf{u}|^2 \right)
\]

momentum equation for ideal fluid

\[
\frac{\partial}{\partial t} \nabla \phi = -\nabla \left( \rho^{-1} p + \chi + \frac{1}{2} |\nabla \phi|^2 \right)
\]

\[
\frac{\partial \phi}{\partial t} + \rho^{-1} p + \chi + \frac{1}{2} |\nabla \phi|^2 = g(t) \quad \text{this does not affect the velocity}
\]

This is the potential flow equation on the surface.

**Do irrotational flows occur?**

Let’s use the momentum equation to derive an expression for how the vorticity evolves. Take the curl of the momentum equation:

\[
\omega_t + \nabla \times (\omega \times \mathbf{u}) = 0
\]

\[
\omega_t = -\mathbf{u} \cdot \nabla \omega + \omega \cdot \nabla \mathbf{u} - \omega (\nabla \cdot \mathbf{u}) + \mathbf{u} (\nabla \cdot \omega)
\]

\[
= -\mathbf{u} \cdot \nabla \omega + \omega \cdot \nabla \mathbf{u} - \omega (\nabla \cdot \mathbf{u}) + \mathbf{u} (\nabla \cdot (\nabla \times \mathbf{u}))
\]

\[
\omega_t + \mathbf{u} \cdot \nabla \omega = \omega \cdot \nabla \mathbf{u}
\]

\[
\frac{D\omega}{Dt} = \omega \cdot \nabla \mathbf{u}.
\]

The LHS is called transport and the RHS is called vortex stretching. In 2D planar, \(\omega \cdot \nabla \mathbf{u} = 0\). Then vorticity is transported by the flow: \(\frac{D\omega}{Dt} = 0\).

**10/21/2013: Irrotational Flows (cont’d)**

Vorticity equation for an ideal fluid: \(\omega_t + \mathbf{u} \cdot \nabla \omega = \omega \cdot \nabla \mathbf{u}\) where the LHS term is called transport and RHS term is vortex stretching.
2D planar flow

Let \( \mathbf{u} = (u(x, y), v(x, y), 0) \) and \( \omega = (0, 0, v_x - u_y) \), then \( \omega \cdot \nabla \mathbf{u} = 0 \). In this case there is no stretching, vorticity is transported by the flow. If there is a patch of fluid that has no vorticity initially, at later times that same patch still has no vorticity. If the flow is irrotational at \( t = 0 \), it remains so for all time. This also holds in 3D for ideal fluids (but is more involved to show).

Kelvin's Circulation Theorem

Let \( C(t) \) be a simple closed, oriented curve that moves with the flow of an ideal fluid with conservative body forces (incompressible - but can be expanded to a larger class). The circulation around the curve is constant in time.

Let \( \Gamma_c(t) = \int_{C(t)} \mathbf{u} \cdot d\mathbf{s} \). Theorem says, \( \frac{d\Gamma_c}{dt} = 0 \).

Proof:

\[
\frac{d\Gamma_c}{dt} = \frac{d}{dt} \int_{C(t)} \mathbf{u} \cdot d\mathbf{s} = \int_0^1 \left[ \frac{D\mathbf{u}}{Dt} \cdot \frac{\partial X}{\partial s} + \mathbf{u} \cdot \frac{\partial (\mathbf{X}(s, t))}{\partial s} \right] ds = \int_0^1 \left[ \frac{D\mathbf{u}}{Dt} \cdot \frac{\partial X}{\partial s} + \frac{1}{2} \frac{\partial }{\partial s} |\mathbf{u}|^2 \right] ds
\]

where \( \mathbf{X}(s, t) \) is the parametrization of the curve. Since the Lagrangian coordinate \( s \in [0, 1) \), then \( \mathbf{X}(0, t) = \mathbf{X}(1, t) \). The periodicity of the loop says integral of second term is zero. Then:

\[
\frac{d\Gamma_c}{dt} = \int_0^1 \frac{D\mathbf{u}}{Dt} \cdot \frac{\partial X}{\partial s} ds
\]

The momentum equation: \( \frac{D\mathbf{u}}{Dt} = -\nabla (p/\rho + \chi) \). Then:

\[
\frac{d\Gamma_c}{dt} = \int_0^1 -\nabla (p/\rho + \chi) \cdot \frac{\partial X}{\partial s} ds
\]

Path integrals of conservative forces are zero. Alternatively,

\[
\frac{d\Gamma_c}{dt} = \int_0^1 -\frac{\partial}{\partial s} (p/\rho + \chi) \cdot \frac{\partial X}{\partial s} ds = \int_0^1 -\frac{\partial}{\partial s} (p/\rho + \chi) ds = 0.
\]

With some more argument we can conclude that if at time \( t = 0 \) the flow of an ideal fluid is irrotational, it remains so for all time.

**Sketch of proof in 3D**: Assume irrotational flow at \( t = 0 \). Suppose that at a later time, there is some point in the domain with nonzero vorticity, \( \nabla \times \mathbf{u} \neq 0 \). Take a 2D disk \( D \) around that point such that \( \omega \cdot \mathbf{n} > 0 \). Then \( \int_D (\nabla \times \mathbf{u}) \cdot \mathbf{n} > 0 \). But this means, \( \int_{\partial D} \mathbf{u} \cdot d\mathbf{s} > 0 \). This contradicts Kelvin’s circulation theorem.
Where does vorticity come from?
Very close to the surface, due to friction the particles that are closest to the surface go slower. Sticking to
the surfaces, causes fluid to roll onto itself. Tangential forces cause local shearing over the surface and this
generates vorticity.

2D irrotational, incompressible, steady flow
Incompressible implies: \( u = \nabla \phi \) so \( u = \phi_x, v = \phi_y \). Irrotational allows me to say \( u = \psi_y, v = -\psi_x \).
\[
\phi_x = \psi_y \quad \phi_y = -\psi_x
\]
are the Cauchy-Riemann equations. Define \( w(z) = \phi + i\psi \) where \( z = x + iy \). This is an analytic function
which means it has a Taylor series, which implies that \( w \) is complex differentiable. Then:
\[
\frac{dw}{dz} = \phi_x + i\psi_x = u - iv.
\]

10/23/2013: Building Airfoil

Example of 2D, steady, incompressible, irrotational flow.
Flow around a circular cylinder of radius \( a \) with a flow at infinity \( u = U e_x \). Given \( \phi = U \cos \theta (r + a^2/r) \).
The corresponding complex potential to \( \phi \) is \( w(z) = U (z + a^2/z) \). It is easy to show that \( \Delta \phi = 0 \). Therefore
\( u = \nabla \phi \) is incompressible. As \( r \to \infty \), \( \phi \to Ux \) then \( u \to U e_x \). Check the boundary condition on the
cylinder [WANT \( u \cdot n = 0 \) on the surface]
\[
\begin{align*}
\mathbf{u} \cdot \mathbf{n} &= \frac{\partial}{\partial r} \phi \\
&= e_x \cdot \nabla \phi \\
&= U \cos \theta (1 - (a/r)^2) \\
&= 0.
\end{align*}
\]
when \( r = a \). We validated that the given flow satisfies the problem: 2D, steady, incompressible, irrotational
flow around a circular cylinder with the given flow at infinity.

Now let’s compute the pressure using the Bernoulli relation. Why care? Understand the forces on the
cylinder (is it generating drag or lift?)
\[
\begin{align*}
u_r &= U \cos \theta \left(1 - \frac{a^2}{r^2}\right) \\
v_\theta &= \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -U \sin \theta \left(1 + \frac{a^2}{r^2}\right) \\
\frac{1}{2} |\mathbf{u}|^2 &= \frac{1}{2} \left[ U^2 \cos^2 \theta \left(1 - \frac{a^2}{r^2}\right)^2 + U^2 \sin^2 \theta \left(1 + \frac{a^2}{r^2}\right)^2 \right] \\
&= 2U^2 \sin^2 \theta
\end{align*}
\]
Then (assuming no gravitational force): \( p + \frac{\rho}{2} |u|^2 = \text{constant} = 0 \). Then: \( p = -2\rho U^2 \sin^2 \theta \).

By symmetry, there is no net force on the cylinder. Surface force is: \( f = -pn \). Then:

\[
f = 2\rho U^2 \sin^2 \theta (\cos \theta \sin \theta)^T
\]

Then there is no lift (or no drag) – bad airplane. **Stagnation points** occur where \( p = 0 \) i.e. \( \sin^2 \theta = 0 \) or front and back of the airplane. At the back there is high pressure but zero flow speed and same at the front. The top (and back) has low pressure but fast flow. When the streamline moves from low pressure to high pressure this is known as **adverse pressure gradient**. If add a tiny amount of tangential forces (i.e. viscosity), get boundary layer separation and the point of separation depends on parameters.

![Vortex-filled wake](image)

**Figure 11:** Vortex-filled wake. Airfoils are designed to avoid the separation.

**Given the same problem statement, is the flow unique?** No, the domain is not simply connected (recall conversation regarding circulation).

10/25/2013: **Building Airfoils (cont’d)**

![Flow around a circular airfoil](image)

**Figure 12:** Flow around a circular airfoil.

The solution is not unique for the potential, \( \phi = U \cos \theta \left( r + \frac{a^2}{r} \right) \). Other solutions are of the form:

\[
\begin{align*}
u_r &= U \left( 1 - \frac{a^2}{r^2} \right) \cos \theta \\
u_\theta &= -U \left( 1 - \frac{a^2}{r^2} \right) \sin \theta + \frac{\Gamma}{2\pi r}
\end{align*}
\]
where $\Gamma$ is the circulation for the flow $u = \frac{\Gamma}{2\pi r} e_\theta$. Recall that the flow $u = \frac{\Gamma}{2\pi r} e_\theta$:

- $\nabla \times u = 0$ for $r \neq 0$;
- $\nabla \cdot u = 0$;
- As $r \to \infty$, then $u \to 0$;
- $u \cdot e_r = 0$ and therefore $u \cdot n = 0$;
- $\int_C u \cdot ds = \Gamma$.

Then $\phi = U \cos \theta \left( r + \frac{a^2}{r} \right) + \frac{\Gamma}{2\pi} \theta$. The associated complex potential is: $w(z) = U \left( z + \frac{a^2}{z} \right) - \frac{i\Gamma}{2\pi} \ln(z)$.

For $r = a$,

$$\frac{1}{2} ||u||^2 = \frac{1}{2} u_\theta^2$$

$$= 2U^2 \sin^2 \theta - \frac{\Gamma}{\pi a} U \sin \theta + c$$

Constants don’t matter because when we integrate across the surface dotted with the normal vector we get zero contribution.

$$p = -2U^2 \rho \sin^2 \theta + \frac{U \rho \Gamma}{\pi a} \sin \theta$$

The second term $(\frac{\rho \Gamma}{\pi a} \sin \theta)$ breaks the up-down symmetry but not the left-right symmetry. There is lift! The lift on the cylinder (the net force on the flow in the vertical direction) is:

$$F_y = \int -p n \cdot e_y ds$$

$$= \int_0^{2\pi} \left( 2U^2 \rho \sin^2 \theta - \frac{U \rho \Gamma}{\pi a} \sin \theta \right) \sin \theta a d\theta$$

$$= -\rho \Gamma U$$

**Negative circulation results in lift in the positive direction.** This lift is independent of shape (due to contour integrals). As the circulation becomes more negative, the stagnation points move lower.
Tear-drop airfoil

The circulation is whatever it needs to be so that the stagnation point is where it needs to be to avoid singularity in the flow. **There is one value of the circulation which puts the stagnation point at the tip.** This is called the **Kutta-Joukowski condition**.

For lift, the circulation needs to be \( \Gamma = -4\pi U a \sin(\alpha) \) where you have a thin symmetric airfoil with a sharp trailing edge and \( \alpha \) is the angle of attack. \( 2a \) is the front-to-tail length.

10/28/2013: Viscosity (frictional forces within the fluid)

Assume viscous stress on this surface: \( \sigma_{12} = \mu u_y \) where \( \mu \) is called the **dynamic (shear) viscosity** and \( u \) is the flow velocity. This is a liquid description of viscosity: as you near the surface you slow down.

Considering layers of a fluid (gas); if particles are allowed to mix at uniform speed then the layers achieve same intermediate flow speed. This swap of particles is related to change in momentum. By Newton’s second law:

\[
\frac{dp}{dt} = f_s
\]

where \( f_s \) is in the horizontal direction and this surface force is related to the viscosity. It is reasonable to propose: \( f_s \approx u_y \). This is an ideal gas (molecular) description of viscosity: swapping particles for two
layers of a gas with different velocity. N.B. For a gas, as temperature increases viscosity increases while for a liquid, as temperature increases viscosity decreases (think warmed up honey).

**Viscous stress**

Viscous stress is a linear function of $\nabla u$. Another assumption is that $\sigma^v$ is a linear function of $D = \frac{1}{2} (\nabla u + \nabla u^T)$ (the symmetric part of the gradient of velocity which is related to the stretching of the blob of fluid).

$$\sigma^v_{ij} = C_{ijkl} D_{kl}$$

where $C_{ijkl}$ are the constants in the linear combination of $D$. This expression has 81 coefficients.

Furthermore, let’s assume the fluid is isotropic. This implies that the tensor $C_{ijkl}$ is independent of the coordinate system. Also we know $\sigma^v_{ij}$ is symmetric. Isotropy requires $C_{ijkl}$ be the product of rank 2 isotropic tensors (i.e. identity is the only rank 2 isotropic tensors). In particular:

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}$$

Now we invoke symmetry of $\sigma$ (i.e. $\sigma_{ij} = \sigma_{ji}$):

$$C_{ijkl} = C_{jikl}$$
$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}$$
$$C_{jikl} = \lambda \delta_{ij} \delta_{kl} + \mu \delta_{jk} \delta_{il} + \gamma \delta_{ji} \delta_{ik}$$
By symmetry, $\gamma = \mu$. Therefore:

$$
C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \\
\sigma^v_{ij} = \lambda \delta_{ij} \delta_{kl} D_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) D_{kl} \\
= \lambda \delta_{ij} D_{kk} + \mu (D_{ij} + D_{ji}) \\
= 2\mu D_{ij} + \lambda \delta_{ij} D_{kk}
$$

Thus, \( \sigma^v = 2\mu D + \lambda \text{tr}(D) \delta = \mu(\nabla u + \nabla u^T) + \lambda \nabla \cdot u \delta \). Nonisotropic materials: liquid crystals or any liquids that have macroparticles (polymers with preferred directions).

\( \mu \) is called the **shear viscosity** and \( \lambda \) is called the **second coefficient of viscosity**.

**Relating the second coefficient of viscosity**

Let’s split the viscous stress tensor for stretching and shearing deformations.

$$
\sigma^v = 2\mu \left(D - \frac{1}{3} \text{tr}(D) \delta\right) + \left(\lambda + \frac{2\mu}{3}\right) \text{tr}(D) \delta
$$

\( \mu_{\text{bulk}} = \lambda + \frac{2\mu}{n} \) is called the bulk (volume) viscosity where \( n \) denotes the dimension of the system (for us, \( n = 3 \)). Fluids for which,

$$
\sigma = -p \delta + 2\mu D + \lambda \text{tr}(D) \delta
$$

is the stress tensor are called **Newtonian fluids**. Ideal fluids are Newtonian fluids with \( \mu = \lambda = 0 \). *How can you get non-Newtonian fluids?* Nonlinear stress-strain relationship or stress relaxation (viscoelastic flows - time history of stress or strain).

### 10/30/2013: Navier-Stokes Equations

**Navier-Stokes equations:**

We know stress in a Newtonian fluid is:

$$
\sigma = -p \delta + \mu (\nabla u + \nabla u^T) + \lambda \nabla \cdot u
$$

The equations of motion for incompressible Newtonian fluids are the **Navier-Stokes** equations. In Cartesian coordinates,

$$
\rho (u_t + u \cdot \nabla u) = -\nabla p + \mu \Delta u + f_b \\
\nabla \cdot u = 0
$$

where \( \mu \) is the **dynamic viscosity** with dimensions \( M/(LT) \) with MKS units \( \text{Pa} \cdot \text{s} \) or CGS units \( \text{P} \) (Poise). For example, \( \mu \) of water \( 10^{-2} \text{P} = 10^{-3}\text{Pa} \cdot \text{s} \). Alternatively, we can express the momentum equation as:

$$
u \cdot (u_t + u \cdot \nabla u) = -\frac{1}{\rho} \nabla p + \nu \Delta u + \frac{1}{\rho} f_b$$

27
where $\nu$ is the **kinematic viscosity** with dimensions $L^2/T$. Same dimensions as a diffusion coefficient (not surprising – basically a diffusion of momentum).

*What’s the difference between Euler and Navier-Stokes equations?* Diffusion term. Tangential forces introduce higher order derivatives of $u$ so we need another boundary condition in the tangential direction.

**No slip boundary condition**

Says the fluid has to move with the solid. For a stationary boundary, on the boundary we require:

$$ u \cdot n = 0 $$
$$ u \cdot \tau = 0 $$

or all components of velocity at the boundary are 0 (here, $n$ denotes the normal vector and $\tau$ is the tangential vector on the boundary).

**Nondimensionalization**

*How “big” is the viscosity?* Depends on the problem. Let’s nondimensionalize the Navier-Stokes equations:

$$ \rho (u_t + u \cdot \nabla u) = -\nabla p + \mu \Delta u $$
$$ \nabla \cdot u = 0 $$
We need to choose a length scale ($L$), time scale ($T$), velocity scale ($U$), and pressure scale ($P$).

- $L =$ radius or diameter of object
- $T =$ $L/U$ if there is no other natural timescale around
- $U =$ flow far off at infinity
- $P =$ two choices: (1) relative to viscous stress $= \mu/T = \mu U/L$ or (2) think Bernoulli: $= \rho U^2$

Introducing dimensionless variables:

$$
\begin{align*}
x &= L \hat{x} \\
t &= T \hat{t} \\
u &= U \hat{u} \\
p &= P \hat{p}
\end{align*}
$$

Then,

$$
\frac{\rho L}{T} \frac{\partial u}{\partial t} + \frac{\rho U^2}{L} u \cdot \nabla u = -\frac{P}{L} \nabla p + \frac{\mu U}{L^2} \Delta u \\
\frac{U}{L} \nabla \cdot u = 0
$$

Simplifying:

$$
\frac{\rho U^2}{L} (\frac{\partial u}{\partial t} + u \cdot \nabla u) = -\frac{\rho U^2}{L} \nabla p + \frac{\mu U}{L^2} \Delta u \\
\nabla \cdot u = 0
$$

Furthermore,

$$
\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + \frac{\mu}{\rho UL} \Delta u \\
\nabla \cdot u = 0
$$

Define the dimensionless parameter $\text{Re} = \frac{\rho UL}{\mu}$ is the Reynolds number. $\text{Re}^{-1}$ is how “big” the viscosity is. The Navier-Stokes equations simplify to:

$$
\begin{align*}
\frac{\partial u}{\partial t} + u \cdot \nabla u &= -\nabla p + \frac{1}{\text{Re}} \Delta u \\
\nabla \cdot u &= 0
\end{align*}
$$

If we take $\text{Re} \to \infty$ do we recover the Euler equations? (Yes if you are being really sloppy but you have too many boundary conditions for the Euler equations.) NO, you do not get Euler equations because you have the tangential boundary condition. What if I am away from the boundary? Yeah, then you are fine.

What about really small lengthscales, i.e. $\text{Re} \to 0$? Multiplying through by the Reynolds number:

$$
\begin{align*}
\text{Re} \frac{Du}{Dt} &= -\text{Re} \nabla p + \Delta u \\
\nabla \cdot u &= 0
\end{align*}
$$
Taking the limit as Reynolds number approaches zero:

\[ \Delta u = 0 \]
\[ \nabla \cdot u = 0 \]

Oops! Overdetermined unless \( u = 0 \). Recall, pressure is whatever it needs to be to enforce the incompressibility condition. But we got rid of the pressure so the incompressibility condition will generally not be satisfied. For low Reynolds number, the pressure is also equally small. We need to use the other choice for nondimensionalization of pressure. For low Reynolds number (Re \( \ll 1 \)), \( P = \frac{\mu U}{L} \), so:

\[ \frac{\rho U^2 D u}{L} \frac{dt}{dt} = -\frac{\mu U}{L^2} \nabla p + \frac{\mu U}{L^2} \Delta u \]

Then simplifying:

\[ \text{Re} \frac{D u}{D t} = -\nabla p + \Delta u \]
\[ \nabla \cdot u = 0 \]

Now take \( \text{Re} \to 0 \),

\[ \Delta u - \nabla p = 0 \]
\[ \nabla \cdot u = 0 \]

these are the Stokes equations.

11/01/2013: Navier-Stokes Equations (cont’d)

Here are Navier-Stokes equations in Cartesian coordinates:

\[ \rho (u_t + u \cdot \nabla u) = -\nabla p + \mu \Delta u + f_b \]
\[ \nabla \cdot u = 0 \]

Some examples of solutions to follow.
Figure 19: Steady uni-directional pressure-driven flow through a pipe.

**Steady uni-directional pressure-driven flow through a pipe.**

In cylindrical coordinates, $u = u_r e_r + u_\theta e_\theta + u_z e_z$. Then, the Navier-Stokes equations become:

\[
\begin{align*}
\frac{\partial u_r}{\partial t} + u \cdot \nabla u_r - \frac{u_\theta^2}{r} &= -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left( \frac{\Delta u_r}{r^2} - \frac{u_r}{r^2} \frac{\partial u_\theta}{\partial r} \right) \\
\frac{\partial u_\theta}{\partial t} + u \cdot \nabla u_\theta + \frac{u_r u_\theta}{r} &= -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \nu \left( \frac{\Delta u_\theta}{r^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right) \\
\frac{\partial u_z}{\partial t} + u \cdot \nabla u_z &= -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \Delta u_z
\end{align*}
\]

\[
\nabla \cdot u = \frac{1}{r} \frac{\partial}{\partial r} \left( ru_r \right) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0
\]

\[
\Delta \phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2}
\]

\[
\nabla \phi = \frac{\partial \phi}{\partial r} e_r + \frac{1}{r} \frac{\partial \phi}{\partial \theta} e_\theta + \frac{\partial \phi}{\partial z} e_z.
\]

Now let’s look for solution of the form: $u = u(r)e_z$. This is divergence free. Assume steady flow. The equations reduce to:

\[
\frac{\partial p}{\partial z} = \mu \Delta u_z
\]

\[
\mu \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) = \frac{\partial p}{\partial z}
\]

The RHS term is independent of $\theta$ and $z$ since we assume a particular form for the flow. Looking at the first equations of the N-S in cylindrical form, $\frac{\partial p}{\partial r} = 0$ so $p$ is independent of the radius. Therefore, combining the two arguments (RHS independent of $\theta$ and $z$ and $p$ independent of $r$), the LHS has to be a constant.

Define $G = -\frac{\partial p}{\partial z}$. Then we can solve the previous equation for the velocity profile:

\[
\mu \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) = -G
\]

and integrate:

\[
r \frac{\partial u}{\partial r} = -\frac{Gr^2}{2\mu} + A
\]

31
Figure 20: Velocity profile for Poiseuille flow.

divide by the $r$ and integrate again:

$$u = -\frac{Gr^2}{4\mu} + A\ln(r) + B$$

Now we need to apply BC (we are dealing with N-S equations and need tangential BCs). Say no slip on the pipe and pipe has radius $R$:

$$u(R) = 0 \Rightarrow B = \frac{GR^2}{4\mu}$$

Assume bounded flow for $r \leq R \Rightarrow A = 0$. Therefore, the solution is:

$$u = \frac{GR^2}{4\mu} - \frac{Gr^2}{4\mu} = \frac{GR^2}{4\mu} \left(1 - \frac{r^2}{R^2}\right)$$

this is named **Poiseuille** flow. Characterized by parabolic velocity profile.

The volume flux:

$$Q = \int_S udS = \int_0^{2\pi} \int_0^R \frac{GR^2}{4\mu} \left(1 - \frac{r^2}{R^2}\right) r dr d\theta = 2\pi \frac{GR^2}{4\mu} \left(\frac{r^2}{2} - \frac{r^4}{4R^2}\right) \bigg|_0^R = \frac{\pi GR^4}{8\mu} = \left(\frac{\pi R^4}{8\mu}\right) G$$

This is the **analog of Ohm's law** (voltage = current times resistance). The analog of voltage is $G$ (pressure drop), $Q$ is current. Then: $G = \left(\frac{8\mu}{\pi R^2}\right) Q$ think $V = RI$.

### 11/04/2013 Solutions to Navier-Stokes:

**Linear shear flow**

The steady solution with no pressure gradient is: $u = \gamma y$ where $\gamma = U/L$ is called the **shear rate**. One of the ways to measure viscosity of the fluid.

**Time-dependent linear shear flow**

The semi-infinite domain $y > 0$ with fluid initially at rest. There is a boundary at $y = 0$ moving to the right with constant velocity, $U$. Assume that the flow is uniform in the x-direction and 2D planar. Furthermore, assume there is no applied pressure gradient. Under these assumptions, NS equations reduce to:

$$\rho u_t = \mu u_{yy} \Rightarrow u_t = \nu u_{yy}$$
where \( u = u(y, t) \). The boundary and initial conditions are: \( u(0, t) = U, u(\infty, t) < M \) for some constant \( M \), and lastly \( u(y, 0) = 0 \). The reason why we study this problem is to learn how to look for similarity solutions (particularly in fluid dynamics).

Let’s nondimensionalize this problem. Scale velocity by \( U \), time by \( T \), length by \( L \).

\[
  u_t = \left( \frac{T \nu}{L^2} \right) u_{yy}
\]

There is no natural \( T, L \) – that’s a clue to look for similarity solutions.

Let \( u(y, t) \) be a solution of the PDE plus boundary conditions. Let \( v(y, t) = u(\alpha y, \alpha^2 t) \) for \( \alpha > 0 \) is also a solution. We can check this is also a solution of the same PDE: \( v_t = \alpha^2 u_t \) and \( v_{yy} = \alpha^2 u_{yy} \), therefore \( v_t = \nu v_{yy} \) and it also satisfied the boundary and initial conditions. Let us assume that there is a unique solution. This implies that \( u(y, t) = u(\alpha y, \alpha^2 t) \). For a fixed \( (y, t) \), let’s pick \( \alpha = 1/\sqrt{t} \). Then we get: \( u(y, t) = u(\alpha y, \alpha^2 t) = u(y/\sqrt{t}, 1) \). I can associate point in time and space with a coordinate \( \eta = y/\sqrt{t} \).

This suggests using a similarity variable, \( \eta = \frac{y}{\sqrt{\nu t}} \). Then \( u(x, t) = f(\nu) \). Plug into the PDE to get an
Figure 23: Velocity profile for time dependent linear shear flow ($\eta = u$).

equation for $f$:

\[
\begin{align*}
\nu_y &= \frac{1}{\sqrt{\nu t}} \\
\nu_t &= \frac{y}{\sqrt{\nu t \nu t}} \left( -\frac{1}{2} \right) = -\frac{\eta}{2t} \\
u_t &= f'(\eta) \left( -\frac{\eta}{2t} \right) \\
\nu_y &= f'(\eta) \frac{1}{\sqrt{\nu t}} \\
u_{yy} &= f''(\eta) \frac{1}{\nu t}
\end{align*}
\]

This yields the following relationship if we plug back into original PDE:

\[
-\frac{\eta}{2t} f' = \nu t f'' \Rightarrow f'' + \frac{\eta}{2} f' = 0
\]

We can solve this and get \( \frac{d}{d\eta} \left( f' e^{-\eta^2/4} \right) = 0 \) then \( f' = B e^{-\eta^2/4} \). Integrate the ODE and get \( f(\eta) = A + B \int_0^\eta e^{-s^2/4} ds \).

**What about boundary conditions?** We let \( u(y, t) = f(\eta) \) with \( \eta = \frac{y}{\sqrt{\nu t}} \). Then \( f(0) = 1 \) (or \( U \)) and \( f(\infty) = 0 \). From the BC’s, we get \( A = 1, B = -1/\sqrt{\pi} \). Then: \( u = 1 - \frac{1}{\sqrt{\pi}} \int_0^\eta e^{-s^2/4} ds \).

**What’s the significance of this similarity parameter?** Fix time. This is the velocity profile ($\eta = u$). As time goes on, for a fixed value of \( \eta \), the value of \( y \) (the place where you hit at a given \( \eta \) (speed)) increases at a rate of \( \sqrt{\eta} \). The velocity profile is always the same shape it just gets stretched as time goes on.

Compute the vorticity of this flow:

\[ \omega = -u_y \]
\[
\omega = -\frac{1}{\sqrt{\pi}} e^{-y^2/4} \eta_y = -\frac{1}{\sqrt{\nu \pi t}} e^{-y^2/(4\nu t)}
\]

This looks like the **fundamental solution to a heat (diffusion) equation**, sometimes called the **heat kernel**. The equation for vorticity is:

\[ \omega_t = \nu \omega_{yy} \]

The initial condition looks like a \( \delta \)-distribution of vorticity on the boundary (**vortex sheet**). (I think)

\[
\int_0^\infty \omega dy = \int_0^\infty -u_y dy = u(0) - u(\infty) = 1.
\]

**Vorticity**

The **vorticity equation**, in general, is:

\[
\frac{D\omega}{Dt} = \omega \cdot \nabla u + \nu \Delta \omega
\]

In ideal fluids we only have transport and stretching. **For viscous fluids, we add diffusion.** Thus, for shear flow any localized initial vorticity will diffuse. In the limit of large Reynolds number, the initial vorticity will fully diffuse.

With viscous 2D flows, the term \( \omega \cdot \nabla u \) vanishes, then we obtain:

\[
\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \nu \Delta \omega
\]

---

**11/06/2013: Solutions to Navier-Stokes (cont’d)**

Last time, we looked at the semi-infinite domain with shear flow on the bottom plane, \( u(0, t) = U \).

**Flow between two semi-infinite plates but with oscillations**

Same setup as before, but \( u(0, t) = U_0 \cos(\omega t) \) solving \( u_t = \nu u_{yy} \). We don’t expect a similarity solution since we have a timescale given by the parameter \( \omega \). Instead let’s look for a time-periodic solution:

\[
u(y, t) = \text{Re} \left( e^{i\omega t} f(y) \right)
\]

Solution:

\[
u = U_0 \exp \left( -\sqrt{\frac{\omega}{2\nu}} \right) \cos \left( \omega t - \sqrt{\frac{\omega}{2\nu}} y \right)
\]

Imagine wiping a hose in an oscillatory fashion but the amplitude is decay with length. Let \( \ell = \sqrt{\frac{2\nu}{\omega}} \) is a lengthscale. Fix a time and plot the velocity profile in Figure 24.
Figure 24: Fixed time, velocity profile for flow between two semi-infinite plates with prescribed oscillatory motion.

Figure 25: Particle paths for pure strain flow.
**Pure strain flow** (stagnation point flow or elongational flow)

\[ u = \alpha x \quad \text{and} \quad v = -\alpha y \]

Recall we have shown (in homework) there is no vorticity (no rotation - just stretching). The particles paths look like shown in Figure 25.

The flow is irrotational and incompressible. This is a steady flow solution for the incompressible Euler equations. The potential is: \( \phi = \frac{\alpha x^2}{2} - \frac{\alpha y^2}{2} \) and the streamfunction is \( \psi = \alpha xy \). Let’s look at the positive domain, \( y > 0 \) with boundary \( y = 0 \), solid wall. Is this still a solution to Euler equations? If we add the boundary, \( u \cdot n = 0 \) is satisfied. So yes, we still have a solution to incompressible Euler equations.

Now, **add a little bit of viscosity**, i.e. \( \mu > 0, \mu \ll 1 (\text{Re} \gg 1) \). Ignoring boundary condition, this is a solution to steady incompressible NS. With BC we don’t satisfy no slip boundary condition. We need to account for no slip condition.

**Streamfunction approach to finding BC:** Start by guessing a solution of the form:

\[ \psi = \alpha x F(y) \]

modify \( F(y) \) by enforcing boundary conditions on the streamfunction. SS-incompressible NS equations are:

\[
\begin{align*}
\rho u \cdot \nabla u &= -\nabla p + \mu \Delta u \\
\nabla \cdot u &= 0.
\end{align*}
\]

Rather than solving for the velocity, let’s replace this with the streamfunction and solve the SS-incompressible NS for the streamfunction. We’ll get the incompressibility condition for “free”. However, we replaced a vector variable \( u \) by a scalar, the streamfunction. Philosophically, this is insufficient to describe the system. A vorticity equation would close the system. Let’s look at the relationship between streamfunction and vorticity. The vorticity equation in 2D is:

\[ \omega_t + u \cdot \nabla \omega = \nu \Delta \omega \]

The expression for the vorticity in 2D is: \( \omega = \psi_x - \psi_y \). Recall that: \( u = \psi_y \) and \( v = \psi_x \). Then:

\[ \omega = -\psi_{xx} - \psi_{yy} = -\Delta \psi \]

Therefore, the SS-incompressible 2D NS equations in terms of vorticity & streamfunction are:

\[
\begin{align*}
\omega_t + u \cdot \nabla \omega &= \nu \Delta \omega \\
-\Delta \psi &= \omega
\end{align*}
\]
The velocity field and the vorticity are:

\[
\begin{align*}
    u &= \alpha x F'(y) \\
    v &= -\alpha F(y) \\
    \omega &= -\alpha x F''(y) \\
    u\omega_x + v\omega_y &= \nu (\omega_{xx} + \omega_{yy}) \\
    \Rightarrow (\alpha x F')(-\alpha F''') + (-\alpha F)(-\alpha x F''') &= \nu \left(-\alpha x F^{(4)}\right) \\
    \Rightarrow F' F'' - F F''' - \frac{\nu}{\alpha} F^{(4)} &= 0
\end{align*}
\]

Let's nondimensionalize this. The dimension of \( F(y) \) is \([F] = L\). Scale \( y \) and \( F \) by \( L \). Then:

\[
\begin{align*}
    \frac{1}{L} F' F'' - \frac{1}{L} F F''' - \frac{\nu}{\alpha L^3} F^{(4)} &= 0 \\
    F' F'' - F F''' - \frac{\nu}{(\alpha L) L} F^{(4)} &= 0 \\
    F' F'' - F F''' - \frac{1}{\text{Re}} F^{(4)} &= 0.
\end{align*}
\]

This is a singular perturbation problem.

11/08/2013: Boundary Layer Problems

**Pure strain flow** (cont’d)

We are trying to naturally arrive at the Euler equations from the Navier-Stokes equations. Recall last time we were looking at pure strain flow as shown in Figure 26 with \( u = \alpha x \) and \( v = -\alpha y \). If we add viscosity, if \( \text{Re} \gg 1 \) for \( y \gg 1 \) we expect \( u \to \alpha x \) and \( v \to -\alpha y \). Look for a streamfunction of the form: \( \psi = \alpha x F(y) \). We arrived at a non-dimensionalized equation for \( F(y) \):

\[
F' F'' - F F''' - \frac{1}{\text{Re}} F^{(4)} = 0
\]
**The boundary conditions** We need 4 BCs and we obtain them by looking at the velocities at each endpoint. The relationship between \( u, v \) and \( F \) is:
\[
\begin{align*}
u &= \psi_y = \alpha y F'(y) \\
v &= -\psi_x = -\alpha y F(y)
\end{align*}
\]
we need at \( y = 0 \), \( u = v = 0 \) then \( F(0) = 0 \) and \( F'(0) = 0 \). At \( y \to \infty \), expect \( u \to \alpha x \) and \( v \to -\alpha y \), then \( F \to y \) and \( F' \to 1 \). For \( \text{Re} \to \infty \), we need to remove one boundary condition; we lose the tangential velocity information at \( y = 0 \) (i.e. \( F'(0) \neq 0 \)).

Integrating once,
\[
\begin{align*}
2F'F''' - F'F'' - FF''' - \frac{1}{\text{Re}} F^{(4)} &= 0 \\
\left((F')^2\right)' - (FF'')' - \frac{1}{\text{Re}} F^{(4)} &= 0 \\
(F')^2 - FF'' - \frac{1}{\text{Re}} F''' &= C \\
(F')^2 - FF'' - \frac{1}{\text{Re}} F''' &= 1 \text{ from BC?}
\end{align*}
\]

Rescale variables: \( F = \frac{1}{\sqrt{\text{Re}}} f(\eta) \) and \( \eta = y\sqrt{\text{Re}} \). Then:
\[
(f')^2 - ff'' - f''' = 1
\]
Solve numerically the equation and we plot \( f' \) as a function of \( \eta \):
Significance of \( f' \to 1 \) implies NS equations becomes Euler equations (we loose viscosity). When \( \eta = 1 \), \( y = \frac{1}{\sqrt{\text{Re}}} \). Really, really small \( y \) as Reynolds number approaches infinity. Then Figure 26 becomes (with the correction on the boundary):
Outside a layer of thickness \( O(\text{Re}^{-1/2}) \) the flow is essentially inviscid and irrotational. All the effects of viscosity are contained in the layer. Lots of vorticity in this layer. The vorticity does not spread out like in the shear example because in this vorticity equation we are missing \( u \cdot \nabla \omega \) term.
Recall, NS equations:
\[
\rho \frac{Du}{Dt} = -\nabla p + \mu (u_{xx} + u_{yy})
\]
How do these terms scale? $T = L/U$ is the time to pass plate moving at the freestream.

$$\rho U^2 / L = -\nabla p + \mu \left( \frac{\mu U}{L^2} + \frac{\mu U}{\delta^2} \right)$$

Viscous effects matter when $\rho U^2 / L = \mu U / \delta^2$. Then: $\delta^2 / L^2 = \frac{\nu}{UL} = 1/Re$. If you have a plate of length 1, you have a scale on which viscous effects matter of $1/\sqrt{Re}$.

**11/13/2013: Boundary Layer Problems (cont’d)**

**History**

These ideas date back to 1904-1905 and are due to Prandtl: **Flow at low viscosity around objects is very different from flow with no viscosity.** Some obvious examples are:

- Start-up flow around airfoil. The circulation around the wing is key to generating lift. Ideal flow theory says if you start with a blob of fluid with no vorticity, it stays with no vorticity. Without viscosity, we cannot generate viscosity $\Rightarrow$ circulation.

- Objects in rapid flows. Without viscosity we can get no wake. Even if the object rotates this does not create wake (in the absence of viscosity) since this rotation is not transmitted to the fluid.

Last time, we looked at a couple examples and concluded that viscosity matters at some length scales. Found these boundary layer arguments we found from similarity solutions or scaling $x,y$ separately that viscous effects matter on a region of thickness $1/\sqrt{Re})$ This is the connection between inviscid flow theory and viscous flow theory. (Acheson Chp. 8 - read a few paragraphs from Prandl’s 1904 paper)

**Deriving Prandtl’s Equations**

Begin with nondimensionalized 2D steady Navier-Stokes with $1/Re = \epsilon$:

\[
\begin{align*}
    uu_x + vv_y &= \epsilon(u_{xx} + u_{yy}) - p_x \\
    uw_x + vv_y &= \epsilon(v_{xx} + v_{yy}) - p_y \\
    u_x + v_y &= 0.
\end{align*}
\]

Let us rescale the two spatial variables separately (as we did in the previous class) keeping in mind flows around object or at a stationary wall. Rescale both $y,v$. 

...
Figure 29: Flow around a semi-infinite plate.

Let $v = \alpha \overline{v}$ and $y = \beta \overline{y}$ where $\alpha$, $\beta$ are to be determined. Plugging these into the equations:

\[
\begin{align*}
 uu_x + \frac{\alpha}{\beta} vu_y &= \epsilon (u_{xx} + \frac{1}{\beta^2} u_{yy}) - p_x \\
 u\alpha \overline{v}_x + \frac{\alpha^2}{\beta} \overline{v} u_y &= \epsilon \alpha (\overline{v}_{xx} + \frac{1}{\beta^2} \overline{v}_{yy}) - \frac{1}{\beta} p_y \\
 u_x + \frac{\alpha}{\beta} \overline{v} &= 0.
\end{align*}
\]

$\alpha = \beta$ to maintain incompressibility. We expect to retain $u_{yy}$ term ⇒ choose $\beta = \sqrt{\epsilon}$ so that the viscous force density in the $x$-direction is of the same order as the convective terms. Therefore,

\[
\begin{align*}
 uu_x + \overline{v} u_y &= \epsilon u_{xx} + u_{yy} - p_x \\
 \sqrt{\epsilon} u\overline{v}_x + \sqrt{\epsilon} \overline{v} u_y &= \epsilon^{3/2} \overline{v}_{xx} + \sqrt{\epsilon} \overline{v}_{yy} - \frac{1}{\sqrt{\epsilon}} p_y \\
 u_x + \overline{v} &= 0.
\end{align*}
\]

The middle equation says that in the limit as $\epsilon \to 0$, we get $p_y = 0$. In the limit of small $\epsilon$, the equations say:

\[
\begin{align*}
 uu_x + \overline{v} u_y &= \epsilon u_{xx} + u_{yy} - p_x \\
 p_y &= 0 \text{ as } \epsilon \to 0 \\
 u_x + \overline{v} &= 0,
\end{align*}
\]

thus $p$ is only a function of $x$.

**Matched Asymptotics**: We require the layer solution to match the outer solution as $\epsilon \to 0$.

Slightly more formal, $\lim_{\overline{y} \to \infty} u(\overline{y}) = \lim_{y \to 0} U(y)$ where $U$ is the solution to Euler equations. Similarly, for the pressure: $\lim_{\overline{y} \to \infty} p(\overline{y}) = \lim_{y \to 0} P(y)$. Because $p_y = 0$, $p$ is the pressure from outer solution.
11/15/2013: Boundary Layer Problems (cont’d)

Prandtl’s Equations

With no viscosity, a boundary condition is unsatisfied. The Prandtl equations are given by:

\[
\begin{align*}
    u_{ux} + v_{uy} &= \epsilon (u_{xx} + u_{yy}) - p_x \\
    u_x + v_y &= 0, \\
    p_y &= 0 \quad \text{as } \epsilon \to 0.
\end{align*}
\]

where the last equation enforces incompressibility and we assume flat plate. The boundary conditions on the plate: \( u = v = 0 \) (no slip condition). Not enough to determine the solution of the problem since we have 2 variable derivative (i.e. \( u_{xx} \), and \( v_y \)). We also enforce the matching condition: we require the layer solution \( (u(\bar{y})) \) to match the outer solution \( (U(y)) \) as \( \epsilon \to 0 \):

\[
\lim_{\bar{y} \to \infty} u(\bar{y}) = \lim_{y \to 0} U(y)
\]

where \( U \) is the solution to Euler equations. Because \( p_y = 0 \), then \( p \) is the pressure from outer solution (given from Euler).

Does a layer exist?

Adverse pressure gradient: low pressure to high pressure (positive pressure gradient); physically we do not expect a boundary layer (analytically maybe there is no solution either).

Flow around cylinder

If turbulent flow, the flow stays attached (this is why golf balls have dimples on them)

Example of matched asymptotics

\[
\begin{align*}
    \epsilon u_{yy} + u_y &= 1 \\
    u(0) &= 0, \quad u(1) = 2.
\end{align*}
\]
Figure 31: Flow around a cylinder (with suggested boundary layer separation).

Figure 32: For the given asymptotic equation, there are two possibilities where the boundary layer can occur: $y = 0$ or $y = 1$. These are representative plots of the solution in either case.

When $\epsilon \to 0$, $u_y = 1 \Rightarrow u = y + c$ where $c$ is an integration constant. Two possibilities:

We can show that possibility 1 cannot happen, i.e. the boundary layer occurs at $y = 0$ (and not at $y = 1$).

Let us assume the boundary layer occurs at $y = 0$. **Outer solution**:

$$u = y + 1$$

holds away from $y = 0$. Now, let us rescale $y$ to get the solution near the boundary layer: $y = \epsilon^\alpha \bar{y}$ and $\bar{u}(\bar{y}) = u(y/\epsilon^\alpha)$. Then, plug in:

$$\frac{\epsilon}{\epsilon^{2\alpha}} \bar{u}_{\bar{y}\bar{y}} + \frac{1}{\epsilon^\alpha} \bar{u}_{\bar{y}} = 1 \quad \text{and} \quad \epsilon^{1-2\alpha} \bar{u}_{\bar{y}\bar{y}} + \epsilon^{-\alpha} \bar{u}_{\bar{y}} = 1$$

Pick $\alpha$ so multiple terms balance and the others are asymptotically smaller (as $\epsilon \to 0$). Without any physical intuition, do a dominant balance:

$$1 - 2\alpha = -\alpha \Rightarrow \alpha = 1 \Rightarrow \text{Dominant}$$

$$1 - 2\alpha = 0 \Rightarrow \alpha = 1/2 \Rightarrow \text{Not dominant}$$

$$-\alpha = 0 \Rightarrow \alpha = 0 \Rightarrow \text{Dominant (outer solution)}$$

Therefore, we choose to rescale $y$ as: $y = \epsilon \bar{y}$ and thus, $\bar{u}(\bar{y}) = u(y/\epsilon)$. In the limit as $\epsilon \to 0$, the inner problem is:

$$\bar{u}_{\bar{y}\bar{y}} + \bar{u}_{\bar{y}} = 0$$

$$\bar{u}(0) = 0.$$
The solution in the layer is: \( u(y) = A \left( 1 - e^{-y} \right) \). The matching condition is:

\[
\lim_{y \to \infty} u(y) = \lim_{y \to 0} u(y)
\]

This yields that the last constant in solution: \( A = 1 \). The inner solution is: \( u(y) = 1 - e^{-y} \) and the outer solution is: \( u = 1 + y \). The uniform solution is given by:

\[
\begin{align*}
\text{uniform} & = u_{\text{inner}} + u_{\text{outer}} - u_{\text{match}} \\
& = \left( 1 - e^{-y/\epsilon} \right) + (1 + y) - 1 \\
& = 1 + y - e^{-y/\epsilon}.
\end{align*}
\]

The exact solution to the ODE is \( u = y + \frac{1 - e^{-y/\epsilon}}{1 - e^{-y/\epsilon}} \).

11/18/2013: Boundary Layer Problems (cont’d)

2D uni-directional flow with a flat plate

At infinity, \( u = U e^x \) and a plate at \( y = 0 \) for \( x \geq 0 \) with no slip condition. We are going to solve the boundary layer equation to find the flow profile near the plate.

**Outer solution:** Just \( u = U e^x \) and therefore the pressure gradient is constant, \( \frac{\partial p}{\partial x} = 0 \).

**Inner solution:** ND equations in the layer:

\[
\begin{align*}
u u_x + v u_y - u_{yy} & = 0 \\
 u_x + v_y & = 0,
\end{align*}
\]

in \( x > 0, y > 0 \). At \( y = 0, u, v = 0 \). **Look for similarity solutions** (there is a velocity scale but no natural lengthscale). *N.B. If you’re wondering how to go from suspicious of similarity solution to yes, it actually is sensible to look for a similarity solution see Acheson!*)
Figure 34: Looking for a similarity solutions where we assume that the $y$-variable for the vertical direction (the thickness of the boundary layer) is changing with the Reynolds number.

\[
\frac{\delta}{L} = \frac{1}{\sqrt{Re}} = \sqrt{\frac{\nu}{UL}}
\]

\[
\frac{\delta}{\sqrt{L}} = \sqrt{\frac{\nu}{U}}
\]

This suggests, $\eta = y \sqrt{x}$. Let’s use streamfunction (cause we have a 2D incompressible flow): $u = \psi_y$ and $v = -\psi_x$. The ND equations in the boundary layer become:

\[
\psi_y \psi_{xy} - \psi_{xx} \psi_{yy} - \psi_{yyy} = 0
\]

Guess: $u = f'(\eta)$, $\eta = y \sqrt{x}$. Then the streamfunction is:

\[
\psi = \int_0^y u \, ds = \int_0^\eta f'(a) \sqrt{x} \, da = \sqrt{x} f(\eta) + c,
\]

where $c$ is solved for by BC. Can check this and you end up with an ODE for $f(\eta)$:

\[
\frac{1}{2} f \psi'' = 0
\]
We need boundary and matching conditions. First, let’s write down boundary conditions:

\[ u, v \text{ are zero on the plate } \Rightarrow f'(0) = 0. \]

\[ \psi_x = \frac{1}{2\sqrt{x}} f(\eta) + \sqrt{x} f'(\eta) \eta_x \]

\[ \psi_x(0) = \Rightarrow f(0) = 0. \]

The matching condition at infinity says:

\[ f' \to 1 \text{ as } \eta \to \infty \]

We can solve this ODE numerically, to obtain:

Figure 35: Similarity solution for flow around a semi-infinite flat plate.

**Observations:**

- The similarity variable is \( \eta = \frac{y}{\sqrt{x}} \). The boundary layer thickens downstream. This breaks down at high Reynolds number and turbulence kicks in.
- What happens to the shear stress on the plate, locally? Shear stress is proportional to \( u_y \). It decreases downstream (because arrows of the velocity profile are the same length near the top downstream).
- Sometimes called the Blasius profile.

**Falkner-Skan solutions**

Let \( U = Ax^m e_x \) is the external stream. By Bernoulli, the pressure gradient is: \( p_x = -mA^2 x^{2m-1} \). Notice if \( m > 0 \), negative pressure gradient (favorable pressure gradient), while if \( m < 0 \), positive pressure gradient (adverse pressure gradient). Can relate this to flow over a wedge with angle \( \alpha = (2m\pi)/(m+1) \) as shown in Figure 36:

**Already did 2 special cases:**

- \( m = 0 \): 2D flow over flat plate problem
- \( m = 1 \): extensional flow

We are going to go over the same exercise as in the flat plate scenario. Can get a similarity solution:

\[ f''' + \frac{1}{2} f f'' + \frac{m}{m+1} (1 - (f')^2) = 0 \]

Solve numerically, and look at velocity profile:

The significance of adverse pressure gradient is that \( u_y(0) = 0 \) for \( m = -0.0904 \).
Figure 36: Flow over a wedge with angle \( \alpha = \frac{2m\pi}{m+1} \).

Figure 37: Similarity solutions for flow over a wedge for different choices of \( m \). Notice that at \( m \approx -0.0904 \), there is an inflection point suggesting that the stream flow separates from the object.

11/20/2013: Boundary Layer Separation

**Falkner-Skan solutions**

Let \( \mathbf{U} = Ax^m \mathbf{e}_x \) is the external stream. By Bernoulli, the pressure gradient is: \( p_x = -mA^2 x^{2m-1} \). **What is the effect of adverse pressure gradient?**

- \( m > 0 \) favorable pressure gradient; accelerating stream over the layer in the stream-direction
- \( m < 0 \) unfavorable (adverse) pressure gradient; decelerating stream over the layer in the stream-direction

**What happens to the stream as you flow out of the layer?** Can get a similarity solution with a similarity variable:

\[
\eta = \frac{y}{x^{(1-m)/2}}
\]

We already did 2 special cases:

- \( \boxed{m = 1} \): extensional flow \( \Rightarrow \eta = y \) (uniform thickness of the boundary layer) see Figure 38. **Perfect balance between convection and diffusion!**
- \( \boxed{m = 0} \): 2D flow over flat plate problem \( \Rightarrow \eta = y/\sqrt{x} \) (increasing thickness of the boundary layer). You create vorticity at the tip and just see it diffuse throughout the boundary (no downward convection, only to the right)
Figure 38: The 2 special cases of Faulkner-Skan flow for $m = 1$ (extensional flow) and for $m = 0$ (flow over semi-infinite flat plate).

\[
f''' + \frac{1}{2}ff'' + \frac{m}{m+1}(1-(f')^2) = 0
\]

Solve numerically, and look at velocity profile:

Figure 39: The 2 special cases of Faulkner-Skan flow for $m = 1$ (extensional flow) and for $m = 0$ (flow over semi-infinite flat plate).

The significance of adverse pressure gradient is that $u_y(0) = 0$ for $m = -0.0904$. Then:

\[
\sigma_{12} > u_y = f''(\eta)\eta_y = f''(\eta)\frac{1}{x(1-m)/2}
\]

For fixed $x > 0$, $\sigma_{12}(0) \approx f''(0)$. When $\sigma_{12}(0) = 0$ at $m = 0$, you get no shear stress on the surface. This is associated with boundary layer separation. The significance of Faulkner-Skan equations is that this critical value of $m \approx -0.0904$ is the place where you lose uniqueness of solution for the velocity profile in the boundary layer (exact value where $f''(0) = 0$; local flow reversal).

Looking at Prandtl equations:

\[
uu_x + vu_y = uu_y - \frac{dp}{dx}
\]
As \( y \to 0 \), both \( u, v \to 0 \), balance between viscous stress and pressure gradient. Near the wall, \( m > 0 \), we expect \( u_{yy} < 0 \) (look at Figure 38) \( \Rightarrow \frac{dp}{dx} < 0 \) (favorable). When \( \frac{dp}{dx} > 0 \) (unfavorable), then \( u_{yy} > 0 \) (flow reversal).

Boundary layer separation is largely responsible for drag on an object in the limit of \( \text{Re} \to \infty \):

How big are these viscous effects? Generically we say,

\[
\sigma_{12} \approx \frac{1}{\text{Re}} \frac{\partial u}{\partial y} = \frac{1}{\text{Re}} \sqrt{\text{Re}} = \frac{1}{\sqrt{\text{Re}}} \to 0.
\]

As \( \text{Re} \to \infty \), viscous stresses are small. This is called skin friction. For big Reynolds number flows, we’re tempted to ignore viscous forces and use only the pressure for calculating lift and drag on an object. However this says you get no tangential forces \( \Rightarrow \) generate no drag.
Figure 41: Flow over semi-infinite flat plate. The viscous effects occur in a boundary layer of thickness $1/Re$.

**D’Alembert’s Paradox**

D’Alembert’s Paradox: Steady, uniform flow of an ideal fluid past a body gives no drag. Can show this generally, but we’ll look at a special case: flow around a cylinder no circulation:

$$p = p_\infty + \frac{\rho}{2} U^2 (1 - 4 \sin^2 \theta)$$

Figure 42: Pressure around a cylinder when there is boundary layer separation. Comparison of the pressure profile for the ideal flow theory and the experimentally measured pressure.

When you have separation you get lower separation at the back behind the separation point.
Figure 43: How to lower the drag force? By increasing pressure at the back you lower the drag force.

11/22/2013:

Helmholtz Vortex Thm

Definitions: A vortex line is a curve tangent to the vorticity field.

\[ \frac{dx(s)}{ds} = \omega \rightarrow \text{this is the analog of streamlines} \]

Let C be a curve transverse (nowhere tangent) to the vorticity field. The set of vortex lines passing through C defines a surface, called the vortex sheet. When C is a simple closed oriented curve, the vortex surface is called a vortex tube.

\[ \Gamma_c = \int_C \mathbf{u} \cdot d\mathbf{s} = \int_S \omega \cdot n dA = 0. \]

because \( \omega \) is tangent to the sheet and \( n \) is normal to the sheet. By Kelvin circulation theorem,

\[ \frac{d\Gamma_c}{dt} = 0 \]
Figure 45: Initial circulation across a cross section of a vortex tube.

if \( C \) is a material curve. Then,

\[
\int_{S(t)} \omega \cdot n \, dA = 0
\]

for all time. By the arbitrary choice of the set \( C \) (and thus \( S \)), \( \omega \) is tangent to all points in \( S(t) \) and therefore the sheet is a material surface.

Proof 2nd part: Let’s note that \( \nabla \cdot \omega = 0 \) (divergence of curl). \( \int_T \nabla \cdot \omega \, dV = 0 \Rightarrow \int_{\partial T} \omega \cdot n \, dS = 0 \).

Then, \( \int_{S_1} \omega \cdot n \, dS + \int_{S_2} \omega \cdot n \, dS + \int_{S_3} \omega \cdot n \, dS = 0 \Rightarrow \int_{S_1} \omega \cdot n \, dS + \int_{S_2} \omega \cdot n \, dS + 0 = 0 \) since \( S_3 \) is a vertex sheet. Can use Stokes’ theorem to get that the two circulations around the two curves are the same, \( \Gamma_{c_1} = \Gamma_{c_2} \).

Figure 46:

**Vorticity equation**

\[
\omega_t + u \cdot \nabla \omega = \omega \cdot \nabla u
\]

where LHS is the transport and RHS is the vortex stretching term (and for 2D planar flow it is zero).

**Vortex tube with a narrow cross-section**

The strength is \( \Gamma = \int_C u \cdot dS = \int_S \omega \cdot n \, dS \). For a narrow tube, \( \omega \cdot n \approx |\omega| \). Then, \( \Gamma \approx |\omega| \). Suppose the tube gets stretched by the flow (elongated). Then by incompressibility, if the tube elongates it means that the cross-sectional area (\( A = |S| \)) must go down. By Helmholtz Vortex Thms, then if \( A \) goes down, \( |\omega| \) must go up. **ONLY A 3D EFFECT!** In 2D planar flows, the vortex tube is perpendicular to the flow so the tube can never get elongated by the flow.
Spindown of a cup of coffee (tea)

Timescale for complete spindown ($\approx 10s$). For the infinitely deep coffee cup, Couette flow theory (separation of variables), you would estimate the timescale to be 2 minutes. The bottom is significant. Slower rotation on near the bottom $\Rightarrow$ get Ekman’s boundary layer. For $u_0 = \Omega r$, $p = \rho \Omega^2 r^2 / 2$. The nature of the pressure, sets up a secondary flow which causes stretching of the vortex tube. Re-estimating (see section 8.5 section in Acheson) you get a timescale of 10 seconds approximately.

![Figure 47: Pressure in a spindown of a cup of coffee.](image)

11/25/2013: Stokes Flow

Stokes Flow

\[
\begin{align*}
\mu \Delta u - \nabla p + f &= 0 \\
\nabla \cdot u &= 0
\end{align*}
\]

Pulling force on a sphere

How much force does it take to pull on a sphere of radius $a$ at Reynolds number zero at a speed $U$? Alternatively (in a change of reference frame), how much drag force is needed to hold the sphere stationary with a given constant flow $U$ at infinity?

Expect that $u = Mf$ because the equations are linear where $M$ is called the mobility matrix. For the sphere problem, the setup is isotropic $\Rightarrow M$ is a constant. Then, we can equivalently ask the question: **What is the flow generated by applying a force at a point?** Zero, unless the force is unbounded. We’re interested in $f = 0$ for $x \neq 0$, but $\int_{\mathbb{R}^3} f dV \neq 0$.

Fundamental solutions to Stokes flow

Let us consider force of the form:

\[ f = f_0 \delta(x) \]

Find the fundamental solution to Stokes equations, i.e. solve,:

\[
\begin{align*}
\mu \Delta u - \nabla p + f_0 \delta(x) &= 0 \\
\nabla \cdot u &= 0
\end{align*}
\]
Expect that the solution will be singular (at least in the derivatives). Look for a solution in the sense of distributions. A **distribution** is a continuous linear functional on the space of $C^\infty_0$ functions. $C^\infty_0$ is $C^\infty$ with compact support. Functional means $F : C^\infty_0 \to \mathbb{R}$ (e.g. any integrable function defines a distribution by: $\langle h, \phi \rangle = \int_\mathbb{R} h(x) \phi(x) \, dx$ for any $\phi \in C^\infty_0$). There are many more distributions, for example $\delta$-distribution:

$$\langle \delta, \phi \rangle = \int_\mathbb{R} \delta(x) \phi(x) \, dx = \phi(0)$$

Can make a definition for derivatives of distributions by relating to integration by parts.

$$\langle h', \phi \rangle = \int_\mathbb{R} h' \phi \, dx = -\int_\mathbb{R} h \phi' \, dx = -\langle h, \phi' \rangle$$

since $\phi$ is differentiable and has compact support. Then, we define: $\langle h', \phi \rangle = -\langle h, \phi' \rangle$. For example, $\langle \delta', \phi \rangle = -\langle \delta, \phi' \rangle = -\phi'(0)$.

Back to the fundamental solution for Stokes flow. However, before Stokes, let’s start by solving (something easier) Laplace’s equation:

$$\Delta G = \delta(x) \text{ in } \mathbb{R}^3$$

Solving this away from the origin, $\Delta G = 0$ for $x \neq 0$. By the symmetry of the RHS we look for symmetric solutions,

$$\Delta G = 0 \Rightarrow \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} G \right) = 0$$

where the solution is $G = c + \frac{d}{r}$. Pick $c = 0$ so that $G \to 0$ as $r \to \infty$. Then $G = \frac{d}{r}$. To solve at the origin, we require:

$$\langle \Delta G, \phi \rangle = \phi(0)$$

Can use integration-by-parts (or Green’s identities) to show that: $d = -\frac{1}{4\pi}$. The fundamental solution to Laplace’s equation is:

$$G = -\frac{1}{4\pi r}$$

More generally, for

$$\Delta \psi = f(x) \text{ in } \mathbb{R}^3$$

we can write $f(x) = \int_{\mathbb{R}^3} f(y) \delta(x-y) \, dy$. The solution then is:

$$\psi(x) = \int_{\mathbb{R}^3} G(x-y) f(y) \, dy := \Delta^{-1} f(y)$$

**Biharmonic equation**: Note that the solution to $\Delta^2 B = G$ is given by $B = -\frac{r}{8\pi}$. Then $\Delta^2 B = \Delta G = \delta(x)$ is known as the **biharmonic equation**. Therefore, $B$ is the fundamental solution to the biharmonic equation.
Finally, let’s go back to Stokes equations:

\[
\mu \Delta u - \nabla p + f_0 \delta(x) = 0 \\
\nabla \cdot u = 0
\]

Take the divergence of the first equation (and commute freely):

\[
0 = -\Delta p + \nabla \cdot (f_0 \delta(x)) \\
\Delta p = f_0 \cdot \nabla \delta(x),
\]

and now apply \( \Delta^{-1} \) and commute as you wish:

\[
p = f_0 \cdot \nabla (\Delta^{-1} \delta) = f_0 \cdot \nabla G = -\frac{1}{4\pi} f_0 \cdot \nabla \left( \frac{1}{r} \right)
\]

Substituting back into Stokes equation, we find:

\[
\Rightarrow 0 = \mu \Delta u - \nabla (f_0 \cdot \nabla G) + f_0 \delta(x) \\
0 = \mu u - \nabla (f_0 - \nabla (\Delta^{-1} G)) + f_0 (\Delta^{-1} \delta(x)) \\
0 = \mu u - f_0 \cdot \nabla \nabla (B) + f_0 (\Delta B) \\
\Rightarrow u = \frac{f_0}{\mu} (\nabla \nabla - I \Delta) B.
\]

Now we have an expression for the flow due to a point force. We compute,

\[
\nabla \left( \frac{1}{r} \right)_i = \frac{\partial}{\partial x_i} \left( \frac{1}{r} \right) = -\frac{1}{r^2} \frac{\partial r}{\partial x_i} = -\frac{x_i}{r^3},
\]

since \( r^2 = x_i x_i \) and \( 2r \frac{\partial r}{\partial x_i} = 2 x_j \delta_{ij} = 2 x_i \). Then: \( p = \frac{f_0 x_i}{4\pi r^3} = \frac{(f_0)_i}{4\pi r^3} \).

**11/27/2013: Stokes Flow (cont’d)**

Last time we almost worked out the expression of the fundamental solution to Stokes flow due to a singular point force. Recall, Stokes equations are:

\[
\mu \Delta u - \nabla p + f_0 \delta(x) = 0 \\
\nabla \cdot u = 0
\]

**Fundamental solution to Stokes equation (cont’d)**

\[
u_i = \frac{1}{8\pi \mu} S_{ij}(f_0)_j
\]
where $S_{ij} = \frac{\delta_{ij}}{r} + \frac{x_ix_j}{r^3}$ and the pressure is $p = \frac{x_i(f_0)i}{4\pi r^3}$. The $S$ tensor is called the Stokeslet (Ossen, Ossen-Burgers tensor).

Here’s the derivation for the Stokeslet (not in class notes):

\[
\begin{align*}
\mathbf{u} &= \frac{f_0}{\mu} (\nabla \nabla - I \Delta) \mathbf{B} \\
(\nabla \nabla B)_i &= \frac{\partial}{\partial x_i} \left( \frac{\partial}{\partial x_j} \left( \frac{-r}{8\pi} \right) \right) \\
&= -\frac{1}{8\pi} \frac{\partial}{\partial x_i} \left( \frac{x_j}{r} \right) \\
&= -\frac{1}{8\pi} \left[ \frac{\delta_{ij}}{r} + x_j \left( -\frac{x_i}{r^3} \right) \right] \\
&= -\frac{\delta_{ij}}{8\pi r} + x_j \left( -\frac{1}{8\pi r^3} \right)
\end{align*}
\]

\[
\Rightarrow u_i = \frac{1}{8\pi \mu} \left( -\frac{(f_0)_j \delta_{ij}}{r} + \frac{(f_0)_j x_i x_j}{r^3} \right) - (f_0)_i \left( -\frac{1}{4\pi r} \right) \\
= \frac{1}{8\pi \mu} \left( \frac{\delta_{ij}}{r} + x_i x_j \right) (f_0)_j.
\]

**Pulling force on a sphere (cont’d)**

Let’s think of the sphere and let $f = F \mathbf{e}_1$. On the surface of the sphere ($r = a$), using the fundamental solution to Stokes equation:

\[
u_i = \frac{F}{8\pi \mu} \left( \frac{\delta_{i1}}{a} + \frac{x_i x_1}{a^3} \right)
\]

The boundary condition on the sphere is that $u_i = U \delta_{i1} = (U 0 0)^T$. Notice that the second term in the parentheses in $u_i$ does not satisfy the boundary condition. We need to come up with a workaround so we can use the Stokeslet to come up with a solution to the sphere problem.

![Figure 48: Dipole force.](image)

Like the method of images, add singular solutions outside the domain to satisfy the boundary condi-
The solution at $x$ due to a singular force $\alpha$ is

$$u_i(x, \alpha) = \frac{1}{8\pi\mu} S_{ij} \alpha_j$$

Derivatives of the solution are again solution to Stokes for $r > 0$. Consider,

$$u_D(x, \alpha, \beta) = -\beta \cdot \nabla u(x, \alpha)$$

This corresponds to a force distribution: $f = -\beta \cdot \nabla (\delta(x)\alpha)$. What is this? We will show this is a dipole force. Apply force $\alpha \delta$ at $x = \beta/2$ (i.e. $\alpha \delta(x - \beta/2)$) and the force $-\alpha \delta$ at $x = -\beta/2$ (i.e. $-\alpha \delta(x + \beta/2)$). Adding these two forces, the resulting flow is:

$$U = u(x - \beta/2, \alpha) - u(x + \beta/2, \alpha)$$

For small $|\beta| << 1$, we can expand the solution to obtain: $U \approx -\beta \cdot \nabla u(x, \alpha) + O(\beta^2)$. $u_D$ is called Stokes dipole. You can break the gradient of $u(x, \alpha)$ into symmetric and anti-symmetric parts. Define the stresslet for the symmetric part and the rotlet for the antisymmetric part. The rotlet corresponds to the flow from a point torque.

![Figure 49: Stresses associated with the rotlet and stresslet.](image)

Obviously, second derivatives are more complicated:

$$u_{PD}(x, d) \propto \Delta u(x, d), \quad p_{PD} = 0$$

is called the potential dipole.

$$\left(u_{PD}\right)_i = C \left( \frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5} \right) d_j, \quad p_{PD} = 0$$

This is nice, because the term with $x_i x_j$ can be used to kill off the “bad boy” in the Stokeslet solution to flow on a sphere due to a singular force. But where does this come from? Consider solutions with constant pressure. Then:

$$\Delta u = 0$$

To make sure it satisfies the boundary condition, consider solutions of the form $u = \nabla G$, where $G = -\frac{1}{4\pi r}$. $\nabla \cdot u = \Delta G = \delta(x)$ is zero for $r > 0$. Then, $p = 0$, and $u = -\frac{1}{4\pi} \nabla \left( \frac{1}{r} \right)$ is a solution to Stokes for $r > 0$. 57
Then,
\[ \frac{\partial r}{\partial x_i} = \frac{x_i}{r} \]
\[ u_i = \frac{1}{4\pi} \frac{x_i}{r^3} \Rightarrow u = \frac{1}{4\pi} \frac{x}{r^3} \]

PIC7
Source of mass at \( x = 0 \), then:
\[ \int_{B_r} u \cdot n dS = \int_{B_r} \frac{1}{4\pi} \frac{x \cdot x}{r^4} dS \]
\[ = \frac{1}{4\pi} \int_{B_r} \frac{1}{r^2} dS = 1. \]

This solution is called **Stokes source**. The gradient of Stokes source gives the **potential dipole**. Let’s compute:
\[ \frac{\partial}{\partial x_j} \left( \frac{x_i}{r^3} \right) = \frac{\delta_{ij}}{r^3} - \frac{3}{r^5} x_i x_j \]
You can think of \( u_{PD} (x, d) \) as a source of strength \( |d| \) at \( x = d/2 \) and a sink of strength \( |d| \) at \( x = -d/2 \):

Now let’s go back to the sphere in a uniform flow field and try a solution of the form:
\[ u_i = \frac{F}{8\pi\mu} \left( \frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3} \right) + A \left( \frac{\delta_{ij}}{r^3} - \frac{3}{r^5} x_i x_j \right) \]
Take \( i = 2 \) (or \( i = 3 \)). Then:
\[ u_2 = \frac{F}{8\pi\mu} \frac{x_2 x_1}{r^3} - 3A \frac{x_2 x_1}{r^5} \]
We want \( u_2 = u_3 = 0 \) for \( r = a \). Then: \( A = \frac{F}{24\pi\mu} a^2 \). For \( i = 1, r = a \): \( u_1 = U = \frac{F}{8\pi\mu} \left( \frac{1}{a} \right) + \frac{F}{24\pi\mu} a^2 \left( \frac{1}{a^2} \right) \)
Solve and get \( F = 6\pi\mu a U \) this is how much force you need to get a certain velocity or vice-versa.

**12/02/2013:**

**Multipole expansion for flow around a sphere**

Last time we found the solution for flow around sphere: \( u = \) Stokeslet + potential dipole. When we try the Stokeslet, we didn’t satisfy boundary condition. The potential dipole is the derivative of a Stokes source. The flow field is:
\[ u_i = \frac{F}{8\pi\mu} \left( \frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3} \right) + \frac{F a^2}{24\pi\mu} \left( \frac{\delta_{ij}}{r^3} - \frac{3}{r^5} x_i x_j \right) \]
Here \( F \) denotes the strength of the Stokeslet at a singular point. To compute force:

1. Compute \( \sigma = \mu (\nabla u + \nabla u^T) - p\delta \) and \( \int_S \sigma \cdot ndS = F \)
   If I put a singular force on the sphere, \( f = F e_1 \delta(x) \), then what is \( \int_S \sigma \cdot ndS =? \) I can use divergence theorem, to say \( \int_S \sigma \cdot ndS = \int_V \nabla \cdot \sigma dV = \int_V \nabla \cdot f = 0 \) (since Stokes equation says: \( \nabla \cdot \sigma + f = 0 \)). We can use the same argument to show that the force on the sphere from the dipole is zero.

2. \( F = 6\pi\mu a U \) from enforcing that \( u = U e_1 \) and relating \( U \) to \( F \) (We’re finding the force here from the boundary conditions.)
Applications of low Reynolds number flows: Charge of an electron - Oil drop experiment

Miliken (1909) and was awarded the Nobel Prize in 1923. We assume \( \text{Re} \ll 1, \rho_0 < \rho_a, \) and \( \mu_0 \gg \mu_a. \)

How fast is this oil droplet falling? It will fall with a velocity, by balance of forces:

\[
6\pi \mu a U = \frac{4\pi a^3}{3} [ (\rho_0 - \rho_a) g ]
\]

Somehow, Milikin could estimate the velocity \( U \) and use Stokes formula and buoyancy to compute the size of the droplet \( a. \) Now charge the plates. Apply strength of electric field to balance the sum of these forces and droplet stops moving. Then:

\[
F_e = F_b
\]

\[
= n e \frac{V}{H}
\]

where \( n \) is the number of electrons in the droplet, \( e \) is the charge of the electron, \( V \) is potential difference and \( H \) is the distance between the plates. This procedure gives me the ability to measure \( ne \) for an oil droplet. Then redo the experiment and since \( e \) is constant, determine it!

Applications of low Reynolds number flows: Stokes-Einstein relation

If you have a particle of radius \( a, \) then the diffusion coefficient is:

\[
D = \frac{k_B T}{6\pi \mu a}
\]

where \( T \) is the temperature (Kelvin) and \( k_B \) is the Boltzmann constant. Recall \( F = \xi U \) where \( \xi = 6\pi \mu a \) is the drag coefficient. Then you can think of the diffusion coefficient as: \( D = \frac{k_B T}{\xi}. \) How to prove this? Many ways, but here is one way.

Let’s do this by assuming we have a sphere moving through a fluid where the applied forces are friction and random thermal forces.

\[
m\ddot{x} = -\xi \dot{x} + F(t)
\]

Let’s assume that \( F(t) \) is Gaussian (coming from a Boltzmann distribution), i.e. \( \langle F \rangle = 0 \) and \( \langle F(t)F(t') \rangle = A\delta(t - t') \). We require thermodynamic equilibrium:

\[
T = \frac{1}{2} \langle m\dot{x}^2 \rangle = k_B T \text{ (thermal energy)}
\]

59
Solve the Langevin equation \((m \ddot{x} = -\xi \dot{x} + F(t))\). In equilibrium, the drag force and the fluctuating force to get:

\[
\frac{1}{2} \langle m \dot{x}^2 \rangle = \frac{A}{2\xi} = k_B T
\]

Then: \(A = 2\xi k_B T\). Compute, \(\langle x^2 \rangle = 2\frac{k_B T}{\xi} t\) for long time. But also,

\[
\frac{\partial p}{\partial t} = D \frac{\partial^2 p}{\partial x^2}
\]

where \(p\) is the probability of being somewhere and \(p(x, 0) = \delta(x)\). Solution to this equation with this initial condition is:

\[
p = \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/(4\pi t)}
\]

and that consequently, \(\langle x^2 \rangle = 2Dt\).

Therefore, you get \(D = \frac{k_B T}{\xi} = \frac{k_B T}{6\pi \mu a}\) for the drag coefficient on the sphere. At statistical equilibrium, the thermal energy is balanced by the energy dissipated.

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**12/04/2013: 2D Stokes**

**Drag in 2D at zero Reynolds number**

We consider an infinitely long cylinder translated at a constant velocity. Can we then find \(F\) (force per unit length) to move the cylinder at velocity \(U\)? Then, the picture is of a translated circle.

Start with the fundamental solution to Stokes flow. Then,

\[
\mathbf{u} = \frac{1}{4\pi \mu} S \mathbf{f}
\]

where the Stokeslet is \(S_{ij} = -\ln(r)\delta_{ij} + \frac{x_i x_j}{r^2}\). A point force in 2D Stokes gives an unbounded flow at infinity – this seems odd. Stokes concluded that there is no solution. A solution is well constructed for the
3D problem but not 2D, this is known as Stokes paradox (1851). In 3D if we seek a correction to Stokes formula for small Reynolds number.

\[ u = u_0 + \text{Re} u_1 + O(\text{Re}) \]

find that \( u_1 \) must be unbounded at \( \infty \). This is known as the Whitehead paradox (1889).

It turns out that \( \text{Re} \to 0 \) is a singular perturbation problem!

\[
\begin{align*}
\rho u \cdot \nabla u &= \mu \Delta u - \nabla p \\
\frac{U^2}{L} &= \frac{U}{L^2}
\end{align*}
\]

To ignore inertia, require \( \frac{U}{\rho} \ll 1 \). Recall, \( \text{Re} = \frac{UL}{\nu} \). What was \( L \) for Stokes problem? We choose \( a \) as the radius of the sphere, then \( \text{Re} = \frac{Ua}{\nu} \). On other lengthscales, \( \frac{Ua}{\nu} \frac{L}{a} = \text{Re} \left( \frac{L}{a} \right) \leftarrow \) this is what we require to be small (not just the Reynolds number). Inertia is not small on lengthscales of order \( O \left( \frac{1}{\text{Re}} \right) \). This is a true characteristic of a singular perturbation problem - lack of uniform approximation.

Stokes was lucky in 3D.... He took a singular perturbation problem and looked at first order only. Had he looked at higher orders he would have seen a blow up in the solution. 1911 Oseen recognized the problem. He instead proposed to look at this problem,

\[ \rho U \cdot \nabla u = \mu \Delta u - \nabla p \]

where \( U \) is the flow at infinity (sphere is stationary). But why does this hold near the sphere? Not fully understood until 1950s using matched asymptotic expansions.

Let’s consider a constant velocity perturbed by the sphere:

- Oseen was solving the outer problem. Flow at infinity. Satisfy the BC at infinity.
- Stokes was solving the inner problem. Flow around the sphere. Satisfy the BC on the sphere.

How do we get Oseen’s equations? Let \( \epsilon = \text{Re} = \frac{Ua}{\nu} \). This is the inner problem now. Then:

\[ \epsilon u \cdot u = \Delta u - \nabla p \]

Rescale to get the outer problem. Let \( Y = \epsilon x \) and \( \epsilon P = p \) where capitalized letters indicate outer variables.

\[ V(Y) = u(Y/\epsilon) = u(x) \]

The outer problem becomes,

\[
\begin{align*}
V \cdot \nabla V &= \Delta V - \nabla P \\
\nabla \cdot V &= 0
\end{align*}
\]

Expand \( V = u + \epsilon V_1 + O(\epsilon) \). Plug in and at order \( \epsilon \), you get Oseen’s equations. Next we need to find the uniform solution and consequently the matching condition:

\[ \lim_{x \to \infty} u \left( \frac{x}{\epsilon} \right) = \lim_{Y \to 0} V(Y) = U \]

This works in 3D, but not in 2D. **This does not work in 2D!** Match at intermediate scale (e.g. \( O(\sqrt{\epsilon}) \)).
12/06/2013: Low Reynolds Number Locomotion

At low Reynolds number, everything instantenously equilibrates \((\text{time-reversibility})\). Consider an object moving in a fluid at zero Reynolds number with a time-periodic motion. If the motion is symmetric in time, then all fluid particles return periodicially to their starting positions. This is called the Scallop Theorem. There is one parameter family of shapes, angle of flapper, which describes the motion. In zero Reynolds number, time doesn’t matter - it’s just a parametrization of shape changes. If I want to move at low Reynolds number I need at least 2 degrees of freedom (2 family parameter of changes) to describe a nonreversible motion.

![Simplest locomotion in zero Reynolds number with two degrees of freedom.](image)

There is a directionality to this swimmer in Figure 52. Many organisms (swimmers) use wave-like shape changes. For example let \(x = (s, a \sin(ks - \omega t))\). See the Acheson book for an analysis of the swimming sheet problem in the case where \(a \ll 1\).

**Helical swimmers**

![Helical swimmer](image)

By turning a helix, swimmers generate velocity since helixes have a right of left-hand rotation. **Here’s a sketch of analysis of helical swimmer.** This is based on resistive force theory, which is a simplification of slender body theory. The idea is to break down the helix into small components which locally approximate the helix as a cylinder and use a force-velocity relationship to determine the swimming speed.

First let’s find the drag on a finite-length (long) cylinder where \(\ell \gg a\). Let \(f = \frac{F}{\ell}\) where this is a force per unit length.
Figure 54: Dragging a rod longitudinally rather than along its length is faster but only twice as fast in fluids at zero Reynolds number.

\[
U_n = \frac{f}{4\pi\mu}(\ln(\ell/a) + c_n)
\]

\[
U_\tau = \frac{f}{2\pi\mu}(\ln(\ell/a) + c_\tau)
\]

where \(c_n, c_\tau\) are \(O(1)\) constants. Let the resistance coefficient in the normal and tangential directions be:

\[
R_n = \frac{f}{U_n} = \frac{4\pi\mu}{\ln(\ell/a) + c_n}
\]

\[
R_\tau = \frac{f}{U_\tau} = \frac{2\pi\mu}{\ln(\ell/a) + c_\tau}
\]

Then, \(R = \frac{R_n}{R_\tau} = 2 + O(1/\ln(\ell/a))\).

Figure 55: Decomposing a force on a rod into the normal and tangential components.

**How much faster does this move and in what direction?**

\[
U = U_n + U_\tau = R_n f_n + R_\tau f_\tau
\]

Take an infinitely long helix and we describe its intrinsic shape as: \(x_0 = (\alpha s, A\cos(\beta s + \omega t), A\sin(\beta s + \omega t))\). \(\alpha\) is the pitch, \(0 \leq \alpha \leq 1\), \(A\) is the amplitude, \(s\) is the arclength. There is this constraint which determines \(\beta, \alpha^2 + \beta^2 A^2 = 1\). In Stokes flow, all forces are balanced. Then we add the turning motion of the helix requires force which is balanced by the drag force of a steady translation.

\[
U = U_0 + U e_1 = \frac{\partial x_0}{\partial t} + U e_1
\]
Determine $U$ so that $R_{\tau}U_{\tau} + R_{\alpha}U_{\alpha} = 0$. The result is:

$$U = -\kappa V$$

where $\kappa = \frac{(1-a^2)(R-1)}{R-a^2(R-1)}$ and $V$ is the phase velocity, $V = -\alpha \omega / \beta$. Recall $R = R_{\alpha} / R_{\tau}$ and therefore the helix is harnessing that asymmetry from dragging a cylinder straight up on its side.