Spaces of Functions

| Topo ⊇ Metric ⊇ Normed Linear ⊇ Banach ⊇ Hilbert |

Metric Spaces.

- **MOTIVATION**: study set of points with a suitable notion of distance between points
- **Properties of distance function**: strictly positive, symmetric, triangle inequality
- **Properties of norm**: strictly positive, homogeneous, triangle inequality
- **Convergence**: A sequence \((x_n)\) of real numbers converges to \(x \in \mathbb{R}\) if for every \(\epsilon > 0\) there is an \(N \in \mathbb{N}\) such that \(d(x_n - x) < \epsilon\) for all \(n \geq N\)
- **Cauchy convergence**: A sequence \((x_n)\) is a Cauchy sequence if for every \(\epsilon > 0\) there is an \(N \in \mathbb{N}\) such that \(d(x_m - x_n) < \epsilon\) for all \(m, n \geq N\)
  - Adjacent terms converging \(\not\to\) Cauchy (e.g. \(\log n\))
- **Unconditionally convergent**: Converges regardless of order \(\iff\) any subsequence converges as well
- **Absolutely convergent**: Series converges in absolute value \((\sum |x_n|)\)
  - Conv \(\not\to\) absolute convergent (e.g. \(x_n = n(-1)^n\) )
- **Complete**: A space is complete if every Cauchy sequence converges to a limit in the space
- **Bounded**: A set \(A\) is bounded iff there is an \(M \in \mathbb{R}\) and an \(x_0 \in X\) such that \(d(x_0, x) \leq M\) for all \(x \in A\)
- **Continuity**: A real function \(f : \mathbb{R} \to \mathbb{R}\) is continuous at a point \(x_0 \in \mathbb{R}\) if for every \(\epsilon > 0\) there is a \(\delta > 0\) such that \(d(x - x_0) < \delta\) implies \(|f(x) - f(x_0)| < \epsilon\)
  - \(f(B_\delta(x_0)) \subseteq B_\epsilon(f(x_0))\)
  - \(f^{-1}(G)\) is open/closed in \(X\) for every open/closed \(G\) in \(Y\)
- **Separable**: iff there exists a countable dense subset
- **Precompact**: Any sequence has a Cauchy subsequence
- **Compact**: Every open cover has a finite subcover
  - Sequentially compact (for metric spaces): Every sequence has a convergent subsequence
    - Compact = precompact + closed
    - For complete spaces, compact = totally bounded + closed
- **Open**: If for any \(x \in U\), there exists \(\epsilon > 0\) such that \(B_\epsilon(x) \subseteq U\).
- **Open map**: maps open sets to open sets (not necessarily continuous)
- **Heine-Borel**: sequentially compact in \(\mathbb{R}^n\) iff closed and bounded
- **Bolzano-Weierstrass**: every bdd sequence in \(\mathbb{R}^n\) has a convergent subsequence
- **Homeomorphism**: iff 1-1, onto, function and inverse function are continuous.
- **Lower semicontinuous**: \(\liminf f(x_n) \geq f(x)\) for all \(x_n \to x\).
- **Upper semicontinuous**: \(\limsup f(x_n) \leq f(x)\) for all \(x_n \to x\).
- **Convex**: A subset \(C\) is convex if \(tx + (1 - t)y \in C\) for any \(x, y \in C\) and \(t \in [0, 1]\).
• Convex function: If \( f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \) with \( x, y \in C \) convex
  - uniformly continuous

• Uniformly convex: If \( ||x|| = ||y|| = 1 \) and \( ||x - y|| > \epsilon \), then: \( ||\frac{x+y}{2}|| < 1 - \delta \)

• Totally bounded: finite \( \epsilon \)-net for every \( \epsilon > 0 \)

• Miscellaneous:
  - A sequentially compact metric space is separable
  - Finite-dimensional linear space are Banach
  - Convergence iff \( \lim \inf x_n = \lim \sup x_n \)
  - Every equicontinuous sequence of functions that converge pointwise must converge uniformly to a continuous function
  - Nice things on a compact set: closed, closed subset is still compact, bounded, continuous functions are bdd and uniformly cont, cont functions take compact to compact, max/min are attained
    - In \( \mathbb{R}^n \), precompact \( \Rightarrow \) bounded
    - If \( K \) is a compact metric space and \( f \) is lower/upper semicontinuous, then \( f \) has a minimizer/maximizer (i.e. bounded from below/above and attains its infimum/supremum).
  - If \( f_n, f \) are continuous and converge pointwise, then they converge uniformly (Dini’s)
    - The uniform limit of continuous functions is continuous

\[ C(X), C^k(X), C^\infty(X). \]

• MOTIVATION: study the smoothness of functions

• Structure: \( ||f||_\infty = \sup \{ f(x) \} \)

• Pointwise convergence: \( f_n(x) \to f(x) \) for all \( x \in X \)

• Uniform convergence: \( \lim_{n \to \infty} ||f_n - f||_\infty = 0 \)

• \( B(X) \): complete, not compact

• \( C(X) \): cannot use \( || \cdot ||_\infty \); if \( X \) is compact \( C(X) = C_B(X) \)

• \( C^*_b(X) = C(X) \cap B(X) \) is complete

• \( C^*_c(X) = \{ f \in C(X) \mid \text{supp}(f) \text{ is compact} \} \) is not closed (therefore not Banach space)

• \( C^0(X) = \overline{C_c(X)} \) is closed and complete

• \( C_c(X) \subset C^0(X) \subset C^*_b(X) \subset C(X) \)

• Dini’s: Spse \( f_n : [0,1] \to \mathbb{R} \) are continuous functions that converge pointwise to \( f \) a continuous function with \( f_n(x) \geq f_{n+1}(x) \). Then \( f_n \to f \) uniformly.

• Dense subsets: polynomials (Weierstrass Approximation), trig polynomials

• Equicontinuity: For every \( \epsilon > 0, \exists \delta(\epsilon, x) \) such that if \( d_X(x, y) < \delta \) implies \( d_Y(f(x), f(y)) \leq \epsilon \) for all \( f \in F \)

• Lipschitz function: If there exists a real constant \( k \geq 0 \) s.t. \( d_Y(f(x_1), f(x_2)) \leq kd_X(x_1, x_2) \)

• Arzela-Ascoli: Let \( K \) be a compact metric space. A subset \( F \subseteq C(K) \) is compact iff it is closed, bounded, and equicontinuous.
  - Alternatively, each family of continuous function is equicontinuous and uniformly bounded on a compact set, then each sequence has a uniformly convergent subsequence.
• **Stone-Weierstrass**: Suppose $X$ is a compact Hausdorff space and $A$ is a subalgebra of $C(X, \mathbb{R})$ which contains a non-zero constant function. Then $A$ is dense in the continuous functions if and only if it separates points.

• **Miscellaneous**:  
  - Uniform conv $\Rightarrow$ pointwise conv ONLY  
  - Pointwise convergence is not a good notion of convergence to use for continuous functions because it does not preserve continuity  
  - If $X$ is compact, then these spaces (in boxed line) are equal  
  - Compact support: For $\epsilon > 0$, $\exists R > 0$ so that $||x|| > R$ implies $|f(x)| < \epsilon$, i.e. $\lim_{||x|| \to \infty} f(x) = 0$  
  - In $\mathbb{R}^n$ every subset is equicontinuous  
  - Closed subset of a complete space is complete

**Topological Spaces.**

• **MOTIVATION**: Some types of convergence (i.e. pointwise) cannot be expressed in terms of a metric on a function space  
• **Structure**: collection of open sets  
• **Topology**: A collection of subsets of $X$, called *open sets* such that:  
  1. The empty set and $X$ are open  
  2. Union of arbitrary collection of open sets is open  
  3. Intersection of a finite number of open sets is open  
• **Neighborhood**: $V \subset X$ is a neighborhood of $x \in X$ if $\exists$ open set $G \subset V$ with $x \in G$ (could be open or closed or neither)  
• **Hausdorff topology**: For distinct points $x, y \in X$ there exist non-intersecting neighborhoods $V_x, V_y$  
• **Convergence**: A sequence $(x_n)$ in $X$ converges to a limit $x \in X$ if for every neighborhood $V$ of $x$, there is a number $N$ such that $x_n \in V$ for all $n \geq N$.  
• **Continuity**: At each $x \in X$ for each neighborhood $W$ of $f(x)$ there exists a neighborhood $V$ of $x$ so that $f(V) \subset W$; $f^{-1}(G) \in T$ for every $G \in S$ where $T$ is the topology on $X$ and $S$ is the topology on $Y$.  
• **Base**: A subset $B$ of a topology $T$ is a base for $T$ if for every $G \in T$ there is a collection of sets $B_\alpha \in B$ so that $G = \bigcup B_\alpha$ (e.g. open intervals in $\mathbb{R}$)  
• **Neighborhood base**: For each neighborhood $V$ of $x$ there is a neighborhood $W \in N$ such that $W \subset V$.  
• **Tychonoff’s**: Spse $X_\alpha$ is compact for each $\alpha \in \Gamma$. Then the product space $\prod_{\alpha \in \Gamma} X_\alpha$ is also compact.  
• **Miscellaneous**:  
  - Homeomorphic spaces are indistinguishable (for e.g. $G$ is open in $X$ iff $f(G)$ is open in $Y$)  
• **Extra Vocab**: trivial topology, discrete topology (power set of $X$, every set is open), metric topology ($G$ is open iff for every $x \in G$ there is an open interval $I$ such that $x \in I$ and $I \subset G$, induced (relative) topology ($S = H \subset Y | H = G \cap Y$ for some $G \in T$), homeomorphic, first countable (countable neighborhood base; e.g. metric), second countable (countable base; e.g. separable metric), finer or stronger (more open sets), coarser or weaker, subbase, Cartesian product, projection map, product topology.
Banach Spaces.

- **MOTIVATION:** Many linear equations may be formulated in terms of a suitable linear operator acting on a Banach space.
- **Structure:** \( d(x, y) = \| x - y \| \) and \( \| x \| = \sqrt{< x, x >} \) where \( < x, x > = \int_{\mathbb{X}} \).
- **Properties:** normed linear space that is complete.
- **\( C^k \) norm:** \( \| f \|_{C^k} = \| f \|_\infty + \| f' \|_\infty + ... + \| f^{(k)} \|_\infty \)
- **Essential supremum:** \( \| f \|_\infty = \inf \{ M : | f(x) | \leq M \text{ a.e. in } [a, b] \} \)
- **Linear map:** A function \( T : X \rightarrow Y \) such that \( T(ax + by) = aT(x) + bT(y) \)
- **Bounded map:** \( \| Tx \| \leq M \| x \| \) for some constant \( M > 0 \)
- **Uniform (Operator) norm:** Defined only for bounded linear operators: \( \| T \| = \sup \frac{\| Tx \|}{\| x \|} = \sup_{\| x \| = 1} \| Tx \| \) ( = FIXED NUMBER)
- **Equivalence:**
  1. \( \| . \|_1 \sim \| . \|_2 \) and \( \| . \|_2 \sim \| . \|_3 \) \( \iff \| . \|_1 \sim \| . \|_3 \)
  2. \( \| . \|_1 \sim \| . \|_1 \)
- **Equivalence relation:**
  1) \( \| . \|_1 \sim \| . \|_2 \) and \( \| . \|_2 \sim \| . \|_3 \) \( \iff \| . \|_1 \sim \| . \|_3 \)
- **Bounded Linear Transformation:** Let \( X \) be a normed linear space and \( Y \) be a Banach space. If \( M \) is a dense linear subspace of \( X \) and \( T \) is a bounded linear map, then there is a unique bounded linear map \( T : X \rightarrow Y \) s.t. \( T(x) = T(x) \) for all \( x \in M \).
- **Open Mapping:** Suppose that \( T : X \rightarrow Y \) is a one-to-one, onto bounded linear map between Banach spaces \( X \) and \( Y \). Then \( T^{-1} : Y \rightarrow X \) is also bounded.
  ! Warning! Cannot drop onto or Banach.
- **Kernel:** \( \ker T = \{ x \in X : Tx = 0 \} \)
- **Range:** \( \text{ran } T = \{ y \in Y : \text{ there exists } x \in X \text{ s.t. } Tx = y \} \)
- **Nullity:** dimension of the kernel
- **Rank:** dimension of the range
- **ODE Solution:** \( Tx = y \) has a unique solution iff \( T^{-1} \) exists
- **ODE Stability:** A solution of \( Tx = y \) is stable iff:
  - \( T^{-1} \) is continuous
  - Any open set \( U \subset X \), \( (T^{-1})^{-1}(U) \) is open in \( Y \)
  - Any open set \( U \subset X \), \( TU \) is open in \( Y \)
- **Compact operator:** iff every bounded sequence \( (x_n) \) in \( X \) is precompact (i.e. has a subsequence \( (x_{n_k}) \) s.t. \( (Tx_{n_k}) \) converges in \( Y \))
- **Strong convergence:** Any \( x \in X \), \( \| T_n x - Tx \|_{op} \rightarrow 0 \)
- **Uniform convergence:** \( \| T_n - T \| \rightarrow 0 \)
  - Uniform conv \( \Rightarrow \) strong conv
  - Strong conv + norm conv \( \Rightarrow \) uniform conv
  - Strong conv \( \nRightarrow \) uniform conv because \( P_n(x_1, ..., x_n, x_{n+1}, ...) = (x_1, ..., x_n, 0, ...) \)
- **Weak convergence:** For any bdd linear functional \( \phi \in X^* \), \( | \phi(x_n) - \phi(x) | \rightarrow 0 \)
- **Weak-* convergence:** \( (\phi_n) \in X^* \) is weak-* convergent if for any \( x \in X \), \( \phi_n(x) \rightarrow \phi(x) \).
• **Mazur's Theorem:** Spse \( x_n \to x_0 \). Then, for any \( \epsilon > 0 \) there exists \( \lambda_i \geq 0 \) with \( \sum_i \lambda_i = 1 \) s.t. \( ||x_0 - \sum_i \lambda_i x_i|| < \epsilon \).

• **Banach-Alaoglu:** Let \( X^* \) be the dual space of a Banach space \( X \). The closed unit ball in \( X^* \) is weak-* compact.

• **Miscellaneous:**
  - Linear operator is continuous \( \iff \) bounded
  - The dual of any normed linear space is Banach
  - Norm on \( L^\infty([a, b]) \) is the essential supremum (upper bound must be of nonzero measure)
  - Closed linear subspace of a Banach space is Banach
  - \( \inf f \leq \liminf f \leq \text{ess inf } f \leq \text{ess sup } f \leq \limsup f \leq \sup f \)
  - If \( T \) is one-to-one and onto, then \( T \) is nonsingular or invertible
  - If \( T \) is a bounded, onto, linear map between Banach then \( T \) is an open mapping.
  - Two norms on a linear space generate the same topology iff they are equivalent
  - Any norm on a finite dim. space is equivalent to its \( \mathbb{R}^n \) norm
  - A map is 1-1 iff ker \( T = \{0\} \) and it is onto iff ran \( T = Y \)
  - The kernel is a closed linear subspace if the linear map is bounded.
  - There exists bounded linear functional s.t. \( ||\Phi|| = 1 \) and \( \Phi(x) = ||x|| \)
  - If \( T \) is a bounded linear map between Banach spaces then: \( c||x|| \leq ||Tx|| \) iff \( T \) has closed range and \( Tx = 0 \) implies \( x = 0 \) only.
  - Finite-dimensional rank operators are compact
  - The uniform limit of compact operators is compact

**Hilbert Spaces.**

• **MOTIVATION:** study the geometry of certain subsets of infinite-dim Banach spaces (inner product - orthogonality - angle)

• **Structure:** \( (\cdot, \cdot) : X \times X \to \mathbb{C} \) where \( ||x|| = \sqrt{<x, x>} \)

• **Properties:** linear in second argument, Hermitian symmetric, positive definite

• **Cauchy-Schwarz Inequality:** \( ||(x, y)|| \leq ||x|| ||y|| \)

How does a normed linear space become an inner product space?

- **Parallelogram Law:** \( ||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2 \)
- **Polarization Formula:** \( (x, y) = \frac{1}{4}((||x+y||^2-||x-y||^2)-i||x+iy||^2+i||x-iy||^2) \)

• **Orthogonal:** \( x \) is orthogonal to \( y \) iff \( (x, y) = 0 \)

• **Orthogonal complement:** \( A^\perp = \{x : x \perp y \text{ for all } y \in A\} \); closed linear subspace

• **Angle:** \( \theta = \frac{\cos^{-1} ||(x, y)||}{||x|| ||y||} \)

• **Orthogonal projection:** Let \( M \) be a closed linear subspace,
  1. For each \( x \in H \) there is a unique closest point \( y \in M \) such that \( ||x - y|| = \min ||x - z|| \) for all \( z \in M \).
  2. The point \( y \in M \) closest to \( x \in H \) is the unique element of \( M \) with the property that \( (x - y) \perp M \).

• **Orthonormal basis:** An orthonormal set such that every vector in the Hilbert space can be expanded in terms of the basis. i.e. \( \forall x \in H, x = \sum_{\alpha \in A} (e_\alpha, x)e_\alpha \)

• **Bessel's inequality:** Spse \( S = \{e_\alpha | \alpha \in A\} \) is an orthonormal set of an inner product space \( X \). Then, \( \sum_{\alpha \in A} |(e_\alpha, x)|^2 \leq ||x||^2 \) for all \( x \in X \). (PF: \( ||x - \sum (e_\alpha, x)e_\alpha||^2 \geq 0 \)

• **Parseval's inequality:** \( ||x||^2 = \sum_{\alpha \in A} |(e_\alpha, x)|^2 \)
• **Generalized Parseval’s inequality**: \((x, y) = \sum_\alpha \overline{a_\alpha} b_\alpha\) (says a Hilbert space with an orthonormal basis is isomorphic to \(l^2\))

• Construct an orthonormal basis: (Gram-Schmidt) 
  \[
  \begin{align*}
  y_1 &= x_1, e_1 = \frac{y_1}{\|y_1\|} \\
  y_2 &= x_2 - (e_1, x_2) e_1, e_2 = \frac{y_2}{\|y_2\|} \\
  y_n &= x_n - \sum (e_i, x_n) e_i, e_n = \frac{y_n}{\|y_n\|}
  \end{align*}
  \]

• **Isomorphic spaces**: if there exists isomorphism \(T : X_1 \to T_2\) s.t. \((T x, T y)_2 = (x, y)_1\)

• **Separable**: iff it has a countable orthogonal basis \(S\). Let \(N = card(S)\). If \(N < \infty\), space is isomorphic to \(\mathbb{C}^n\). If \(N = \infty\), space is isomorphic to \(l^2(\mathbb{N})\).

• **Best approximation of Hilbert spaces**: 
  - Finite-dim subspace \(M\): \(x_0\) is the best approximation \(\|x - x_0\| = \text{inf}_{y \in M} \|x - y\|\)
  - Closed convex subset \(C\): \(x_C\) is the unique best approx \(\|x - x_C\| = \text{inf} \|x - y\|\)
  - \(x - x_C \perp C\) or \((x - x_C, y - y_C) = 0\)

• **Miscellaneous**: 
  - Inner product is a continous map
  - If \(M\) is a closed subspace of a Hilbert space \(H\), \(H = M \otimes M^\perp\)
  - If \(x \perp y\), then \(\|x + y\|^2 = \|x\|^2 + \|y\|^2\) (Pythagorean theorem)
  - Every Hilbert space has an orthonormal basis which may be finite, countable, or uncountable
    - A series of non-negative terms is convergent iff it is bounded from above.
    - For \(x_\alpha\) in an orthogonal subset of \(H\), the sum \(\sum x_\alpha\) converges iff \(\sum \|x_\alpha\|^2 < \infty\) and in that case: \(\|\sum x_\alpha\|^2 = \sum \|x_\alpha\|^2\)
    - An orthonormal basis of a Hilbert space is complete
    - An orthonormal set \(U\) is a basis if \(U^\perp = 0\)
    - If \(T\) is a bounded then \(\|T\|^2 = \|TT^*\| = \|T^*T\| = \|T^*\|^2\).

• **Vocab**: standard inner product, Hilbert-Schmidt norm, orthogonal direct sum, quasiperiodic (a function that is a sum of finitely many periodic functions), isomorphism (bijective linear map)

**Fourier Series.**

• **MOTIVATION**: Allows to express a problem in terms of a suitable orthonormal basis

• **Structure**: \((e_n) = \frac{1}{\sqrt{2\pi}} e^{inx}\) is an orthonormal basis of \(L^2(\mathbb{T})\)

• **Approximate identity**: \(\{\Phi_n \in C(\mathbb{T}) | n \in \mathbb{N}\}\) is a family of approximate functions if:
  1. \(\Phi_n(x) \geq 0\)
  2. \(\int \Phi_n(x) dx = 1\)
  3. \(\lim_{n \to \infty} \int \Phi_n(x) dx = 0\) for \(\delta \leq |x| \leq \pi\)

• **Convolution**: \((f \ast g)(x) = \int_T f(x-y)g(y)dy\)

• **Examples of approximate identity**:
  1. \(g_n = e^{-|g(x/e)|}??\)
  2. \(\phi_n = \alpha_n(1 + \cos x^n)\)

• **Fourier coefficients**: \(\hat{f}_n(\xi) = (e_n, f) = \frac{1}{(2\pi)^{1/2}} \int T f(x) e^{-inx} dx\)

• **Fourier expansion**: \(f(x) = \frac{1}{(2\pi)^{1/2}} \int \hat{f}_n(\xi) e^{inx} d\xi\)

• **Parseval’s identity**: \(\|f\|^2 = \|\hat{f}_n\|^2\)

• **Generalized Parseval’s identity**: \(\langle f, \hat{g}\rangle = \sum \hat{f}_n \hat{g}_n\)
• If \( f \) is even **Fourier cosine expansion**: \( f(x) = \frac{1}{2}a_0 + \sum a_n \cos nx \)
• If \( f \) is odd **Fourier sine expansion**: \( f(x) = \sum b_n \sin nx \)
• **Convolution Theorem**: \( (f \ast g)_n = \sqrt{2\pi} \hat{f}_n \hat{g}_n \)
• **Young’s inequality**: For \( \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r} \) then \( \|f \ast g\|_r \leq \|f\|_p \|g\|_q \)
• **Special Young’s inequality** \( \|f \ast g\|_\infty \leq \|f\|_2 \|g\|_2 \)
• **Weak \( L^2 \)-derivative**: \( f'(x) = \frac{1}{\sqrt{2\pi}} \sum (in) \hat{f}_n e^{inx} \)
  - Alternatively, \( \int f' \phi \, dx = -\int f \phi' \, dx \) for any \( \phi \in C^1(\mathbb{T}) \)
• **Compact support**: A function has compact support if the space where the function is nonzero is compact
• \( H^1(\mathbb{T}) = \{ f(x) = \frac{1}{\sqrt{2\pi}} \sum \hat{f}_n e^{inx} \text{ s.t. } f \in L^2(\mathbb{T}) \text{ and } \sum n^2 |\hat{f}_n|^2 < \infty \} \)
  - Alternatively: \( f \in H^1 \) if \( \int |f \phi' dx| \leq M ||\phi||_2 \)
• **Special Sobolev Embedding**: If \( f \in H^k(\mathbb{T}) \) for \( k > 1/2 \), let \( S_N(x) \) be the partial sum of the Fourier series of \( f \). Then, \( ||S_N - f||_\infty \leq \frac{C_k}{N^{k-1/2}} ||f^{(k)}||_2 \) and \( S_N \) converges uniformly to \( f \).
• **Sobolev Embedding**: If \( f \in H^k(\mathbb{T}) \) for \( k > 1/2 \), then \( f \in C(\mathbb{T}) \).
• **Generalized Sobolev Embedding**: If \( f \in H^k(\mathbb{T}) \) for \( k > m + 1/2 \), then \( f \in C^{(m)}(\mathbb{T}) \).
• **Many-dimensions Sobolev Embedding**: If \( f \in H^k(\mathbb{T}^d) \) for \( k > m + d/2 \), then \( f \in C^{(m)}(\mathbb{T}^d) \).
• **Dirichlet Problem**: \( f(x) = a_0/2 + \sum (a_n \cos nx + b_n \sin nx) \) and,

\[
\begin{align*}
\Delta u &= 0, \\
|u| &= f, \text{ on } \partial D
\end{align*}
\]

then, \( u(r, \theta) = a_0/2 + \sum r^n(a_n \cos n\theta + b_n \sin n\theta) = \frac{1}{2}\pi P_r \ast f \).
• **Poisson kernel**: \( P_r(\theta) = \sum r^n |e^{i\theta}| = \frac{1-r^2}{1-2r\cos \theta + r^2} \)
• **Poisson formula**:

\[
u(z) = \begin{cases} 
1/2\pi \text{Re} \left( \int_0^{2\pi} e^{is} f(e^{is})ds \right) & \text{when } z \in D \\
|f(z)| & \text{when } z \in \partial D
\end{cases}
\]
• **Mean Value Property**: Let \( U \) be a harmonic function on a disk \( D \) centered at zero. Then \( u(0) \) equals the average of \( u \) on its circumference

\[
u(0) = \frac{1}{2\pi a} \int_{|y| = a} u(y)ds
\]
• **Maximum Value Principle**: Let \( D \) be a connected, bounded open set and \( u \) a harmonic function, continuous everywhere. Then the max and min values of \( u \) are obtained on \( \partial D \) and nowhere inside (unless \( u = \text{constant} \))
• **Isoperimetric Inequality**: For any \( \Gamma, 4\pi A \leq l^2 \) with equality iff \( \Gamma \) is a circle.
• **Solving PDEs**: We can use Fourier series to solve various constant coefficient, linear PDEs.
• **Miscellaneous**:
  - To prove orthonormal basis: orthonormal + span(\( e_n \)) is dense in the space
  - Any continuous function on \( \mathbb{T} \) can be approximated by trigonometric polynomials
  + Since uniform convergence on \( \mathbb{T} \) implies \( L^2 \) convergence, and continuous functions are dense in \( L^2(\mathbb{T}) \), it follows that the trig polynomials are dense in \( L^2(\mathbb{T}) \).
Need to obtain a trig polynomial approx of a continuous function $f$ by taking
the convolution of $f$ with an approximate identity that is a trig polynomial.
- For large $n$, convolution of $f$ with approx identity, yields local average of $f$
- Periodic Fourier transform is a Hilbert space isomorphism
- To prove dual space: 1. $\exists$ isomorphism (1-1, onto, norm preserving) OR 2. double inclusion
  - The smoother a function is (the more differentiable it is), the quicker its Fourier coefficients decay
  - $(f^{(k)})_n = (in)^k \hat{f}_n$
  - About weak $L^2$ derivative: derivative of a function whose derivative is square integrable but not necessarily continuous. Satisfies integration by parts.
- Harmonic function: $\Delta f = 0$

**Bounded Linear Operators on a Hilbert Space.**

- **MOTIVATION:** Characterize the bounded linear functional on a H.S. using Riesz’s representation
- **Important examples:** projections, unitary, self-adjoint, or compact operators
- **Direct sum:** $M \oplus N = \{ x \in X : x \text{ can be written uniquely as } x = y + z \text{ with } y \in M, z \in N \}$
- **Projection:** Linear map $P : X \to X$ s.t. $P^2 = P$.
  - If $X = M \oplus N$, then $\text{ran}P = M$ and $\text{ker}P = N$
- **Orthogonal direct sum:** If $M$ be a closed subspace of a Hilbert space, then $H = M \oplus M^\perp$.
- **Orthogonal projection:** If $P$ is a projection and $P$ is self-adjoint: $(Px, y) = (x, Py)$.
  - $||P|| = 1$
- **Linear functional:** A linear map from $H$ to $\mathbb{C}$.
  - Bounded (continuous): $||\phi(x)|| \leq M||x||$
  - Norm: $||\phi|| = \sup_{||x||=1} ||\phi(x)||$
  - Bounded functional norm: For bounded $\phi_y = (y, x)$, then $||\phi_y|| = ||y||$.
- **Dual space:** $X^* = \{ \phi : X \to \mathbb{C} \text{ s.t. } \phi \text{ is linear and bounded} \}$
  - For Hilbert spaces: $\phi_y(x) = \langle y, x \rangle$ for all $x \in H$.
  - For Banach spaces: $\phi(x) = \phi(\sum_i x_i e_i) = \sum_i \phi_i x_i$ for all $x \in X$ where $\phi_i = \phi(e_i)$.
- **Riesz’s representation:** If $\phi$ is a bounded linear functional on a Hilbert space $H$, then there is a unique vector $y \in H$ s.t. $\phi(x) = \langle y, x \rangle$ for all $x \in H$.
- **Adjoint operator:** $A^*$ is a bounded linear map on a Hilbert space s.t. $(x, Ay) = (A^*x, y)$ for any $x, y \in H$.
- **Self-adjoint operator:** $(x, Ay) = (Ax, y)$ for any $x, y \in H$.
  - Alternatively, $||A^2|| = ||A||^2$
  - In $\mathbb{R}^n$, self-adjoint = symmetric $\iff A^\top = A$
  - In $\mathbb{C}^n$, self-adjoint = Hermitian $\iff \overline{A^\top} = A$
- **Fredholm operator:** If $\text{ran}A$ is closed and $\text{ker}A$ and $\text{ker}A^*$ are finite dimensional.
- **Unitary map:** $U : H_1 \to H_2$ is unitary if:
  1 $U$ is linear
  2 $U$ is bijective
  3 $(Ux, Uy)_{H_2} = (x, y)_{H_1}$ for all $x, y \in H_1$ (i.e. preserves norm)
- Alternatively, show $U^*U = UU^* = I$

- **Isomorphic:** Two Hilbert spaces are isomorphic if there is a unitary map.

- **Normal operator:** $U^*U = UU^*$

- **Weak convergence in Hilbert space:**
  - Weak conv: $\lim_{n \to \infty} (x_n, y) = (x, y)$ for all $y \in H$.
  - Strong conv: $||x_n - x|| \to 0$ as $n \to \infty$
    + Strong $\Rightarrow$ weak
    + Weak $\not\Rightarrow$ strong since $e_n \rightharpoonup 0$, but not strongly (PF by Bessel's inequality)
    + Weak $\Rightarrow ||x|| \leq \lim\inf ||x_n||$
    + Strong $\iff$ weak + convergence in norm

- **Uniform Boundedness Principle:** Spse $(\phi_n)$ is a set of linear bounded functionals on a Banach space $X$. If $\sup |\phi_n(x)| < \infty$ for any $x \in X$, then $||\phi_n||$ is bounded (i.e. $\sup ||\phi_n|| \leq M$).

- Let $(x_n) \rightharpoonup x$ iff:
  - $(x_n)$ is bounded
  - $(x_n, y) \to (x, y)$ in $D$, dense subset of $H$ OR $(x_n, e_\alpha) \to (x, e_\alpha)$ where $e_\alpha$ is an orthonormal basis of $H$

- **Important Examples of weakly $\not\Rightarrow$ strongly convergent & $||x|| < \lim ||x_n||$**
  - **Oscillations:** $f_n(x) = \sqrt{2} \sin n\pi x$ on $L^2([0, 1])$
    + $||f_n - 0||_2 = ||f_n||_2 = 1$ strongly but weakly goes to 0
  - **Concentrations:** $f_n(x) = \sqrt{n} \chi_{[0,1/n]}$ on $L^2([0, 1])$
    + PF: Just like before, first using polynomials then density argument
  - **Escape to infinity:** $f_n(x) = \chi_{[n,n+1]}$ on $L^2([0, 1])$
    + PF: Approximate $g$ in $<f_n, g>$ by $C^\infty$ by density. Choose $N$ large enough so $\text{supp}(f_n) \cap \text{supp}(g) = 0$

- Let $(x_n)$ be a **bounded** sequence. Then:
  1. In a finite dimensional H.S. $H = \mathbb{C}^n$ so by Bolzano-Weierestrass there exists a strongly convergent subsequence
  2. In an infinite dimensional H.S. you may not have a strongly convergent subsequence (e.g. orthonormal basis $e_n$) but you always have a weakly convergent subsequence

- **Weakly precompact:** If any sequence has a weakly convergent subsequence.
  - weakly precompact iff bounded

- **Weakly compact:** weakly precompact (OR bounded) + weakly closed (OR closed)

- **Weak Bolzano-Weierestrass:** Every bounded sequence in a Hilbert space has a weakly convergent subsequence.
  - PF: Start with a bounded sequence, expand it in terms of orthonormal basis (only proved for separable Hilbert space) where coefficient live in $\mathbb{C}$. Using the Fourier coefficients, by B-W bounded sequences have convergent subsequences. Construct appropriate subsequences & use diagonal argument to form $(x_{k,k})$. Show $(x_{k,k})$ is bounded and weakly convergent to some $x$ constructed appropriately.

- **Banach-Alaoglu:** The closed unit ball in a Hilbert space is weakly precompact.

- **Minimization Problems:**
- If \( K \) is a weakly precompact subset of a Hilbert space and \( f \) is weakly lower semicontinuous (i.e. \( \liminf f(x_n) \geq f(x) \) whenever \( x_n \to x \)), then \( f \) has a minimizer.
- If \( K \) is convex precompact subset of a Hilbert space and \( f \) is strongly lower semicontinuous and convex, then \( f \) has a minimizer. If furthermore, \( f \) is strictly convex, then the minimizer is unique.

**Miscellaneous:**
- If \( X = X^{**} \), then \( X \) is reflexive
- If \( X \) is a Hilbert space, \( X = X^* \)
- If \( \text{ran} A \) is closed (i.e. \( c||x|| \leq ||Ax|| \)) and \( \ker A^* = \{0\} \) (i.e. \( y \) is orthogonal to \( \ker A^* \)), then the solution of \( Ax = y \) exists for all \( y \in H \).
- \( \ker A^* = \{0\} \) iff the solution of the adjoint problem \( A^* x = y \) is unique
- For a bounded self-adjoint operator, \( ||A|| = \sup_{||x||=1} |(x,Ax)| \)
- \( L^2(\mathbb{T}) \) is isomorphic to \( l^2(\mathbb{Z}) \)
- Any \( n \)-dimensional complex Hilbert space is isomorphic to \( \mathbb{C}^n \)
- If \( A \) is a bounded self-adjoint operator, then \( e^{iA} = \sum \frac{1}{n!}(iA)^n \) is unitary

**Spectrum of Bounded Linear Operators.**

**MOTIVATION:** Spectral theory provides a way to attack the problem of diagonalizing (finding complete set of eigenvectors) a linear map on an infinite-dimensional space.

**Diagonalization of \( n \times n \) matrices:**
- The matrix \( A \) is **diagonalizable** if you can do this: \( Au_k = \lambda_k u_k \) for \( k = 1, ..., n \)
- **Spectrum:** set of eigenvalues of \( A \)
- **Eigenvalues** are given by roots of the characteristic polynomial: \( p_A(\lambda) = det(A - \lambda I) \)
- **Singular** \( \iff \) zero \( \det \iff \ker(A - \lambda I) \neq 0 \iff \) not invertible \( \iff \) eigenvalues.

**Resolvent set:** For a bounded operator on a Banach space: \( \rho(A) = \{ \lambda \in \mathbb{C} \text{ s.t. } (A - \lambda I) \text{ is invertible} \} \)

**Spectrum set:** \( \sigma(A) = \mathbb{C} / \rho(A) \)

**Point spectrum:** \( = \{ \lambda \in \mathbb{C} \text{ s.t. } A - \lambda I \text{ is not 1-1} \} = \{ \text{eigenvalues of } A \} \)

**Continuous spectrum:** \( = \{ \lambda \in \mathbb{C} \text{ s.t. } A - \lambda I \text{ is 1-1 but not onto with } \text{ran}(A - \lambda I) = X \} \)

**Residual spectrum:** \( = \{ \lambda \in \mathbb{C} \text{ s.t. } A - \lambda I \text{ is 1-1 but not onto with } \text{ran}(A - \lambda I) \neq X \} \)

**Resolvent:** \( R_\lambda = (\lambda I - A)^{-1} \)

**Spectral radius:** \( r(A) = \sup \{|\lambda|\}, r(A) \leq ||A|| \)

**Invariant subspace:** A subspace \( M \) of a Banach space is called invariant if \( Ax \subseteq M \) where \( A \) is a bounded linear op. (e.g. \( \{0\}, X \))

**Compact operator:** \( T(B) \) is precompact in \( Y \) when \( B \) is a bounded subset of \( X \)
- For every bounded sequence \( (x_n) \), \( T(x_n) \) has a convergent subsequence in \( Y \)
- Every compact operator is bounded
- If \( T \) has finite dimensional range, \( T \) is compact
- Maps every weakly convergent sequence to a strongly convergent sequence

**Spectrum results:**
- A bdd, self-adjoint: \( \sigma_r(A) = \emptyset \), \( \sigma_c(A) \text{ is real} \), \( \sigma(A) \subseteq [-||A||, ||A||] \)
- A infinite dimensional compact then three possibilities:
1. \( \sigma(A) = \{0\} \)
2. \( \sigma(A) = \{0, \lambda_1, \ldots, \lambda_n\} \)
3. \( \sigma(A) = \{\lambda_1, \lambda_2, \ldots, 0\}, n \to 0 \)

- \( A \) is compact and self-adjoint:
  - At least one \( \pm||A|| \) is an eigenvalue
  - \( A \) has at most countable nonzero eigenvalues of real numbers (with 0 the only possible accumulation point)
  - There is a complete orthonormal set of eigenvectors \( Ax = \sum \lambda < e_i, x > e_i \)
  - \( A = \sum_{\lambda_k \in \sigma_p/\{0\}} \lambda_k P_k \) where \( P_k \) is an orthogonal projection
  - \( |\lambda_2| = \sup_{||x||=1, x \perp \text{span}(e_1)} |<x, Ax>| \) and so on

- **Fredholm Alternative:** Let \( K \) be a compact operator on H.S. Let \( T = I - K \). Then:
  1. \( \text{dim}(\ker T) = \text{dim}(\ker T^*) < \infty \)
  2. \( \ker T \) is closed
  3. \( \text{ran } T = \text{ran } (\ker T^*) \)
  4. \( \ker T = \{0\} \iff \text{ran } T = H \)

- **Functions of operators:**
  - **Polynomials:** Let \( q(x) = c_0 + c_1 x + \ldots + c_d x^d \) be a polynomial and \( A \) is a compact self-adjoint operator. Then \( q(A) = c_0 I + c_1 A + \ldots + c_d A^d \).
  - **Continuous on \( \sigma(A) \):** By spectral theory: \( A = \sum \lambda_n P_n \).
    1. \( \sigma(A) \) is finite: \( q(A) = \sum q(\lambda_n) P_n \)
    2. \( \sigma(A) \) is infinite: If \( f(0) = 0 \), \( f(A) = \sum f(\lambda_n) P_n \)

- **Spectral Mapping:** If \( A \) is compact, self-adjoint op. on H.S. and \( f \) is a continuous function on \( \sigma(A) \), then \( \sigma(f(A)) = f(\sigma(A)) \)

- **Right-shift operator:**
  - 1-1 \( \Rightarrow \) invertible \( \Rightarrow \) point spectrum is empty
  - Spectrum consists of the unit circle and is purely continuous

- **Left-shift operator:**
  - Point spectrum is the interior of the unit disk
  - Continuous spectrum is the unit circle
  - Residual spectrum is empty

- **Miscellaneous:**
  - \( \sigma(A) \subseteq \{ \lambda \text{ s.t. } |\lambda| \leq ||A|| \} \)
  - If \( \lambda \in \sigma(T) \) then \( \lambda^k \in \sigma(T^k) \)
  - If \( \lambda \in \sigma_r(T) \) then \( \lambda \in \sigma_p(T^*) \)
  - \( r(A) = \lim ||A^n||^{1/n} \)
  - For self-adjoint bounded linear op, \( r(A) = ||A|| \)
  - \( r(A) = 0 \not\Rightarrow A = 0 \)
  - If \( A \) is self-adjoint bounded linear op on H.S.
    - eigenvalues of \( A \) are real.
    - eigenvectors with different eigenvalues are orthogonal
  - If \( T, S \) are bounded and either one compact, then \( S \circ T \) is compact
  - For compact self-adjoint \( A \): \( |\lambda_1| \geq |\lambda_2| \geq \ldots \)
  - \( e^A = \sum A^n/n! \)
  - \( (I - A)^{-1} = \sum A^n \)
Bochner Integrals.

- **MOTIVATION**: Extends definition of Lebesque integrals to values in Banach spaces
- **Simple function**: $S(x) = \sum b_i \chi_{E_i}(x)$ where $b_i \in B$
- **Bochner measurable**: If there exists a sequence of simple functions such that $S_n(x) \to f(x)$ $\mu$ for almost every $x$. (i.e. $\|S_n(x) - f(x)\|_B \to 0$)
- **Bochner integrable**: If there exists a sequence of simple functions such that $\lim \int_X \|f_n(x) - S_n(x)\|_Bd\mu = 0$ and moreover, $\int_X f(x)d\mu = \lim \int_X S_n(x)d\mu$.
- A Bochner measurable function is Bochner integrable $\iff$ $\int_X \|f\|_Bd\mu < \infty$
- **Derivative**: A function is differentiable at $x \in U$ if there exists a bounded linear operator s.t.
  \[ \lim_{h \to 0} \frac{\|f(x+h) - f(x) - Ah\|_Y}{\|h\|_X} = 0 \]
- **Directional derivative**: $\delta f(x; h) = \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}$
- **Gateaux derivative**: For any $h$, it is a map: $Df_G(x): h \to \delta f(x; h)$
  - If $f$ is Frechet differentiable, then: $Df_G(x) = Df(x)$
- **Miscellaneous**:
  - If $T$ is a bounded linear operator on Banach spaces, then $Tf$ is also Bochner integrable and $\int_X Tf(x) = T \int_X f(x)$
  - Dominated Convergence, MVT, FTC, Hahn-Banach
  - If Gateaux derivative is continuous at $x$, then $f$ is differentiable at $x$ and Gateaux derivative is Frechet derivative.

Measure Theory.

- **$\sigma$-algebra**: A collection $A$ of subsets of $X$ s.t.:
  1. $\emptyset \in A$
  2. complements of subsets are still in $A$
  3. countable unions (and intersections) are still in $A$
- **Measurable space**: $(X, A)$ is a set $X$ with a $\sigma$-algebra $A$ on $X$
- **Borel $\sigma$-algebra**: A $\sigma$-algebra generated by open sets in $X$.
  - Contains all closed sets also.
  - Collection of countable unions of closed sets does not form a Borel $\sigma$-algebra
- **Measure**: A map $\mu: A \to [0, \infty]$ on a $\sigma$-algebra s.t.:
  1. $\mu(\emptyset) = 0$
  2. $\mu(\bigcup A_i) = \sigma(\mu(A_i))$ for disjoint $A_i$
- **$\sigma$-finite**: If there is a countable family of measurable subsets of $X$ s.t. each set has finite measure and $X = \bigcup A_i$
- **Examples of measures**: counting measure (= the # of elements) and delta measure supported at $x_0$
- **Lebesque measure**: The measure on the Borel $\sigma$-algebra on $\mathbb{R}^n$ s.t. $\lambda(C) = \text{Vol}(C)$.
  - If for every $\epsilon > 0$, there is a closed set $F$ and open set $G$ with $\lambda(G/F) < \epsilon$.
  - $\lambda(A) = \inf\{\lambda(U) \text{ for } U \text{ open}\} = \sup\{\lambda(K) \text{ for } K \text{ compact}\}$
  - Translationally invariant, rotationally invariant, scaling: $\lambda(tA) = t^n\lambda A$
- **Complete**: If every subset of a set of measure zero is measurable.
- **Essential supremum**: $\text{ess sup} A = \inf\{C: x \leq C \text{ for all } x \in A/N, \mu(N) = 0\}$
- **Measurable function**: A mapping $f: X \to Y$ s.t. $f^{-1}(B) \in \sigma(X)$ for every $B \in \sigma(Y)$
- **Measure preserving**: $\mu(T^{-1}(A)) = \mu(A)$
• Simple functions: $\psi = \sum c_i \chi_{A_i}$
  - For any nonnegative measurable function there exists a monotone increasing sequence of simple functions that converge pointwise to $f$!

• Integration: $\int f d\mu = \sup \{ \int \psi d\mu : \text{simple function } \psi \}$
  - $\int_A f d\mu = \int_X f \chi_A d\mu$

• Convergence Theorems:
  - [Lusin’s]: Let $f : [a, b] \to \mathbb{C}$ be a measurable function. Then for every $\epsilon > 0$ there exists a compact $E \subset [a, b]$ such that $f$ restricted to $E$ is continuous.
  - [Egorov’s]: Let $E \subseteq X$ be a measurable set with finite measure. Let $(f_n)$ be a sequence of measurable functions on $E$ such that each $f_n$ is finite a.e. and converges to a finite limit in $E$. Then for every $\epsilon > 0$, there is a subset $A$ with $\mu(E - A) < \epsilon$ such that $f_n$ converges uniformly on $A$.
  - [Monotone Convergence]: Spse $(f_n)$ is a monotone decreasing sequence of nonnegative measurable functions. Let $f$ be the pointwise limit, then: $\lim_{n \to \infty} \int f_n d\mu = \int \lim f_n d\mu = \int f d\mu$
  - [Fatou’s]: If $(f_n)$ is a sequence of nonnegative measurable functions then: $\int (\lim \inf f_n) d\mu \leq \lim \inf \int f_n d\mu$ (strict inequality for $f_n = n$ on $0 < x < 1/n$)
  - [Lebesgue Dominated Convergence]: Spse $(f_n)$ is a sequence of integrable functions that converge pointwise in $X$. If there is a nonnegative integrable function $g$ s.t. $|f_n(x)| \leq g(x)$, then $f$ is integrable and $\lim_{n \to \infty} \int f_n d\mu = \int \lim f_n d\mu = \int f d\mu$
  - [Fubini’s & Tonelli’s]

• Littlewood’s Principles:
  - Every measurable function is nearly continuous
  - Every pointwise convergent sequence is nearly uniformly convergent
  - Every set is nearly a finite sum of intervals

• Holder spaces: $C^{k,\alpha}(\Omega)$.
  - $C^{0,\alpha}(\Omega) = \{ f : [f]_\alpha = \sup \frac{|f(x) - f(y)|}{d(x, y)^\alpha} < \infty \}$ for $\Omega \subseteq \mathbb{R}^n$
  - $\|f\|_{C^{0,\alpha}} = \|f\|_\infty + \|f\|_\alpha$
  - $\|f\|_{C^{k,\alpha}} = \sum_{i \leq k} \|D^i f\|_\infty + \sum_{i \leq k} [D^i f]_\alpha$

• Miscellaneous:
  - Banach space
  - Smoothness between $C^k$ and $C^{k+1}$
  - $C^{0,\alpha} \hookrightarrow C$
  - (smoother) $C^{0,\alpha_1} \hookrightarrow C^{0,\alpha_2}$ if $\alpha_1 > \alpha_2$

• $L^p$ spaces.

• MOTIVATION: $(C^\infty_c, \|\cdot\|_p)$ is not complete for $p < \infty$. Thus we consider its completion.

• Properties:
  - Banach spaces, only $L^2$ is Hilbert
  - Dual spaces:
    a) Dual of $L^p$ is $L^q$
    b) Dual of $L^1$ is $L^\infty$ if $(X, \mu)$ is $\sigma$-finite
  - $L^p$ is reflexive for $1 < p < \infty$
  - $L^p$ is uniformly convex for $1 < p < \infty$
• Relations between $L^p$ spaces: If $\mu(X) < \infty$, then $L^\infty \subseteq L^q \subseteq L^p \subseteq L^1$.
  - In general, there is no relation of the type $L^q \subset L^p$.
  - If $1 \leq p < q \leq \infty$, then $L^q \subset L^p + L^r$ and $L^p \cup L^r \subset L^q$.

• Convergence in $L^p$:
  - Strong convergence: $\|f_n - f\|_p \to 0$.
  - Weak convergence: For any $g \in L^q$, $\lim \int f_n g d\mu = \int f g d\mu$.
  - Weak-* convergence: When $p = \infty$ and $q = 1$.
  - Weak $\not\Rightarrow$ strong (see examples in Bounded Linear Op. in H.S.)
  - Strong $\not\Rightarrow$ pointwise a.e. ($\Rightarrow c$.ex. $f_n = \{\chi_{[0,1/n]}, \ldots, \chi_{[n-1/n,1]}\}$ and $f = 1$).
  ($\Leftarrow c$.ex. pointy hat functions on domain $\frac{1}{n}$).
  - Weak compactness: If $1 < p < \infty$, and $(f_n)$ is a bounded sequence in $L^p$, then there exists a subsequence $(f_{n_k}) \to f \in L^p$.
  - Weak-* compactness: If $p = \infty$ and $X$ is $\sigma$-finite.
  - $\|\cdot\|_p$ is not continuous wrt weak convergence ($f_n \to f$ but $\|f_n\|_p \not\to \|f\|_p$), but it is lower semicontinuous.

• Relations between different convergence:
  - Pointwise a.e. then strong $\Leftrightarrow \lim \|f_n\|_p = \|f\|_p$ for $p \neq \infty$.
  - $(\|f_n\|_p)$ is bounded and $f_n \to f$ a.e. Then $f_n \to f$ in $L^p(X)$ for $\mu(X) < \infty$.
  - If $f_n \to f$, then $\|f\|_p \leq \lim \inf \|f_n\|_p$.

• Approximations using simple functions, $p \neq \infty$:
  - Simple functions, $\Psi(x) = \sum a_i \chi_{E_i}$, with $\mu(E_i) < \infty$, are dense in $L^p$.
  - $C_c$ functions are dense in $L^p$ wrt $\|\cdot\|_p$ (PF: using Egorov’s show $\|\Psi - f\|_p \to 0$).
  - $C^\infty$ functions are dense in $L^p$ wrt $\|\cdot\|_p$.

• Approximations using standard mollifier:
  - Mollifier: $\eta(x) = ce^{1/|x|^2-1}$ if $|x| < 1$ and 0 otherwise ($\in C^\infty_c$).
  - Standard mollifier: $\eta_\epsilon(x) = 1/\epsilon^n \eta(x/\epsilon)$.
  - $f^\epsilon = \eta_\epsilon * f \in C^\infty_c(\Omega_\epsilon)$ when $f \in L^1$.
  - If $f \in L^1$, then $f^\epsilon \to f$ pointwise a.e.
  - If $f \in L^p_{loc}$, $1 \leq p < \infty$, then $f^\epsilon \to f$ in $L^p_{loc}$.

• Lebesque Differentiation Theorem: If $f \in L^p_{loc}$, then $\lim 1/|B(x,\epsilon)| \int f(y) dy = f(x)$ a.e. The point $x^*$ where the above holds is called the Lebesque point of $f$.

• Important Inequalities for finite measure spaces:
  1. $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ by convexity of $x^p$ with $p \geq 1$.
  2. Jensen’s: If $\phi : \mathbb{R} \to \mathbb{R}$ is convex, then for any $f : X \to \mathbb{R}$ with $f \in L^1$:
    $\phi \left( \frac{1}{\mu(X)} \int_X f d\mu \right) \leq \frac{1}{\mu(X)} \int_X \phi \circ f d\mu$.
  3. Holder’s: $\|f\|_p \|g\|_q \leq \|f\|_r \|g\|_r$.
  4. Cauchy-Schwartz: $\|f\|_2^2 \leq \|f\|_2 \|g\|_2$.
  5. Generalized Holder’s: If $f_i \in L^{p_i}$, then $\|f_1 f_2 \ldots f_n\|_1 \leq \|f_1\|_{p_1} \ldots \|f_n\|_{p_n}$.
  6. Minkowski: $\|f + g\|_p \leq \|f\|_p + \|g\|_p$. 

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7. **Young’s**: If $1/p + 1/q = 1 + 1/r$, $\|f \ast g\|_r \leq \|f\|_p \|g\|_q$

8. **Tchebyshev’s**: For $p \neq \infty$, $\epsilon > 0$, $\mu(x \in X : |f(x)| > \epsilon) \leq \frac{1}{\epsilon^p} \|f\|_p^p$

**Distributions, $D'(\Omega)$**.

- **Test functions**: $C^\infty$ functions with the following notion of convergence:
  $\phi_m \rightarrow \phi$ in $D(\Omega)$ if there exists a fixed compact set $K$ s.t. $\text{supp}(\phi_m - \phi)(x) \subseteq K$ and $D^n \phi_m \rightarrow D^n \phi$ on $K$ uniformly.

- **Distributions**: The dual of the space of test functions.
  - **Structure**: $T_f(\phi) = \int f \phi dx$
  - **Convergence of distributions**: $T_n \rightarrow T \in D(\Omega)$ if $T_n(\phi) \rightarrow T(\phi)$ for any test function $\phi(x)$
  - **Examples**: delta functions $(\delta_{x_0}(\phi) = \phi(x_0))$, $L^1_{\text{loc}}$, differentiation ($d\phi = \phi'(x_0)$)

- **$\alpha^{th}$ derivative of distributions**: $(D^n T)(\phi) = (-1)^\alpha T(D^n \phi)$
- **$\alpha^{th}$ weak derivative of distributions**: $f(D^n \phi)(x) = (-1)^\alpha \int g(x) \phi(x) dx$

- **Miscellaneous**:
  - If $\int f \phi dx = \int g \phi dx$ for any test function, then $f = g$
  - How to show something is a distribution? Linear + continuous (i.e. convergence as a distribution)
  - The weak derivative exists if no jump discontinuity; counterexample:

$$u(x) = \begin{cases} 
  x & 0 \leq x < 1 \\
  2 & 1 \leq x \leq 2 
\end{cases}$$

- Weak derivatives of $L^1_{\text{loc}}$ exist if the distributional derivative is also in $L^1_{\text{loc}}$
- $D^n T$ is still a distribution
- $H' = \delta_0$

**Sobolev Spaces: $W^{k,p}(\Omega)$**.

- **MOTIVATION**: Holder spaces are not often suitable for elementary PDE theory since the analytic estimates of the solutions do not necessarily belong to such spaces. We would rather study spaces containing less smooth functions.

- **Definition**: $W^{k,p}(\Omega) = \{ f \in L^p(\Omega) : D^n f \in L^p(\Omega), |\alpha| \leq k \}$

- **Structure**: $\|f\|_{W^{k,p}} = (\sum_{|\alpha| \leq k} \|D^n f\|_p^p)^{1/p}$

- **Properties**:
  - $W^{k,p}$ are Banach spaces
  - If $p = 2$, then: $H^k = W^{k,2}$ are Hilbert spaces wrt $\|\cdot\|_{W^{k,2}}$
  - $W^{0,p}_0(\Omega)$ is the closure of $C^\infty_c(\Omega)$ in $W^{k,p}(\Omega)$

- **Convergence**: We say $u_m \rightarrow u$ in $W^{k,p}(\Omega)$, if $\lim \|u_m - u\|_{W^{k,p}} = 0$

- **Leibniz’s Formula**: If $g$ is a test function, then $gu \in W^{k,p}(\Omega)$ and:

$$D^n (gu) = \sum_{\beta + \alpha = n} \frac{n!}{\alpha! \beta!} D^\beta g D^{\alpha-\beta} u$$

- **Interior approximation using smooth functions, $p \neq \infty$**: $D^n(\eta \ast f) = \eta \ast D^n f = (D^n f)^\epsilon$

- **Exterior approximation using smooth functions, $p \neq \infty$**: $\text{If } d\Omega \in C^1$, then $C^\infty$ is dense in $W^{k,p}(\Omega)$
• Morrey’s Inequality: Assume $n < p \leq \infty$. Then there exists $C$ depending on $p$ and $n$, s.t. $|u(x) - u(y)| \leq Cr^{1-n/p}||Du||_p$ for any $u \in C^1$.

• Sobolev conjugate: $p^* = \frac{np}{n-p}$

• Poincare Inequality:
  1. If $u \in W_0^{1,p}(\Omega)$, $||u||_q \leq C||Du||_p$ for $1 \leq q \leq p^*$
  2. $||u - (u)_{\Omega}||_p \leq C||Du||_p$

• Rellich Compactness Theorem: Let $\Omega$ be a regular bounded domain in $\mathbb{R}^n$. Suppose that $1 \leq p < n$ and $1 \leq q < p^*$. Then the bounded sets in $W^{1,p}(\Omega)$ are precompact in $L^q(\Omega)$.

• Embedding:
  1. When $n < p \leq \infty$, $W^{1,p}(\Omega) \hookrightarrow C^{0,\gamma}$ for $\gamma = 1 - \frac{n}{p}$
  2. Gagliardo-Nirenberg-Sobolev: $W^{1,p}(\Omega) \hookrightarrow L^{p^*}$
  3. Rellich-Kondrachov: $W^{1,p}(\Omega) \hookrightarrow L^q$ for $1 \leq q < p^*$
  4. $W^{1,p}(\Omega) \hookrightarrow L^p$

• Trace: $T : W^{1,p}(\Omega) \rightarrow L^p(d\Omega)$ is a bounded linear operator

• Extension: $E : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^n)$ for $C^1$ boundary

• Miscellaneous:
  - $L^2$ is the completion of $C(\mathbb{T})$ under $||.||_2$ norm
  - $H^1$ is the completion of $C^1(\mathbb{T})$ under $||.||_{H^1}$ norm
  - For which $\alpha, n, p$ does $u$ belong to $W^{1,p}(\Omega)$? Need $\alpha < (n-p)/n$. In particular, when $p > n$, $u \not\in W^{1,p}(\Omega)$
  - $u \in W_0^{1,p}(\Omega)$ iff $Tu = 0$ on $d\Omega$

Schwartz Space.

• MOTIVATION: The study of smooth functions with rapid decay in both the function and the Fourier transform of the function.

• $S(\mathbb{R}^n) = \{ \phi \in C^\infty(\mathbb{R}^n) : x^\beta D^\alpha \phi \in L^\infty(\mathbb{R}^n) \}$

• Structure: $||\phi||_{\alpha,\beta} = \sup |x^\alpha D^\beta \phi|$

• Convergence: $\phi_k \rightarrow \phi \in S$ if $||\phi_k - \phi||_{\alpha,\beta} \rightarrow 0$

• $|D^\alpha \phi| \leq \frac{C_{\alpha,\beta}}{(|1 + |x|^2|)^\beta}$

• Tempered distribution: the dual of the Schwartz space
  - $|T(\phi)| \leq \sum C_{\alpha,\beta}||\phi||_{\alpha,\beta}$

• Miscellaneous:
  - $C^\infty_c(\mathbb{R}^n) \subseteq S(\mathbb{R}^n) \subseteq C^\infty(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$
  - Example: Gaussian
  - It is a vector space closed under multiplication, differentiation and

Fourier Transform NOT on $L^2$.

• Fourier transform on $S$, $F$: $\hat{\phi}(\xi) = (\frac{1}{\sqrt{2\pi}})^n \int \phi(x)e^{-ix\xi}dx$
  - $F^{-1} = F^*$
  - $F(D^\alpha \phi) = (i\xi)^\alpha F(\phi)$
  - $D^\alpha F(\phi) = F((-i\xi)^\alpha \phi)$
  - F.T. of a Gaussian is still a Gaussian
  - $F^*F(x) = (\frac{1}{\sqrt{2\pi}})^n \int f(\xi)e^{ix\xi}dx$
  - $FF^* = F^*F = I$
- $F$ is an isomorphism with inverse $F^*$ and preserves the $L^2$ inner product (i.e. $\|F f\|_2 = \|f\|_2$)

**Fourier transform on $L^1$:** $\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ix \xi} dx$

**Fourier transform on $S'$, $\hat{T}$:** $\langle \hat{T}, \phi \rangle = \langle T, \phi \rangle$ for all $\phi \in S$

- Inverse Fourier transform: $\langle \hat{T}, \phi \rangle = \langle T, \hat{\phi} \rangle$

**Miscellaneous:**
- Fourier transform is bounded, uniformly continuous, and tends to zero.
- F.T. of delta function: $\hat{\delta} = \frac{1}{\sqrt{2\pi}} \delta$
- F.T. of 1 function: $\hat{1} = \frac{1}{\sqrt{2\pi}} \delta$

**Sobolev Space:** $H^s(\mathbb{R}^n)$.

- $H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n) = \{ f \in S'(\mathbb{R}^n) : \xi^s \hat{f}(\xi) \in L^2(\mathbb{R}^n) \}$ where $\xi^s \hat{f}(\xi) \in L^2(\mathbb{R}^n)$

- Structure: $\|f\| = \|<\xi^s \hat{f}(\xi)\|_2$

- Trace Embedding: Let $s > 1/2$. Then: $T : H^s(\mathbb{R}^n) \to H^{s-1/2} (\mathbb{R}^{n-1})$ with $\|Tu\|_{H^{s-1/2}} \leq c \|u\|_{H^s}$

**Properties:**
- A Hilbert space
- $S(\mathbb{R}^n) \hookrightarrow H^1 \hookrightarrow H^{1/2} \hookrightarrow H^0 \hookrightarrow H^{-1} \hookrightarrow H^{-1000} \hookrightarrow S'(\mathbb{R}^n)$
- $S(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$
- $(H^s)' = H^{-s}$
- $H^s(\mathbb{R}^n) \hookrightarrow C^r(\mathbb{R}^n)$ if $s > r + n/2$

**Tricks**

- **Cantor’s Diagonal Argument**
  - Let $x_n < C_N(L + \epsilon)^n$ where $C_N = \frac{x_0}{(L+\epsilon)^n}$ where $N \in \mathbb{N}$

- **Intermediate Value Theorem** $\int_a^b f'(x)dx = f(b) - f(a)$

- **Fundamental Theorem of Calculus** If $\int_a^b f(x)dx = 0$, then $\exists x_0 \in [a, b]$ so that $(b-a)f(x_0) = 0$.

  Say you are looking for a **min or max in a set of points**, $I(x)$. If $I$ is compact, then there exists a finite cover for this set and therefore finitely many points that satisfy the condition.

- **Proving** $x = \inf A$ or $x = \sup A$:
  1) $\forall \epsilon > 0, \exists x_\epsilon \in A$ s.t. $x \leq x_\epsilon + \epsilon$
  2) Let $x = \inf A$; take a minimizing (maximizing) sequence s.t. $A_n \to x$; show $A_n$ Cauchy and if space is complete $x \in X$.

- **Zorn’s Lemma**: Let $S$ be a nonempty partially ordered set. If every totally ordered subset has an upper bound, then $S$ has a maximal element.

- **Odd Inequalities**:
  - For any $a, b \geq 0$, $a^x b^{1-x} \leq ax + (1-x)b$, $\forall x \in [0, 1]$.
  - PF: $f(x) = (a/b)^x b$ is concave up. By convexity: $f(x) \leq xf(1) + (1-x)f(0)$. 
- $\phi(z) = 0 \iff z = 0$ is proved by Hahn-Banach. Let $t(z)$ be a linear subspace. Define some functional $\lambda t(z) = t|z|$. Extend $\lambda$ to all space by Hahn-Banach. Then $\lambda(z) = ||z|| = 0$.
- **Calculus limits:** $0 \over 0$ indeterminant but $\frac{e^x}{x}$ and $\frac{x}{0}$ DNE
- **Linear Fredholm integral operator:** $Kf(x) = \int_0^1 k(x, y) f(y) dy$
- **Measurable function** Then we define: $\int_X f d\mu = \sup \{ \int_X \phi d\mu : \phi \text{ is simple and } \phi \leq f \}$. In other words, there exists a simple function $\phi$ s.t. $\phi \leq f$ and $\int_X |\phi(x) - f(x)| \leq \epsilon$.

**Bag o’ Examples**

- Cauchy $\nRightarrow$ Convergence (e.g. $\frac{1}{n}$ in $(0, 1)$)
- Adjacently Close $\nRightarrow$ Cauchy (e.g. $\log(n)$)
- Complete set $\nRightarrow$ sequentially compact subset (e.g. $\mathbb{R}$)
  - How does one make a complete set into sequentially compact?
    - complete + bounded $\Rightarrow$ sequentially compact for $\mathbb{R}^n$ only
    - complete + totally bounded $\Rightarrow$ sequentially compact
- Bounded $\nRightarrow$ totally bounded (e.g. discrete metric space is not $\epsilon$-finite)
- Pointwise convergence $\nRightarrow$ uniform convergence (e.g. $f_n = x^n$, $f = 0$ if $x \in [0, 1)$ and 1 if $x = 1$)
- $C^\infty (X) \subset C(X) \subset C_0(X) \subset C^0(X)$

<table>
<thead>
<tr>
<th>$f \in C^\infty (X)$</th>
<th>$f \in C_0(X)$</th>
<th>$f \not\in C^\infty (X)$</th>
<th>$f \not\in C_0(X)$</th>
<th>$f \in C(X)$</th>
<th>$f \not\in C_0(X)$</th>
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<tbody>
<tr>
<td>$f(x) = 1 - x^2$ if $</td>
<td>x</td>
<td>\leq 1$ and $f(x) = 0$ otherwise</td>
<td>$f(x) = e^{-x^2}$</td>
<td>$f(x) = 1$</td>
<td>$f(x) = x^2$</td>
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- Pointwise convergence $\nRightarrow$ convergence of integrals (e.g. triangles with constant area)
- $f$ is continuous bijection $\nRightarrow$ $f^{-1}$ si continuous (e.g. $f : [0, 2\pi] \rightarrow \mathbb{T}$ where $f(\theta) = e^{i\theta}$)
- $K$ a compact subspace of a Hausdorff space $X$ is also closed. If $X$ is not Hausdorff the result does not hold. For e.g.
- $C^k([a, b])$ is an inner product space, but not complete
- Examples of orthonormal basis of Hilbert spaces:
  - Finite-dim $\mathbb{C}^n$: $e_1 = (1, 0, ..., 0)$, $e_2 = (0, 1, 0, ..., 0)$, ..., $e_n = (0, 0, ..., 1)$
  - $l^2(\mathbb{Z})$: $e_n = (\delta_{kn})_{k=-\infty}^{\infty}, n \in \mathbb{Z}$
  - $L^2(\mathbb{Z})$: $e_n = \frac{1}{\sqrt{2\pi}} e^{inx}$ called **Fourier basis**
- Examples of Banach spaces:
  - $C^k(K)$ on a compact metric space $K$ equipped with the sup-norm
  - $C^k([a, b])$ with the $C^k$ norm
- Examples of non-Banach spaces:
  - $(C^k([a, b]), ||,||_\infty)$ because the uniform limit of continuously differentiable functions need not be differentiable
  - $C(K), ||,||_p$ is not complete
• \( f(t) = e^{it} + e^{int} \) is quasiperiodic but not periodic
• \( f_n \) continuous bounded function that converge pointwise to \( f \) not necessarily continuous.
• If \( f : X \to Y \) is a map between topological spaces where \( X \) has the discrete topology, then \( f \) will be continuous.
• The differential operator is an unbounded linear map.
• We know that every finite dim. subspace of a normed linear space is closed. However this may not hold for infinite dimensional subspaces. For example, let \( X = l^\infty \) and let \( a_n = (1, 1/2, ..., 1/n, 0, ...) \) converges to \( a = (1, 1/2, ..., 1/n, 1/(n+1), ...) \) as \( n \to \infty \) but is not in \( A = \cup_i A_i \) where \( A_i = (x_1, x_2, ..., x_i, 0, 0, ...) \). Therefore \( A \) is not closed.

Questions

• Suppose \((X,d)\) is a metric space. For any function \( f : \mathbb{R} \to \mathbb{R} \), we define \( d_f(x,y) = f(d) \). For which conditions of \( f \) will \((X,d_f)\) become a metric space?
  - Prev examples: \( ln(1 + d) \), \( d/(1 + d) \), \( \lambda d \)
  - Answer: onto and 1-1
• (Good problem converting between continuity and open sets) Show the following statements are equivalent:
  1. \( f \) is continuous
  2. if \( U \subseteq Y \) is open, then \( f^{-1}(U) \) is open in \( Y \)
  3. if \( \lim(x_n) = x \), then \( \lim(f(x_n)) = f(x) \)
• Can’t think of a counterexample for: \( B_r(y) \subseteq B_R(y) \implies d(x,y) \leq R - r \)
• (Fair test of e-covers) Suppose \( S \subset X \) is an \( \epsilon \)-net of \( A \). Show there exists a subset \( K \) of \( A \) so that \( K \) is an \( 2\epsilon \)-net of \( A \) while its cardinality is no greater than the cardinality of \( S \).
• Show \( F = C(X) \).
• Problem 2 homework set 5 (201A).
• Problem 3 homework set 5 (201A).
• (BLT’) Let \( X \) and \( Y \) be the same and \( M \) be a linear subspace of \( X \) and \( T \) is a bounded linear operator \( T : M \to Y \). Then there exists a unique bounded linear map \( \bar{T} \) s.t. \( \bar{T}|_M = T \) and \( ||\bar{T}|| = ||T|| \).
• \( T : X \to Y \) is bounded. Then:
  1. \( T \) has closed range and 1-1, iff
  2. bdd below