One-to-Many Matching Auctions in Platforms

Hemant K. Bhargava ∗ Gergely Csapó† Rudolf Müller ‡

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Abstract

Platforms create value by matching participants on alternate sides of the marketplace. While many platforms practice one-to-one matching (e.g., Uber), others can conduct and monetize one-to-many simultaneous matches (e.g., lead marketing platforms). Ideally, the choice between the two modes of matching should be made not ex ante, but rather based on the relative premium that participants perceive for exclusive matches and the nature of heterogeneity in these two sets of valuations. This paper studies the problem of designing an auction format for such platforms, i.e., a set of rules for allocation and pricing of matches. For the sake of practicality, we require deterministic auctions that incentivize truthful bidding, and we formulate the optimal incentive-compatible (IC) auction as a mixed integer mathematical program. Although the optimal IC auction is notoriously hard to solve, its value is that it leads to a heuristic design that is simple to implement, provides good revenue, and has speedy performance, all critical in practice. Specifically, we develop multiple relaxations of the optimal auction to obtain upper bounds on the (unknown) optimal revenue and, conversely, refinements that produce heuristic auctions whose optimal revenue is a lower bound. By demonstrating a tight gap between the two bounds for one such design, RM, we prove that it has excellent revenue performance and places low information and computational burden on the platform and participants.

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1 Introduction

Technology-enabled platform marketplaces have stormed into business in the last couple of decades. Platforms facilitate multiple groups of trading partners (say, shoppers and merchants) to congregate, discover, and transact with each other (Choudary et al., 2016). Because platforms focus on enabling value creation and exchange, rather than value production itself, a vital function is to match consumers and providers (Evans and Schmalensee, 2016). One such platform is the lead marketing firm BuyerLink.com (formerly Reply.com) which matches merchants selling specific products (such as automobiles, real estate, insurance, etc.) with shoppers who have expressed interest in purchasing these products. While many platforms seek to support the entire shopper-merchant transaction (e.g., Uber participates in discovery, matching, fulfillment, payment and post-sales issues), others primarily facilitate matchmaking (e.g., Craigslist, AirBnB, match.com, and eBay). In the middle, matching platforms such as Beebell, BuyerLink.com, CreditKarma, and Google Search operate a mechanism through which they match a shopper and a merchant. This matching role is crucial from a monetization perspective as well, because the platform gets paid for the match, either the connection itself or some metric of success in the interaction between the shopper and merchant.¹

In many platforms such as BeeBell, CreditKarma or Uber, matching is executed as a one-to-one process, where one shopper is matched with one merchant out of several interested ones. For instance, a platform that serves video advertisements against some news or entertainment content can only show one video ad at a time. Many such platforms employ digital online auctions, picking one winning merchant based on winner and price determination algorithms applied to merchants’ bids against the shopper’s attributes. Advertising auctions

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¹The terms shopper and merchant are used as a placeholder for two roles, which could be patient-provider, content consumer-content producer, app developer-smartphone operating system, etc. Specifically, the shopper could be a consumer or a buyer firm, and the merchant could be a firm or an individual.
are a popular example of this approach.\textsuperscript{2} The theoretical and technological infrastructure for such one-to-one matching auctions is relatively well studied (Varian, 2009; Milgrom, 2004).

This paper pushes deeper into the matching algorithms of platforms when multiple merchants can potentially be matched with each shopper. One-to-many matches are relevant to platforms such as BeeBell, BuyerLink and CreditKarma, because they provide merchants the possibility to sell something to a potential shopper (rather than make the sale, as in iTunes). A shopper may intrinsically want to connect with multiple merchants in order to find the best price-quality match for herself. This is common in industries served by BuyerLink, such as home loans and automobile sales. Conversely, some platforms pick one “best” merchant, but only imperfectly. For instance, CreditKarma matches a shopper interested in financial products to a single merchant by attempting to predict the likelihood that a shopper would be approved for a loan, credit card or other product, by each merchant. This imperfection could be addressed by making 1:n matches, with n generally a single-digit integer (e.g., BuyerLink’s policy is to match a shopper with up to 3 merchants). Such low-n matching is also increasingly relevant today in location-based advertising on small mobile devices, exemplified by Beebell which connects an event visitor with a handful of restaurants and other merchants in proximity to the event.

When a platform has committed to making either always-shared or always-exclusive allocations, the auction design problem to find the revenue maximizing auction is well-understood, and involves setting a particular reserve price that depends on the distribution of merchants’ willingness to pay for being matched exclusively or shared. While many platforms practice these simple approaches, the diversity and large scale of their matching opportunities suggests that such platforms might be better off mixing shared and exclusive

\textsuperscript{2}Superficially, advertising systems such as Google Adwords match a shopper to multiple advertisers by filling multiple ad slots at once. However, these slots are ranked, hence in this case multiple vertically ordered products are sold to multiple buyers, and each product only to one buyer. Moreover, only one ad is clicked at a time, i.e., the search engine connects and monetizes a web user to one single advertiser at a time.
allocations. For example, **BuyerLink** implements this hybrid allocation strategy. But, then, how exactly should the platform choose the mode of matching and the corresponding allocations and prices? One simple possibility is to act opportunistically and choose based on observed bids, combined with pay your bid pricing, as done by **BuyerLink**. However such opportunistic rules might cause merchants to bid differently. Example 1 illustrates the challenge in specifying a set of rules that allows for both kinds of outcomes.

**Example 1.** Consider a shopper in zip code 60173, who has expressed interest in a BMW mini. The platform has bids from 5 dealers in this area who are interested in such shoppers, and can connect the shopper simultaneously with up to 3 dealers. Table 1 lists two alternate scenarios of reservation values of the 5 dealers for shared and exclusive purchase. If dealers actually bid these values, and the platform employed pay-your-bid pricing, then it would make a shared allocation (to dealers D3, D4 and D1) in Scenario 1, and exclusive allocation (to D1) in Scenario 2. However, with knowledge of these rules, the winning bidder D3 in Scenario 1 would keep winning even if she would have bid 10 instead of 14. If the platform employs one-to-many matching only, it misses revenue in Scenario 2 as it cannot incorporate the high exclusivity margin of merchant D1. If it employs one-to-one matching only, it misses revenue in Scenario 1.

<table>
<thead>
<tr>
<th>Valuation</th>
<th>D1</th>
<th>D2</th>
<th>D3</th>
<th>D4</th>
<th>D5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shared purchase ($s_i$)</td>
<td>10</td>
<td>8</td>
<td>14</td>
<td>11</td>
<td>9</td>
</tr>
<tr>
<td>Exclusive purchase</td>
<td>25</td>
<td>20</td>
<td>20</td>
<td>15</td>
<td>9</td>
</tr>
<tr>
<td>Exclusivity margin ($m_i$)</td>
<td>15</td>
<td>12</td>
<td>6</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>

As Example 1 illustrates, one-to-many platforms may, *ex interim*, desire both exclusive and shared matches. However, opportunistic revenue maximization with pay-your-bid pricing over the bids can lead to a messy auction process with bid shading behavior that may be opaque to the platform. Alternately, the platform can specify the rules in advance with respect to an *ex ante* distribution of merchant valuations for shared and exclusive matches,

<table>
<thead>
<tr>
<th>Scenario 1</th>
<th>D1</th>
<th>D2</th>
<th>D3</th>
<th>D4</th>
<th>D5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario 2</td>
<td></td>
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without knowing the draws of buyer valuations, but this way revenue might be missed for particular bid configurations.

While these two pure forms, always one-to-one or always one-to-many, carry a revenue disadvantage, the structure makes it possible to set optimal reserve prices, as will be explained later. This allows to specify a parameterized design where each auction is configured with respect to known category-specific information. For example, a platform which trades many similar “products” (e.g., millions in the automobile lead marketing case, covering shoppers interested in one of over 300 models across one of about 40,000 zip codes, and with different demand profiles for shared vs. exclusive allocations) would announce the same general format across all, but customize the auction rules (e.g., setting a reserve price) based on priors for how car sellers value customer leads for a particular class of models within a particular range of neighborhood zip codes. Taking this category information, an ex ante choice for shared and exclusive matches can be made, combined with the optimal reserve price for the distribution of valuations in the category. Still, this approach can miss revenue opportunities due to bid shading and lack of ex interim flexibility.

To avoid bid shading and seeking an auction design that uses observed bids in choosing the allocation format while imposing minimal computational and informational burden on participants, our design objective is to develop a deterministic auction which encourages truthful bidding. As we discuss in §2, finding the revenue maximizing design within the space of all such auctions is notoriously hard to solve. A key contribution of our paper is to develop a practical and high-revenue design that intelligently switches ex interim between shared and exclusive matches. We do so by extending the optimal reserve price approach, which underpins one-to-one matching, to one-to-many matching platforms. Our design is based on maximizing, for each category, the expected revenue computed over a distribution of shared values and exclusivity margins which is ex ante known to the platform. Our formula produces reserve prices that are configured based on the prior information, and it produces
allocation rules (who wins), and price rules (what they pay) for each category, where the type of matching (one vs. multiple winners) depends on the bids. An additional “soft constraint” could be to obtain auction rules which do not vary significantly across categories. The solutions we provide turn out to satisfy this constraint without adding the constraint as a hard constraint to the design problem. Furthermore, the auction rules are equivalent to a posted-price mechanism whenever only a single buyer happens to be interested in the item.

The practical contributions that this paper makes with regard to matching platforms are enabled by our theoretical contributions in modeling and solving auction problems with multi-dimensional private information. We note that the essential complexity in our setting is that a merchant has a base valuation for being served (called shared value) and a premium for being served exclusively (so, the total value for being served exclusively is the shared valuation plus an exclusivity margin). Unlike single-dimensional auctions, there is no general apparatus for solving optimal auction problems with two-dimensional private information. We provide a heuristic solution to this challenging problem that satisfies our posted goals of practical simplicity. We also provide a theoretical and empirical comparison of the heuristic against simplistic designs, for example the optimal auction for allocating always exclusively, as well as against the revenue of the unknown, optimal mechanism.

Our key theoretical contribution is an innovative three-step approach for addressing the two-dimensional auction design problem. First, we formulate the search for an optimal mechanism in the space of mechanisms as a mixed integer optimization program. Second (because of the challenge in identifying the optimal solution within this space), we develop approaches that generate specialized, parameterized mechanisms whose parameters can be efficiently computed, and the mechanism can be easily implemented. Third, we develop techniques for estimating tight bounds for the original problem and use these bounds to demonstrate the revenue performance of the proposed specialized mechanisms. Conversely, we also demonstrate that our proposed design (RM) outperforms alternative and naïve mech-
anisms (only exclusive, only shared, or VCG based). Furthermore, one notable aspect of our work is to show that the simplest auction designs in this setting, namely selling always exclusively or always non-exclusively, produce revenue within a constant factor approximation of the optimal revenue. Finding simple mechanisms that provide constant approximations constitute an important line of research in mechanism design as it can explain salient features of optimal auctions and shed light on the practical success of simple mechanisms in complicated environments (see, e.g., Hartline, 2012). Finally, we show that the bounds that we develop are tighter than bounds for this class of auctions based on standard methods in literature. Our results require mild conditions, that distributions of shared valuations and exclusive margins are independent, and have monotone hazard rate.

In the next section relevant works from the literature are discussed. Then we introduce the main concepts and notation for our model in §3, where we specify a mathematical program to represent the optimal dominant strategy incentive compatible auction (DSA). In section 4 we set the foundation for deriving an approximately optimal auction by constructing restricted variants of the original problem; optimal solutions for these designs constitute lower bounds on revenue of the optimal DSA. These bounds are used in section 5 to prove that simple mechanisms can approximate the optimal solution. We conclude that section by constructing a heuristic from the affine maximizer family of mechanisms. Finally, section 6 presents a simulation study on the expected revenue of the discussed mechanisms for various settings.

2 Related Literature

The problem of designing the revenue maximizing auction when private information is given by a single parameter has been solved in the seminal (Myerson, 1981) paper. His approach provides a closed form characterization of the optimal mechanism under mild assumptions for
most of the cases. In contrast, for settings with multi-dimensional types, the optimal mechanism design problem quickly becomes intractable and there hasn’t been a general framework so far to treat these problems. Some researchers use linear programs to compute approximations (e.g., Cai et al., 2011), but the solutions are non-practical as they must be described by an explicit table of inputs (bids) and outputs (allocations and prices) that is exponential in the number of bidders and items. Thus, even if one finds the optimal mechanism, it is generally too complicated to implement in practice. The optimal mechanism is usually tailored in a complex manner to specific details of the distribution on agent preferences.

Some researchers choose to fold the two-dimensional problem into a single-dimensional model (e.g., Deng and Pekeč, 2013), by assuming that the exclusivity margin is either public information or a fixed relative mark-up on, or a constant multiple of their value for shared allocation. Using this assumption the techniques of (Myerson, 1981) can be applied. However, this approach is insufficient for matching platforms because it eliminates the case where the ability to dynamically choose between shared and exclusive allocations (vs. predetermined exclusive-only or shared-only) is most relevant (i.e., when exclusivity margins are heterogeneous, distributed differently across problems instances, and not correlated with shared valuations). In such settings, a single-dimensional model fails to capture information relevant to making efficient matches and is doomed to miss out on revenue opportunities.

These undesirable properties are reflected in the works of (Rochet and Choné, 1998), (Armstrong, 1996) and (Manelli and Vincent, 2007). For more critiques and thoughts on optimal mechanism design see (Hartline, 2012).

There are different attempts to circumvent the inherent hardness in multi-dimensional mechanism design. One approach is to assume that, apart from a single parameter of private information, the rest of the parameters are public information. This captures relevant information while maintaining the computational advantage of a single-dimensional model. This path was chosen in (Deng and Pekeč, 2013) with a focus on exclusivity contracts. In that
paper an agent is allocated exclusively if nobody else from his direct neighbourhood receives it. The neighbourhood is defined on a publicly known network. Our setting covers the case when this neighbourhood graph is a clique. To get their main result they restrict themselves to clique neighbourhood graphs and single-dimensional valuations, where the value for the shared allocation is private information, while the exclusivity margin is public. The rest of the assumptions are similar to ours: private values are distributed according to a monotone hazard rate distribution and they search for deterministic mechanisms. Due to the single-dimensional assumption they can apply the techniques of Myerson and derive the optimal mechanism, which is the one that maximizes the sum of virtual valuations for each type report. In contrast, our model keeps multi-dimensionality where the Myerson framework doesn’t have a bite.

Along this line of research, (Salek and Kempe, 2008) and (Pei et al., 2014) study the problem of selling a digital good with unlimited supply of copies to bidders whose value for the good is decreasing in the number of bidders obtaining it. The function according to which the valuation depreciates is public information, therefore their setting is single-dimensional. (Salek and Kempe, 2008) provides the revenue maximizing Bayes-Nash implementable mechanism based on the Myerson techniques, while (Pei et al., 2014) adapts the prior-free “single-sample” mechanism from (Dhangwatnotai et al., 2010) that yields a constant approximation for that setting. Their theorems hinge on the assumption that types are single-dimensional and are independently distributed according to monotone hazard rate distributions. Moreover, besides the fact that the “single-sample” mechanism is not deterministic, it does not even extend directly to our setting as applying reserve prices combined with the Vickrey-Clark-Groves mechanism (VCG, see (Vickrey, 1961), (Clarke, 1971) and (Groves, 1973)) is not incentive compatible in general, demonstrated by Example 2 on page 25.

Jerath and Sayedi (2012) study an extension of GSP for sponsored search advertising
patented by Yahoo!. This auction takes two bids as input: one for being displayed among other ads and a second bid for exclusive display. As truth-telling is not dominant strategy in that auction the authors aim to identify and analyse bidding strategies that lead to a Bayes-Nash equilibrium. To render their analysis tractable they restrict themselves to a highly stylised setting including only three agents: two having no exclusivity margin and one whose exclusivity margin is a fixed portion of the shared valuation. They find that allowing advertisers to bid for exclusivity usually increases the search engine’s revenue (because of an increased competition effect) but that revenue may fall under certain conditions. Note that for such a single-dimensional setting the optimal auction format is characterised by (Deng and Pekeć, 2013).

Cai et al. (2011) and Cai et al. (2013) weaken the requirement on implementability by looking for multi-dimensional mechanisms that are Bayes-Nash incentive compatible. They also admit randomized mechanisms, which allows them to use linear programming to acquire a polynomial-time approximation scheme. Their method is not applicable for our problem as it doesn’t handle allocational externalities. Moreover, if we go for deterministic and dominant strategy incentive compatible mechanisms, then we have to face an integer linear program with exponential number of variables and constraints, which renders this approach impractical. Nonetheless, our constant approximation result holds even for the optimal Bayes-Nash incentive compatible mechanism.

Another line of research focuses on heuristic mechanisms, which possess a succinct description and might exhibit a guaranty on its expected revenue. In relation to our problem the work of Devanur et al. (2011) and Dhangwatnotai et al. (2010) bear relevance. They deal with multi-parameter mechanism design involving unit-demand bidders, regular type distributions and matroid or downward closed feasibility constraints. They derive simple mechanisms that achieve constant approximations of the optimal revenue. The key point of their proof is to employ the solution from a single-dimensional analogue as an upper bound
for the optimal revenue. Then the revenue of their simple mechanism is compared to that upper bound. In spite of the similarities, these results cannot be utilized directly for our problem as their setting cannot accommodate allocational externalities. Even if one treats the different allocations as items, the corresponding feasibility constraints are not downward closed, which is a necessary assumption in their model. The way we derive the second upper bound is reminiscent to the technique developed in Devanur et al. (2011) in the sense that we also split the multi-dimensional agents to single-dimensional representatives. The twist in our method is that in order to handle allocational externalities we endow the representatives with interdependent valuations.

Allocational externalities are similar in nature to interdependent valuations and identity dependent externalities in the sense that they all try to catch the impact of the agents on each other. The implications of externalities have been studied in various settings, see, e.g., Jehiel et al. (1996) and Segal (1999) and Figueroa and Skreta (2011). In Aseff and Chade (2008) the optimal auction for selling two identical goods is considered, where bidders impose externalities on each other. Their model can be parametrized such that it coincides with a special case of Deng and Pekeč (2013), but it is limited to selling two items, multiplicative externality and single-dimensional valuations.

The idea of using affine maximizers to improve on the revenue of the VCG mechanism has appeared mostly in connection with combinatorial auctions. Likhodedov and Sandholm (2004) specifies a special class of affine maximizers and tries to fine-tune its parameters. Tang and Sandholm (2012) considers the case of two bidders and two items and searches for the best parameters for a given affine maximizer. One of our heuristics is also an affine maximizer, but its parametrization idea differs from that of the two previous papers. Moreover, with the help of simulation and the derived upper bound mechanisms, we can demonstrate in a novel way how small the gap is between the heuristic mechanisms and the optimal revenue.
3 Model

Suppose an item can be sold simultaneously to \( k \) out of \( n \) bidders (in the set \( N = \{1, \ldots, n\} \)) who each have unit demand for the item.\(^3\) The value of bidder (or agent) \( i \) for obtaining the item is characterized by his *type* \( t^i = (s^i, m^i) \), where \( s^i \) is the valuation for getting the item shared with some other agent, while \( m^i \) is the margin for exclusive possession, i.e., \( s^i + m^i \) is the valuation for receiving the item exclusively. The shared valuation \( s^i \) is drawn from the set \( S^i \) with cumulative distribution \( F \) (with density \( f \)), while the exclusivity margin \( m^i \) is drawn from \( M^i \) with cumulative distribution \( G \) (density \( g \)). The set of possible types for agent \( i \) is \( T^i = S^i \times M^i \). We make the following assumptions. \( F \) and \( G \) are independent and strictly increasing. Agent characteristics are i.i.d, i.e., \( T^i = T^j = [0, \tilde{s}] \times [0, \tilde{m}] \) for all \( i, j \in N \), where \( \tilde{s}, \tilde{m} \in \mathbb{R}^+ \). Types are distributed independently among agents, so the distribution of type \( t^i \) is \( F \times G \) and the distribution of type-tuples is \( (F \times G)^n \). The basic notation used in the paper is summarised in Table 3 on page 46 in the appendix.

Following the Revelation Principle (Myerson, 1981) we restrict our attention to direct auctions. Each direct auction \( (x, p) \) can be characterized by its allocation rule \( x : T \to \{0, 1\}^n \) and its payment scheme \( p : T \to \mathbb{R}^n \). Let \( a(t) = \{i : x^i(t) = 1\} \) be the set of agents \( (\subseteq N) \) who receive the item for bid \( t \). Note that \( a(t) \) would be a singleton under exclusive allocation, and \( |a(t)| \geq 2 \) represents shared allocation. Given an auction \( (x, p) \) and report profile \( \hat{t} = (\hat{s}, \hat{m}) \), the realized valuation \( v \) of agent \( i \) having type \( t^i = (s^i, m^i) \) is

\[
v^i(x(\hat{t}), t^i) = \begin{cases} 
    s^i & \text{if } i \in a(\hat{t}) \text{ and } |a(\hat{t})| \geq 2, \quad \text{(agent i receives a shared allocation)} \\
    s^i + m^i & \text{if } a(\hat{t}) = \{i\}, \quad \text{(agent i receives exclusive allocation)} \\
    0 & \text{otherwise.}
\end{cases}
\]

---

\(^3\)The limit \( k \) is often set by policy, and a value of 3 or 4 provides sufficient competition and variety without overloading customers with unwanted sales calls. For instance, lead generation platforms for automobile trades typically match a customer with 3 car dealers.
while the net utility (after subtracting payment) is
\[ u^i(x(\hat{t}),p(\hat{t}),t^i) = v^i(x(\hat{t}),t^i) - p'(\hat{t}). \]
Because \( v \) depends only on the allocation \((a)\) component of \( x \), we will occasionally write
\( v^i(a(\hat{t}),t^i) \) for expositional clarity rather than \( v^i(x(\hat{t}),t^i) \).

When selling a single item to a buyer with unknown value \( v \) drawn from a distribution \( D \), profit is maximized with the well-known “inverse elasticity = optimal markup” pricing rule. Written differently, the rule is that the optimal price is the value \( v^* \) which satisfies
\[ v - \frac{1 - D(v)}{d(v)} = 0, \tag{1} \]
where the buyer’s virtual valuation \( v - \frac{1 - D(v)}{d(v)} \) is null. The equation yields a unique, optimal, solution when \( F \) has monotone increasing hazard rate (IHR). IHR ensures non-decreasing elasticity and log-concavity of \( D \), and this special structure enables insightful qualitative implications, making it attractive in many areas of economics (see, e.g., Bagnoli and Bergstrom, 2005). Distributions in this class have tail not fatter than that of the exponential distribution, and include the normal, exponential, some parameterization of gamma, Pareto and the uniform distribution. The concept of virtual valuation also plays a central role in designing revenue maximizing single-item auctions when valuations for the item are identically, independently distributed. The virtual valuation of agent \( i \) for real-valued random variable \( r^i \in \mathbb{R} \), that is distributed as \( D \), is defined as
\[ \phi_D(r^i) = r^i - \frac{1 - D(r^i)}{d(r^i)}. \tag{2} \]
In our two-dimensional context, \( r^i \) above could be either \( s^i \), \( m^i \) or \( s^i + m^i \), so that \( D \) would be \( F \), \( G \) or \( F \times G \). For IHR distribtuions \( D \), the unique point where the corresponding virtual valuation hits zero is called the reserve value \( r_D \) of \( D \), and will play a crucial role in our proposed auction design.
3.1 Design Goals and Choices

Consider a particular instance of the auction design problem faced by a one-to-many matching platform, i.e., with specific distributions \( F \) and \( G \) that describe shared valuations and exclusivity margins. Ideally, the platform chooses a revenue-maximizing design with respect to these distributions, meaning that it chooses, among all imaginable allocation and pricing rules, a combination that maximizes expected revenue, assuming agents follow a utility maximizing bidding strategy. As outlined in Section 2, finding such rules is in our case not only a notoriously difficult problem given the multi-dimensional nature of valuations, but also may give complex designs that are difficult to execute, communicate to bidders, and last but not least to play. Therefore, our design objectives are a combination of the profitability objective and practicality, i.e., maximizing auction revenue subject to designs that are simple enough and impose low costs on all participants.

From an agent’s point of view, the choice is between mechanisms that promote truthful bidding vs. those under which agents should scheme and shade their bids in equilibrium. In general, the best among the latter kind of auctions has the potential to produce higher revenue, but at higher participation costs for agents because agents must compute how to shade their bids while making assumptions and anticipating behavior of other agents. These costs may be substantial (relative to the simpler “bid your true value” rule), and can impose substantial costs on the auctioneer as well. If agents cannot reliably identify their optimal bidding strategies, then realized revenue may fall short of predicted revenue. Hence our joint objective of profitability and simplicity steers us in the direction of incentive compatible mechanisms, i.e, rules that maximize profits while strongly encouraging truthful bidding, and thereby being easy for agents to participate in. Formally, this defines a feasible space of auction designs in which truth-telling is a dominant strategy for agents. Another natural requirement from a mechanism that we impose is that truth-telling leads to non-negative utility for every agent.
Definition 1 (DSIC). A direct auction is dominant strategy incentive compatible (DSIC) if truth-telling is a dominant strategy for each agent: given the other agents’ bids, every agent’s utility is maximized by bidding truthfully. Formally, a direct auction \((x, p)\) is DSIC if for every \(i, t^{-i}, t^i\) and \(\hat{t}^i\) it holds that

\[
u^i (x(t), p(t), t^i) \geq u^i (x(\hat{t}^i, t^{-i}), p(\hat{t}^i, t^{-i}), t^i).
\]

Definition 2 (EPIR). A direct auction is ex-post individual rational (EPIR) if a truth-telling agent has non-negative utility for every report of other agents. Formally, a direct mechanism \((x, p)\) is EPIR if for every \(i\) and \(t = (t^i, t^{-i})\) it holds that

\[
u^i (x(t), p(t), t^i) \geq 0.
\]

3.2 Mathematical Programming Formulation

The requirements described above can be summarized into the following requirement: auction design problem:

\[
\text{Given distributions } F \text{ and } G, \text{ find a direct auction (allocation and pricing rules) which maximizes expected revenue while preserving individual rationality (EPIR) and truthful bidding (DSIC).}
\]

This statement concisely represents our design goal with respect to revenue optimality and practicality. We call this problem the dominant strategy auction (DSA) problem, and model it as a mathematical program where the parameters in the objective are determined by the priors of the distributions, and the constraints represent incentives and allocation feasibility. Once solved, the decision variables tell for each report of types who receives the item and prices to be paid. Because it is DSIC, agents’ reports are simply their true values.
The mathematical program is stated as follows.

\[
\max_{x(t), p(t)} \mathbb{E}_t \left[ \sum_i p^i(t) \right] \quad \text{(DSA)}
\]

subject to

\[
\begin{align*}
    u_i \left( x(t), p(t), t^i \right) &\geq u_i \left( x(t^i, t^{-i}), p(t^i, t^{-i}), t^i \right) \quad \forall i, \forall t^{-i}, \forall t^i, \forall \hat{t}^i, \quad \text{(DSIC)} \\
    u_i \left( x(t), p(t), t^i \right) &\geq 0 \quad \forall i, \forall t, \quad \text{(EPIR)} \\
    \sum_i x^i(t) &\leq k \quad \forall t, \quad \text{(SUPPLY)} \\
    x^i(t) &\in \{0, 1\} \quad \forall i, \forall t. \quad \text{(BI)}
\end{align*}
\]

Constraint (DSIC) is responsible for dominant strategy incentive compatibility, while (EPIR) enforces ex-post individual rationality. (SUPPLY) sets the upper bound on the number of copies that can be sold, and (BI) ensures that the mechanism is deterministic. Note that with DSA we refer to the whole mathematical program. The optimal solution \((x, p)\) for DSA will be called DSA*.

We note that the DSA formulation is, for us, primarily a theoretical construct. It has a huge number of variables and constraints (exponential size in the input). Even if we could solve it, that solution would not be practical as it would only provide a full list of \(x(t)\) and \(p(t)\) vectors, one for each \(t\). For us, the program serves a different purpose. On one hand, it enables us to generate alternative practical mechanisms, either by adding certain constraints to this program (see §4.1) or via a special class of DSIC two-dimensional auctions (presented in §4.2). Optimal solutions to these refined programs are therefore feasible solutions to the original DSA program, and provide lower bounds to the optimal DSA revenue. On the other hand, we develop relaxations of this program which readily can be optimized, thereby generating upper bounds for the optimal solution of DSA, thereby enabling performance
Table 2: VCG allocation and prices for the example from Table 1. Allocation type (shared or exclusive) and allocations \((x_i)\) optimize profit given the bids. Prices charged are a difference of column \(D_i\) values (for the two rows labeled “Others’ value...”), representing \(\sum_{j \neq i} v_j\) under efficient allocation, when bidder \(i\) is not present, and present, respectively.

<table>
<thead>
<tr>
<th>Allocation ((x_i))</th>
<th>(D1)</th>
<th>(D2)</th>
<th>(D3)</th>
<th>(D4)</th>
<th>(D5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Others’ value if (i) not present</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Others’ value if (i) present</td>
<td>34</td>
<td>35</td>
<td>30</td>
<td>33</td>
<td>35</td>
</tr>
<tr>
<td>Prices ((p_i))</td>
<td>9</td>
<td>0</td>
<td>9</td>
<td>9</td>
<td>0</td>
</tr>
</tbody>
</table>

Scenario 1

<table>
<thead>
<tr>
<th>(D1)</th>
<th>(D2)</th>
<th>(D3)</th>
<th>(D4)</th>
<th>(D5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>34</td>
<td>36</td>
<td>36</td>
<td>36</td>
<td>36</td>
</tr>
<tr>
<td>0</td>
<td>36</td>
<td>36</td>
<td>36</td>
<td>36</td>
</tr>
</tbody>
</table>

Scenario 2

<table>
<thead>
<tr>
<th>(D1)</th>
<th>(D2)</th>
<th>(D3)</th>
<th>(D4)</th>
<th>(D5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>34</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Considering the difficulty of identifying the optimal DSA auction, a one-to-many platform provider could start by accepting heuristic solutions of the design problem. One appealing candidate design is to combine a VCG pricing rule with allocations that maximize agent value (i.e., to allot exclusive or shared based on whether the highest exclusive bid exceeds the sum of the \(k\)-highest shared bids). Formally, the allocation rule assigns to agent \(i\) exclusively if \((\hat{s}_i + \hat{m}_i)\) exceeds (or equals) the sum of the highest-\(k\) \(\hat{s}\) values, or otherwise to the first \(k\) agents in descending order of \(\hat{s}_j\) values (using some tie-breaking rule when necessary).\(^4\) The VCG pricing rule charges every bidder the externality she poses on other bidders, thereby eliminates bid-shading incentives that will arise under a pay-your-bid pricing rule. More precisely, the price bidder \(i\) has to pay equals the total value of the efficient allocation for other bidders, would she not have participated in the auction, minus the value that all other bidders get when she is participating. Note that for bidders who do not win the item this results in zero payments.

\(^4\)Shouldn’t the \(s\)’s and \(m\)’s have “hats” indicating they are bids rather than values? It is the same due to DSIC property, but in principle ... ?? Do we even have a notation? We have \(\hat{t}\) but not \(\hat{s}\) or \(m\). I added it.
Table 2 gives the VCG allocations and prices corresponding to Scenarios 1 and 2 of our example. While this design maximizes social value while being easy to implement, it is unlikely to have strong revenue performance. It is well-known that VCG auctions may yield very low revenue (cite the lovely but lonely Vickrey auction from CA book) and that, due to Revenue Equivalence, there is no way to adjust payments without losing DSIC and EPIR while maintaining allocation efficiency. We therefore need to give up allocation efficiency and include the allocation rule in our search space.

![Figure 1: A map of comparative performance for alternative mechanisms. OE and OS (only-exclusive and only-shared) are one-dimensional auctions; MaxSimple is (ex ante) the better of the two. VCG and RM are two-dimensional auctions. Actual comparisons, determined through theoretical and simulation techniques, will be presented in Section 5.]

4 Approximate optimal auctions (Revenue Lower Bound)

The DSA problem statement primarily serves the purpose of a theoretical rendering of the optimal DSIC auction. To obtain and evaluate some practical auction designs, this section develops five different feasible solutions to this problem by constructing five, solvable, restrictions of the original DSA problem. By construction, each method will provide a lower bound for the optimal DSA. Note that the optimal revenue $DSA^*$ is not known, hence the performance of these lower bounds will be evaluated by constructing and computing upper
bounds on DSA*, which we do in the next section by relaxing DSA.

Without loss of generality or performance, we can focus on allocation rules that satisfy a monotonicity condition. The condition is obtained by adding up two (DSIC) constraints such that the first constraint is applied to agent \(i\) having type \(t^i\) and reporting \(\hat{t}^i\), while the second one is expressed for the same agent having \(\hat{t}^i\) and reporting \(t^i\), to yield

\[
v^i \left(x(t^i, \hat{t}^i), \hat{t}^i\right) - v^i \left(x(\hat{t}^i, t^{-i}), t^i\right) \geq v^i \left(x(t), \hat{t}^i\right) - v^i \left(x(t), t^i\right).
\]

(\text{MON})

Allocation rules satisfying (MON) will be called monotone. The significance of this property is that designs which fail it cannot have a combination of allocation and pricing rules that together satisfy (DSIC). Hence such designs can be eliminated without loss of revenue, and our search can be limited to the space of monotone allocations.

A few monotone designs are obvious and by construction provide lower bounds on the revenue from the optimal auction. These designs are (a) only-exclusive (OE): sacrifice the possibility that shared allocation is better, and always allocate to the bidder with the highest bid for exclusive purchase, (b) only-shared (OS): sacrifice the possibility that exclusive allocation is better, and always allocate to bidders with highest shared valuations, and (c) pick the optimal shared vs. exclusive allocation auction based on priors about distribution of valuations. All these designs require only one-dimensional bids, would be simple to implement and would not impose an information burden meaningfully higher than that for a standard one-to-one matching auction. If any of these designs produced sufficiently good revenue performance across the problem parameters, then we would have an acceptable design. Unfortunately, as we show in the next section, none does.

With the limitations of these obvious designs, we aim to develop non-obvious and theoretically-motivated designs that exceed the performance of these obvious solutions and still are attractive in practice, easy for merchants to participate in and for the platform to execute in
real-time. Our fourth and fifth candidate designs allow for two-dimensional bids and have allocation and pricing rules that are easy to implement. The first of the two is the VCG mechanism discussed before, but it fails the high-revenue test. The second two-dimensional auction is developed by applying theoretical principles of affine maximizer designs that guarantee truthful bidding and extends the optimal reserve price approach to two-dimensional bids. Accordingly, we call this mechanism RM (for reserve-price mechanism). Section 6 demonstrates that RM outperforms VCG and the single-dimensional optimal designs, and we argue that its revenue must be a close approximation to the optimal revenue from DSA* by constructing a tight upper bound for DSA* and showing that RM revenue is robustly close to this upper bound across a variety of parameter settings.

### 4.1 One-dimensional auctions

OE and OS commit the platform to either exclusive-only or shared-only allocations for all instances or problem categories, and the platform solicits the corresponding one-dimensional bid. Analysis of these two cases is useful for platforms that prefer a consistent allocation policy, hence this analysis will provide guidance regarding which one to choose. Intuitively, this will depend on the magnitude and distribution of $s$ and $m$ sets across multiple instances. In other cases, a platform might wish to exploit category-specific prior information (e.g., leads for BMW buyers in zip code 93940 during June) and then pick the better of OE and OS based on expected revenue performance given the prior, and then commit to it for that category in advance. This is the MaxSimple mechanism in our framework. For each category, participants need to provide only one bid (for exclusive or shared allotment, as specified by the platform) but which kind it is can vary across categories. We prove that this auction yields a constant factor approximation of the optimal revenue.

The three one-dimensional auctions have an advantage that the optimal revenue mechanism can be found in each case, hence they provide useful lower bounds for the original
optimal auction problem (DSA). For OE (always exclusive allocation) and OS (always shared), the DSA requirement is met by adding respectively, the constraint \( \sum_i x_i(t) \leq 1 \) (exclusive only) or \( \sum_i x_i(t) \geq 2 \) (shared only). In the first case we can use the optimal single item auction, i.e., the best out of those mechanisms that allocate to at most one agent. The second is the other extreme: it is the optimal mechanism among those which never allocate exclusively. Notice that these optimal auction problems are single-dimensional as in each of them only one valuation counts (shared or exclusive). Therefore we can utilize for both of them the framework of Myerson, 1981 that gives a characterization of the optimal single-dimensional mechanism. As we need this framework as well for our upper bounds in Section 5, we repeat how these solutions look like in a general setting with unit demand agents and homogeneous, indivisible items. For our purposes, we adapt a generalization of the original result of Myerson, 1981 from Devanur et al., 2011.

**Theorem 1** (Myerson, 1981). Let \( NU \) be a set of unit-demand agents, and \( A \subseteq 2^{NU} \) the set of feasible allocations, i.e., the set of agents who can be served simultaneously. Assume that agent \( i \)'s type \( t_i \) is single-dimensional and drawn from set \( T_i \subset \mathbb{R} \), with regular distribution \( D_i \). Let \( T = \times_i T_i \), and let \( D = D_1 \times \ldots \times D_n \) be the distribution over \( T \). Then the single-parameter environment \( (NU,A,T,D) \) has the following properties.

1. For every DSIC and EPIR mechanism \( (x,p) \), where \( p \) is the maximal revenue given \( x \) and \( x_i(t) \) is the probability that agent \( i \) is served, the expected value of total payments can be written as
   \[
   \mathbb{E}_t \left[ \sum_i p_i(t) \right] = \mathbb{E}_t \left[ \sum_i \phi_{D_i}(t^i) x^i(t) \right],
   \]
   the expected total virtual valuation.

2. A revenue-maximizing DSIC and EPIR auction design is given by the allocation rule
   \[
   a(t) = \operatorname{arg \ max}_{b \in A} \left\{ \sum_{i \in b} \phi_{D_i}(t^i) \right\},
   \]
and payment rule

\[ p^i(t) = \begin{cases} 
\inf \{ \hat{\ell}^i \mid i \in a(\hat{\ell}^i, t^{-i}) \} & i \in a(t) \\
0 & \text{otherwise}
\end{cases} \]  \hspace{1cm} (3)

Let \( C \) denote the cumulative density function of the valuation for exclusive allocations, that is, the convolution of \( F \) and \( G \). The first part of Theorem 1 states that the platform’s total expected revenue equals, in expectation, the total virtual valuations of winning merchants, and it applies to both OS and OE with \( D \) corresponding to \( F \) and \( G \) respectively. But it should not be interpreted as specifying a point-wise mapping between prices paid and virtual valuations. Instead, prices paid are the incentive-compatible payments given in the second part of the theorem ((3)): each winning merchant pays the amount she would have strategically bid (after shading) given that all other merchants bid truthfully. With the help of Theorem 1 we can directly define the optimal designs for the first two one-dimensional auctions.

**Definition 3** (Optimal exclusive auction (OE)). The optimal auction is characterized by the allocation rule (with ties broken arbitrarily in case of multiple optimal arguments)

\[ a_{OE}(t) = \arg \max_{b \subseteq N} \left\{ \sum_{i \in b} \phi_C(s^i + m^i) \right\} \]

and payment rule

\[ p^i_{OE}(t) = \begin{cases} 
\inf \{ \hat{s}^i + \hat{m}^i \mid i \in a_{OE}((\hat{s}^i, \hat{m}^i), t^{-i}) \} & i \in a_{OE}(t) \\
0 & \text{otherwise}
\end{cases} \]  \hspace{1cm} (4)

**Theorem 2.** If \( F \) and \( G \) have MHR, then OE is feasible for (DSA) and achieves the highest expected revenue among those mechanisms which allocate to at most one agent. Moreover, its expected revenue can be expressed as

\[ \text{Rev}(OE) = \mathbb{E}_t \left[ \sum_{i \in a_{OE}(t)} \phi_C(s^i + m^i) \right]. \hspace{1cm} (5) \]

**Proof.** It follows directly from Theorem 1 given that \( C \) has MHR as the set of MHR.
distributions is closed under convolution (see Barlow et al., 1963).

Note that OE does not allocate at all the item if all virtual values happen to be strictly less than 0.

**Definition 4 (Optimal shared auction (OS)).** The optimal auction is given by the allocation rule (with ties broken arbitrarily in case of multiple optimal arguments)

\[
a_{OS}(t) = \arg \max_{\substack{b \subseteq N \mid |b| \leq k, |b| \neq 1 \mid \sum_{i \in b} \phi_F(s^i)}} \left\{ \sum_{i \in b} \phi_F(s^i) \right\}
\]

and payment rule

\[
p^i_{OS}(t) = \begin{cases} \inf \{ \hat{s}^i \mid i \in a_{OS}((\hat{s}^i, m^i), t^{-i}) \} & \text{if } i \in a_{OS}(t) \\ 0 & \text{otherwise.} \end{cases}
\]

**Theorem 3.** OS is feasible for (DSA) and achieves the highest expected revenue among those mechanisms which never allocate exclusively. Moreover, its expected revenue can be expressed as

\[
\text{Rev}(OS) = \mathbb{E}_t \left[ \sum_{i \in a_{OS}(t)} \phi_F(s^i) \right].
\]

**Proof.** It follows directly from Theorem 1. Note that Myerson requires only that for each agent the allocation is monotone non-decreasing in its own type given the others’, therefore the fact that OS never allocates for singletons is not an issue for invoking Theorem 1.

The allocation rule of OS deserves some explanation. By the requirement that we always allocate to at least two bidders, bidders might win an item even if their virtual value is below 0, or in other words, their value is below the reserve price. Whenever this happens, bidders will also pay prices that are below the reserve prices. While this sounds counter intuitive, never allocating to a single bidder is necessary to preserve incentive constraints. Indeed, a bidder who happens to get an exclusive allocation would enjoy extra value from this allocation. In a setting where she and all other bidders have shared value below the
reserve price, she would have incentives to report a higher shared value than his true one, securing an exclusive allocation. This demonstrates that platforms who reduce for the sake of simplicity the expressiveness of bids need to design allocations carefully in order to not invite strategic bidding related to information that is not revealed in the auction.

As noted earlier, OE and OS both provide useful lower bounds for (DSA). However, as we shall show later, the quality of these bounds varies considerably across the parameter regions. Because the quality variation generally moves in opposite directions, it becomes worthwhile to combine the two auction formats into the \texttt{MaxSimple} defined below.

**Definition 5** (\texttt{MaxSimple} mechanism). For any instance of the exclusivity auction problem \texttt{MaxSimple} calculates the expected revenue of OS and OE, then commits the mechanism with the highest value.

Note that the choice in \texttt{MaxSimple} is made upfront for given priors $F$ and $G$ and not for every type realization, therefore the actions of the agents don’t influence which of the two single item auctions is executed. Despite the fact that \texttt{MaxSimple} capitalizes only on one type of bid and allocation, a later result (Theorem 6, which can be established only after setting up additional results) reveals that the seller can still capture a constant fraction of the optimal revenue by implementing it. This is useful for practitioners because they can keep the bidding language and the rules of the auction simple for only a small sacrifice in the revenue. Furthermore, there is only one parameter (the reserve price) to change when priors change. Note that while OE and OS are the revenue-maxizing auctions within their classes of allocations (always assign to just one or always adjust to multiple merchants), the benefit of \texttt{MaxSimple} is that it enables the platform to make a category-specific choice between the two.
4.2 A two-dimensional auction: RM

Although MaxSimple is intuitive and simple, it is a combination of two different mechanisms. Therefore if a platform provider has multiple similar products to offer, then changing the rules and the type of allocation product by product, in order to derive revenue advantage from the alternate rule,\(^5\) can create confusion among the buyers. Besides, MaxSimple is only single-dimensional, thus it doesn’t take advantage of the extra information and the wider offer menu it is possible to utilize by taking into account both types of valuations, and allowing for both types of allocations. The next natural step accordingly, is to devise mechanisms that consider two types of bids: one for shared and one for exclusive allocations.

One obvious choice for an auction that takes both dimensions of types into account would be VCG, discussed earlier in §3 and illustrated in Table 2. It is DSIC and achieves the highest revenue among those mechanism that maximize welfare. However, the drawback is that its expected revenue can be quite low. We seek here to improve on the revenue performance of VCG. We propose a new mechanism (RM) along the lines developed in the OE and OS methods, while employing both sets of valuations and preserving truthful bidding.

In order to define the auction we recall the definition of an affine maximizer, also called Generalized VCG auctions.

**Definition 6** (Affine maximizer auctions). Let \(A_N\) be a set of feasible allocations for agents in \(N\), and let the index \(b\) range over this set, i.e., each \(b\) is an \(n\)-element vector. A mechanism which makes allocation \(a(t)\) is an affine maximizer if there is a vector of \(\gamma^b\)'s (each \(\gamma^b\) is a scalar) and a vector of \(\lambda^i\)'s (across all agents \(i \in N\)) such that for each \(t\),

\[
\left(\gamma^{a(t)} + \sum_{i \in N} \lambda^i v^i(a(t), t^i)\right) \geq \left(\gamma^b + \sum_{i \in N} \lambda^i v^i(b, t^i)\right), \forall b \in A_N. \tag{8}
\]

\(^5\)Note: Even for the same “product” the demand profiles or the number of bidders can vary when the product sale is repeated at a different date, day of week, time, location etc., potentially motivating switching the auction format.
The payment rule of the affine maximizer mechanism is equal to
\[ p^i(t) = \frac{1}{\lambda^i} \left[ \left( \gamma^a(t^{-i}) + \sum_{j \in N \setminus i} \lambda^j v^j(a(t^{-i}), t^j) \right) - \left( \gamma^a(t) + \sum_{j \in N \setminus i} \lambda^j v^j(a(t), t^j) \right) \right] \]
(9)

where \( a(t^{-i}) \) has
\[ \left( \gamma^a(t^{-i}) + \sum_{j \in N \setminus i} \lambda^j v^j(a(t^{-i}), t^j) \right) \geq \left( \gamma^b + \sum_{j \in N \setminus i} \lambda^j v^j(b, t^j) \right), \forall b \in A_{N \setminus i}. \]

**Fact 1.** (Roberts, 1979) Affine maximizer mechanisms are DSIC and EPIR.

This means that any affine maximizer is a feasible solution of DSA. Choosing \( \lambda^i = 1 \) for all \( i \in N \) and \( \gamma^b = 0 \) for all \( b \in A_N \) yields VCG.

Using these observations we define the Reserve Mechanism (RM) as a mechanism that for each bid combination decides on a shared or an exclusive allocation depending on which maximizes a particular affine function.

**Definition 7** (Reserve mechanism (RM)). Let \( r_F \) and \( r_C \) denote the shared and exclusivity reserve values respectively (see Equation 2). Then RM is defined as an affine maximizer, where \( \lambda^i = 1 \) for all \( i \), \( A = \{ a \subseteq N \mid |a| \leq k \} \), \( \gamma^{(i)} = -r_C \) for all \( i \), \( \gamma^b = 0 \) and \( \gamma^a = -r_F |a| \) for all \( a \subseteq N \) such that \( 2 \leq |a| \leq k \).

Observe that RM either allocates as OE or as OS, depending which of the ones yields more welfare. Prices paid are also higher because the first term in the price determination is also higher.

We close this section by observing that we cannot simply set reserve prices for exclusive and shared bids, eliminate bids below the reserves, and then allocate efficiently with respect to the remaining bids. This rule would not be incentive compatible in general, demonstrated by Example 2.

**Example 2.** Let \( N = \{1, 2\} \), \( k = 2 \), \( t^1 = (3, 3) \), \( t^2 = (2, 1) \) and \( r_F = 0, r_C = 5.5 \). Then the efficient allocation that meets the reserve is that agent 1 receives the item exclusively. Now, set \( \tilde{t}^1 = (1, 4) \) and note that the efficient allocation that satisfies the reserve is to share the item between agent 1 and 2. This allocation violates monotonicity condition (MON).
RM is an appealing choice for platform providers because it is simple and fast enough to be implemented, but complex enough to be able to capitalize on the particularity of the multi-dimensional valuations. The reserve prices $r_F$ and $r_C$ are determined uniquely for each “product” based on knowledge about the distribution of reservation prices for that product. For instance, for BuyerLink.com’s lead marketing auctions for automotive sales, this would mean computing these values for each “make” and “model” combination for groups of geographical locations which are similar in their distribution of reservation prices. Notably this requires no more information than that needed for designing a single-dimensional mechanism. Moreover, as the experimental evaluations in the next section highlight, its expected revenue is very close to the optimal one regardless of the number of agents.

5 Theoretical Upper Bounds on Revenue

The previous section has developed, for the one-to-many matching problem, several potential auction formats that are easy to implement and computationally tractable. As we show later, the revenue performance of these auctions can be readily compared by simulations, providing a basic understanding under which each design outperforms the others. This however does not inform whether the best design performs well relative to the optimal solution DSA* of the auction design problem (DSA). As there are no known techniques for identifying the optimal design for this two-dimensional problem, our approach to estimate the distance from the optimal solution is to establish and compare against upper bounds to (DSA). A straightforward upper bound is the optimal expected welfare, as it is always larger or equal to the optimal revenue due to the EPIR assumption. The problem is that the gap between the two can be significant. See for example graph 8 on page 38 where the optimal revenue is less than 60% of the optimal welfare. In order to find tighter bounds we propose a different, novel approach. We show in this section two ways of relaxing the conditions of (DSA) such
that the resulting optimization problem admits a closed form optimal solution. The idea is to relax the problem such that the resulting setting is single-dimensional, and then use the framework of Myerson, 1981 to solve the relaxed problem exactly. For notational convenience the expected revenue achieved by any mechanism $\text{MECH}$ will be symbolized by $\text{Rev}(\text{MECH})$.

### 5.1 Relaxation 1: Public exclusivity margin

We establish the first upper bound by solving (DSA) under the assumption that the exclusivity margin is public knowledge. That is, $s^i$ and $m^i$ are still distributed according to $F$ and $G$, but the realization of $m^i$ is observable. Therefore, any incentive constraint involving two different $m^i$ can be omitted from (DSA). This yields the following relaxed mathematical program.

$$\max \mathbb{E}_t \left[ \sum_i p^i(t) \right] \quad (\text{UBM})$$

subject to

$$u^i(x(s^i, m^i, t^{-i}), p(s^i, m^i, t^{-i}), t^i) \geq u^i(x(\hat{s}^i, m^i, t^{-i}), p(\hat{s}^i, m^i, t^{-i}), t^i) \quad \forall i, \forall t^{-i}, \forall m^i, \forall s^i, \forall \hat{s}^i,$$

$$u^i \left( x(t), p(t), t^i \right) \geq 0 \quad \forall i, \forall t,$$

$$\sum_i x^i(t) \leq k \quad \forall t,$$

$$x^i(t) \in \{0, 1\} \quad \forall i, \forall t.$$

We will refer to the optimal mechanism for (UBM) as $\text{UBM}^\ast$. (UBM) is clearly a relaxation of (DSA), which means that each feasible solution for (DSA) is feasible for (UBM), moreover, $\text{Rev}(\text{UBM}^\ast) \geq \text{Rev}(\text{DSA}^\ast)$. In order to identify $\text{UBM}^\ast$ succinctly some additional notation
is introduced. For given type profile \( t = (s, m) \) define \( a_s(t) \) and \( a_e(t) \) to comprise the set of agents with the highest virtual values with respect to \( F \).

\[
\begin{align*}
a_s(t) &= \arg \max_{a \subseteq N} \left\{ \sum_{i \in a} \phi_F(s^i) \right\} \quad \text{(set of agents with k-highest shared virtual values),} \\
a_e(t) &= \arg \max_i \left\{ \phi_F(s^i) + m^i \right\} \quad \text{(agent with highest sum of shared virtual value and exclusivity margin).}
\end{align*}
\]

Ties are broken arbitrarily in case of multiple optimal arguments. Note that \( \sum_{i \in a_s(t)} \phi_F(s^i) \geq 0 \) as the empty set is also a solution. The next Theorem shows that the optimal expected revenue can be expressed in terms of virtual valuations with respect to \( F \). Moreover, the set of winners for report \( t \) in the optimal auction is equal to either \( a_s(t) \) or \( a_e(t) \), dependent on which one provides higher virtual value.

**Theorem 4.** If \( F \) has MHR and the exclusivity margins are public information, then the allocation rule of \( \text{UBM}^* \) given type profile \( t = (s, m) \) can be characterized as

\[
x^i(t) = 1 \text{ for } \begin{cases} 
   i \in a_s(t) & \text{ when } \sum_{j \in a_s(t)} \phi_F(s^j) \geq \phi_F(s^{a_e(t)}) + m^{a_e(t)} \\
   i = a_e(t) & \text{ when } \sum_{j \in a_s(t)} \phi_F(s^j) < \phi_F(s^{a_e(t)}) + m^{a_e(t)}
\end{cases}
\]

(10)

and \( x^i(t) = 0 \) otherwise. The expected revenue under this allocation rule is equal to

\[
\mathbb{E}_t \left[ \sum_i p^i(t) \right] = \mathbb{E}_t \left[ \sum_i \left( \phi_F(t^i) + m^i \prod_{j \neq i}(1 - x^j(t^i, m^i, t^{-i})) \right) x^i(t^i, m^i, t^{-i}) \right].
\]

(11)

The proof can be found in the appendix. It builds on results in Deng and Pekeč, 2013 for a similar model. Using the framework of Myerson, 1981, they provide the optimal auction when shared valuations are private information and exclusivity margins are simple functions of other bidder’s shared value.

In general, the optimal solution of (UBM) might yield strictly higher revenue than (DSA). This is due to the fact that the optimal allocation in (UBM) is not always feasible for (DSA).
as the following example shows.

**Example 3.** Let $N = \{1, 2\}$, $s^i \sim U(0,1)$, $m^i \sim U(0,1)$ for all $i \in N$. Assume that $t^1 = (0.3, 0.3)$, $t^2 = (0.4, 0.1)$. Then we have that $\phi_U(0,1)(s^1) + m^1 = -0.1$, $\phi_U(0,1)(s^2) + m^2 = -0.1$, $\phi_U(0,1)(s^1) = -0.4$ and $\phi_U(0,1)(s^2) = -0.2$, therefore according to UBM* nobody gets anything. Now, change the valuations of agent 1 such that $\hat{t}^1 = (0.5, 0.05)$. As $\phi_U(0,1)(\hat{s}^1) + \hat{m}^1 = 0.05$ and $\phi_U(0,1)(\hat{s}^1) = 0$ agent 1 receives the item exclusively. This allocation rule violates monotonicity condition (MON), which means that there is no payment scheme such that the incentive compatibility constraints are satisfied. Therefore it cannot be part of a solution for (DSA).

### 5.2 Relaxation 2: Bound with representatives

The derivation of the second upper bound mechanism is more convoluted. For every instance of (DSA) we define an instance of an alternative setting by introducing two representatives for each agent: one for each dimension of his type. To distinguish this setting in the notation, we add a “bar” to each item, as in $\overline{x}$.

**Definition 8** (Representative environment). For any instance of (DSA) given by the set of agents $N$, the set of types $T$ and distributions of types $F,G$, the representative environment is defined as follows:

- For each $i \in N$ introduce $i_s$ and $i_e$, such that the set of agents is $\overline{N} = \overline{N}_s \cup \overline{N}_e$, where $\overline{N}_s = \{1_s, \ldots, i_s, \ldots, n_s\}$ and $\overline{N}_e = \{1_e, \ldots, i_e, \ldots, n_e\}$.
- Let $\overline{A} = \{a \subseteq \overline{N}_s \mid |a| \leq k\} \cup \{i_e\}_{i_e \in \overline{N}_e}$, the set of feasible allocations.
- Let $\overline{T}^{i_s} = S^i$ for all $i_s \in \overline{N}_s$, $\overline{T}^{i_e} = M^i$ for all $i_e \in \overline{N}_e$, $\overline{T} = \times_{i \in \overline{N}} \overline{T}^i$, the set of type profiles.
- Let the valuations for all $i \in \overline{N}$, for all $t \in \overline{T}$, for all $\hat{t}^i \in \overline{T}^i$ and for allocation rule $\overline{x}$ be:

$$
\overline{v}^i(\overline{x}(\hat{t}^i, t^{-i}), t) = \begin{cases} 
  s^{i_s} & \forall i_s \in \overline{N}_s, \text{ if } \overline{x}^{i_s} = 1, \\
  s^{i_s} + m^{i_e} & \forall i_e \in \overline{N}_e, \text{ if } \overline{x}^{i_e} = 1, \\
  0 & \forall i \in \overline{N} \text{ otherwise.}
\end{cases}
$$

and for payment function $\overline{p}$

$$
\overline{p}(\overline{x}(\hat{t}^i, t^{-i}), \overline{p}(\hat{t}^i, t^{-i})) = \overline{v}^i(\overline{x}(\hat{t}^i, t^{-i}), t) - \overline{p}^i(\hat{t}^i, t^{-i})
$$
the utility function.

- Let $t^i \sim F$ for all $i_s \in N_s$, $t^i \sim G$ for all $i_e \in N_e$

Note that agent $i_e$ exhibits informational externality in his valuation, meaning that his value for a particular allocation may depend on the true type of other agents. More specifically, the value that an agent $i_e$ observes if assigned an item depends on the value of agent $i_s$. In such settings dominant strategy incentive compatibility is too demanding as it requires truthfulness for every possible report of the other agents, leaving little freedom for non-trivial mechanisms (see for example Roughgarden and Talgam-Cohen, 2013). Therefore we relax the incentive compatibility condition to ex-post incentive compatibility (EPIC), which requires truthfulness only for every truthful report of the other agents.

**Definition 9 (EPIC).** A direct mechanism is ex-post incentive compatible (EPIC) if for each agent truth-telling is a dominant strategy given that the others report truthfully. Formally, a direct mechanism $(x, p)$ is EPIC, if for every $i$, $t^{-i}$, $t^i$ and $\hat{t}^i$ it holds that

$$u^i(x(t), p(t), t) \geq u^i(x(\hat{t}^i, t^{-i}), p(\hat{t}^i, t^{-i}), (t^i, t^{-i})).$$  

Another difference to (DSA) is that we allow for allocations where only one shared agent receives the item. This increases the set of feasible allocations, hence bears the possibility to generate more revenue.

The revenue optimization problem for the representative environment can be stated as

$$\max \mathbb{E}_t \left[ \sum_i \overline{p}_i^i(t) \right]$$  \hspace{1cm} (UBR)
subject to

\[ \bar{w}^i (\bar{x}(t), \bar{p}(t), t) \geq \bar{w}^i (\bar{x}(\bar{t}^i, t^{-i}), \bar{p}(\bar{t}^i, t^{-i}), (t^i, t^{-i})) \quad \forall i, \forall t^i, \forall \bar{t}^{-i} \quad \text{(EPIC)} \]

\[ \bar{w}^i (\bar{x}(t), \bar{p}(t), t) \geq 0 \quad \forall i, \forall t \quad \text{(EPIR2)} \]

\[ (1 - \bar{x}^{j_e}(t))k \geq \sum_{i_s \in \overline{N}_s} \bar{x}^{i_s}(t) \quad \forall t, \forall j_e \in \overline{N}_e \quad \text{(FeasA)} \]

\[ \sum_{i_e \in \overline{N}_e} \bar{x}^{i_e}(t) \leq 1 \quad \forall t \quad \text{(FeasB)} \]

\[ \bar{x}^i(t) \in \{0, 1\} \quad \forall i, \forall t. \]

(EPIC) is responsible for ex-post incentive compatibility, while (EPIR2) for ex-post individual rationality. (FeasA) ensures that at most \( k \) items are allocated to agents in \( \overline{N}_s \), and that there cannot be allocations where agents both from \( \overline{N}_s \) and \( \overline{N}_e \) receive an item. (FeasB) states that at most one agent from \( \overline{N}_e \) can be served. We will refer to the optimal mechanism for (UBR) as \( \text{UBR}^* \).

Strictly speaking, (UBR) is not a relaxation of (DSA) as it has more variables than (DSA). Still, the following proposition shows that the optimal solution of (UBR) can be used as upper bound for (DSA).

**Proposition 1.** For every feasible mechanism \((x, p)\) for (DSA), there is a feasible mechanism \((\bar{x}, \bar{p})\) for (UBR) such that \(\text{Rev}(\bar{x}, \bar{p}) \geq \text{Rev}(x, p)\).

Here, only the intuition behind the proof is given, for technical details we refer to the appendix. As each representative corresponds to a particular dimension of an original agent’s valuation, any allocation rule in a feasible solution \((x, p)\) for (DSA) induces an allocation rule \(\bar{x}\) for (UBR). As each agent in \(\overline{N}\) has a one-dimensional type, \(\bar{x}\) is part of a feasible solution of (UBR) if and only if it satisfies (MON) (this is a well-known fact of one-dimensional mechanisms). It can be shown that feasibility of \(x\) implies (MON) except for a null set of
types in the original problem, and that except for that null set, the same prices can be used to make \( x \) truthful. On such null sets, \( x \) can be adjusted in a way that firstly, expected revenue from \((x,p)\) does not change, and secondly the corresponding \( x \) is part of a feasible solution of \((UBR)\), using the same prices.

It is immediate from Proposition 1 that the optimal expected revenue of \((UBR)\) serves as another non-trivial upper bound for \( Rev(\text{DSA}^\ast) \). As \((UBR)\) is single-dimensional, it is possible to provide a closed form solution for \( UBR^\ast \) in a similar way as before. For given type profile \( t \) let

\[
\bar{a}_s(t) = \arg \max_{a \subseteq N_s, |a| \leq k} \left\{ \sum_{i \in a} \phi_F(t^i) \right\}
\]

\[
\bar{a}_e(t) = \arg \max_{i e \in N_e} \left\{ t^i + \phi_G(t^i) \right\}.
\]

Ties are broken arbitrarily in case of multiple optimal arguments. Note that \( \sum_{i \in \bar{a}_s(t)} \phi_F(t^i) \geq 0 \) as the empty set is also a solution.

**Theorem 5.** When \( F \) and \( G \) have MHR, the allocation rule of \( UBR^\ast \) represented by \( \pi \), for type profile \( t \), is computed as follows,

\[
\bar{x}(t) = 1 \text{ for } \begin{cases} \bar{a}_s(t) & \text{when } \sum_{j s \in \bar{a}_s(t)} \phi_F(t^j) \geq \max_{j e \in N_e} \{ t^j + \phi_G(t^j) \} \\ \bar{a}_e(t) & \text{when } \sum_{j s \in \bar{a}_s(t)} \phi_F(t^j) < \max_{j e \in N_e} \{ t^j + \phi_G(t^j) \} \end{cases}
\]

and \( \bar{x}(t) = 0 \) otherwise. The optimal revenue under this allocation is

\[
Rev(UBR^\ast) = E_t \left[ \sum_{i s \in N_s} \phi_F(t^i) \bar{x}(t) + \sum_{i e \in N_e} (t^i + \phi_G(t^i)) \bar{x}(t) \right]
\]

and it holds that \( Rev(UBR^\ast) \geq Rev(\text{DSA}^\ast) \).
5.3 Significance of Relaxations

The two relaxations provide us an alternative to using the expected maximum welfare as an upper bound. To get a sense of the quality of these bounds, we calculated by simulation the revenue of UBR* and UBM* along with expected maximal welfare for different instances by means of simulation (more details on the technique are given in Section 6). The results are depicted in Figure 2 and 3. In any comparison between welfare-maximizing and other solutions, note that for a “neutral” case of optimizing the revenue from a linear demand function with uniform pricing, the optimal monopoly revenue is no more than half the welfare-maximizing value.

UBR denotes the upper bound with representatives, whilst UBM stands for the bound with public exclusivity margin. As the displayed revenue ratios are taken over the optimal welfare, it is apparent that both relaxation bounds are much tighter than the welfare upper bound. In particular, in Figure 2 UBM is around half of the optimal welfare. Furthermore, dependent on the support of the two valuations, the difference between UBR and UBM might be significant. In general, when exclusivity margins are small relative to shared valuations, UBM* is tighter, while UBR* is closer to the optimal revenue when exclusivity margins are relatively large.
Figure 2: Revenue over optimal welfare of the two upper bounds ($s^i \sim Exp(1), m^i \sim Exp(2)$).

Figure 3: Revenue over optimal welfare of the two upper bounds ($s^i \sim Exp(1), m^i \sim Exp(0.5)$).

UBM can as well be used to prove that MAXSIMPLE, introduced previously as a lower
bound, achieves a constant factor approximation of $DSA^*$ (see the appendix for proof). Note that, as stated earlier, the monopoly optimal revenue under a “neutral” demand setting (linear demand, zero marginal cost) is $\frac{1}{2}$ of optimal welfare, hence a heuristic that attains, in its worst case, $\frac{1}{4}$ of optimal welfare actually has a worst-case revenue performance of 50%.

**Theorem 6.** Take an instance of the exclusivity auction problem, where $F$ and $G$ are MHR distributions. Then expected revenue of $\text{MaxSimple}$ is at least
\[
\left(1 - \frac{1 - 1/H_n}{1 - 1/H_n}\right)
\] the expected revenue of the optimal mechanism. The minimum of this approximation ratio is $1/4$, attained in case of two bidders. Moreover, this approximation ratio holds for the optimal Bayes-Nash implementable mechanism as well.

### 6 Performance Analysis

This section describes and compares the performance of the discussed mechanisms across a spectrum of type distributions. The expected revenues are acquired via computational simulations in the following way. We sample from the type distribution many times and calculate the payment for each type report. The expected revenue is then estimated by the average of the simulated payments. To show the robustness of the results we also derive the 99% confidence intervals for each point estimate. The upper (lower) limit of the confidence interval is calculated as

\[
\beta + (-)t_\alpha \frac{\sigma}{\sqrt{n^*}}.
\] (12)

where $n^*$ is the number of simulations, $\beta$ is the average and $\sigma$ is the standard deviation of the simulated values, while $t_\alpha$ is Student’s $t$-distribution value for the given critical level $\alpha$ and degrees of freedom $n^* - 1$. For our experiments we set $\alpha = 99\%$ and $n^* = 20000$, therefore the corresponding t-value is $t_{99\%} \approx 2.6$. $\bar{a}$ and $\sigma$ are estimated from the samples for each simulation. For each setting we included the upper bound on the optimal revenue (UB), which is calculated as the minimum of $Rev(UBM^*)$ and $Rev(UBR^*)$. 

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Figure 4: Comparing the expected revenues of different mechanisms and the optimal welfare to the expected revenue of RM. Comparisons are made for different number of agents, different expected value of shared values and exclusivity margins ($exp_s$ and $exp_m$ respectively). The number of items is assumed to be 3 and that $s^i \sim \text{exponential}$, $m^i \sim \text{exponential}$.
Figure 5: Revenue to upper bound ratios \((s^i \sim U(0, 1), m^i \sim U(0, 1))\).

Figure 6: Revenue to upper bound ratios \((s^i \sim \text{Exp}(0.5), m^i \sim \text{Exp}(0.5))\).
Figure 7: Revenue to upper bound ratios ($s^i \sim Exp(0.5)$, $m^i \sim Exp(0.25)$).

Figure 8: Revenue to optimal welfare ratios ($s^i \sim Exp(0.5)$, $m^i \sim Exp(0.5)$).
7 Conclusion

This paper has developed results for a new kind of one-to-many matching auction format which is relevant for many of today’s platforms. Many platform applications already exist in a setting where one-to-many matches are possible (e.g., a single lead can be assigned to multiple interested sellers, or a single web surfer could be shown multiple simultaneous ads) but such matches are either done crudely or not at all because of the complexity in running such auctions. The complexity arises from the fact that some auction participants could potentially have very high value for exclusive purchase, and this causes the platform to have a method for choosing between exclusive vs. shared (i.e., multiple) allocation. The model and results developed in this paper can advance the practice of matching platforms through a proposed auction format that has high revenue performance, is easy for bidders to participate in and is predictable (because truth-telling is the dominant strategy for bidders).

Our proposed mechanisms and results have several implications for practice. From a practical perspective, the mechanism provides the matchmaking platform with an algorithm for computing allocations and prices after receiving participants’ bids as inputs. The platform owner requires only a reasonable prior about the distributions of valuations in order to configure the algorithm (specifically, by setting reserve prices). Once configured, the algorithm is very efficient at computing the actual allocations and prices for any particular configuration of bids. Moreover, as the platform owner obtains improved information about the distribution, the algorithm can efficiently be reconfigured with new reserve prices. Finally, the mechanism is extremely straightforward for bidders because the optimal equilibrium strategy for them is simply to bid their true valuations.

The choice of deterministic, ex-post individual rational and dominant strategy implementable mechanism might seem to be limiting. Indeed, it is folklore knowledge that in multi-dimensional settings randomization improves revenue and that the class of Bayes-Nash
implementable mechanisms allows more flexibility for the designer and admits different solving techniques (e.g., Cai et al., 2011). However, dominant truth-telling setting is highly relevant for two reasons: the deep need in practice for speed and simplicity, and (as we demonstrate) little loss in revenue from doing so. Most of the applications, especially lead marketing and sponsored search, involve real-time auctions where participants exhibit varied levels of sophistication. Therefore, both the auctioneer and bidders desire fast, predictable and simple mechanisms that require minimal user interaction and that make the available strategies clear for the users. Moreover, the dominant strategy implementation is prior-free, hence it is more robust from a practical perspective. In contrast, Bayes-Nash implementation requires a lot more of the participants: all of them should have the same prior about the type distributions, and they have to be able to compute expected utilities based on their actions. The usage of lotteries might be acceptable in certain applications, but it is not yet desired in everyday business. Paying more than your bid is likely to alienate users from a system as they may regret their participation in the auction, therefore ex-post individual rationality can be seen as a natural property of an applied mechanism. Moreover, our main theorem demonstrates that focusing on dominant strategy implementable, ex-post individual rational, simple therefore robust mechanisms costs only a small fraction of the optimal revenue. This means that the trade-off between optimality and simplicity is small.

Our research can benefit platforms that already practice one-to-many matching, and enables a new auction format for platforms that presently use one-to-one matching. For instance, television advertising has primarily been a one-to-one matching process (one time slot on a program sold to one advertiser), but modern technology allows a one-to-many match. Video serving platforms and publishers could display a static of 4 simultaneous video ads, waiting for the viewer to click on one. Television advertising is ripe for such transformation, owing to recent changes in consumption habits and consequent demand in
decline for traditional TV advertising. Publishers could therefore offer advertisers a choice between an exclusive sale and sharing an impression with a handful of other advertisers, while giving the viewer choice on which ad to click. Moreover, the underlying construct (multiple sales, when buyers have a margin for exclusive purchase) is applicable to many other information goods because of their non-rivalrous property. Higher value for exclusive purchase can be fueled by the threat of competition, by a sense of privilege, or by special customer preferences (e.g., luxury goods). For example, a prospector would derive higher value from exclusive possession of information regarding a natural resource repository. A retailer may perceive greater value from exclusive right to sell a good because it avoids competition with other shops. A newspaper advertiser could buy exclusive access on a page, or split the audience by purchasing a fraction of the advertising space. A computing platform can entice more users when a marqué software application is solely available on that platform and not on competing platforms. Mobile search advertisers may, due to the device’s limitations in display size and navigation, be willing to pay substantially more for exclusive promotion, and major search engines have examined designs where advertisers can place two-dimensional bids for receiving the click exclusively or shared (Sayedi, 2012; Jerath and Sayedi, 2012). Given such wide relevance of one-to-many matching, this paper contributes by creating insights and an implementable apparatus that can improve profits and welfare.

References


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Manelli, Alejandro M. and Daniel R. Vincent (2007). “Multidimensional mechanism design: Revenue maximization and the multiple-good monopoly”. In: Journal of Economic The-


A Technical Details and Proofs

<table>
<thead>
<tr>
<th>Notation</th>
<th>Meaning</th>
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<tbody>
<tr>
<td>$k$</td>
<td>Maximum number of sold items</td>
</tr>
<tr>
<td>$N = {1, \ldots n}$</td>
<td>Set of agents</td>
</tr>
<tr>
<td>$A$</td>
<td>Set of possible allocations</td>
</tr>
<tr>
<td>$S^i, M^i$</td>
<td>Set of shared valuations and exclusivity margins of agent $i$ respectively</td>
</tr>
<tr>
<td>$T^i = S^i \times M^i$</td>
<td>Set of types of agent $i$</td>
</tr>
<tr>
<td>$T = T^1 \times \ldots \times T^n$</td>
<td>Set of type profiles</td>
</tr>
<tr>
<td>$t^i = (s^i, m^i)$</td>
<td>Type of agent $i$</td>
</tr>
<tr>
<td>$t^{-i} = \times_{j \neq i}^j$</td>
<td>Type profile excluding the type of agent $i$</td>
</tr>
<tr>
<td>$F, G$</td>
<td>CDF of shared valuations and exclusivity margins respectively</td>
</tr>
<tr>
<td>$C$</td>
<td>CDF of the exclusive value, i.e., the convolution of $F$ and $G$</td>
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<tr>
<td>$\phi_D(t^i)$</td>
<td>Virtual valuation of agent $i$ for random variable $t^i$ distributed as $D$</td>
</tr>
<tr>
<td>$x : T \rightarrow {0, 1}^N$</td>
<td>Allocation rule</td>
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<td>$p : T \rightarrow \mathbb{R}^N$</td>
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<tr>
<td>$v^i(x(\hat{t}), t^i)$</td>
<td>Valuation of agent $i$ for allocation $x(\hat{t})$ having actual type $t^i$</td>
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<tr>
<td>$u^i(x(\hat{t}), p(\hat{t}), t^i)$</td>
<td>Utility of agent $i$ for allocation $x(\hat{t})$, payment $p(\hat{t})$ having actual type $t^i$</td>
</tr>
<tr>
<td>$SW_N(a \mid t)$</td>
<td>Total welfare of agents in $N$ for allocation $a$ and type profile $t$</td>
</tr>
<tr>
<td>$a(t) = {i : x^i(t) = 1}$</td>
<td>Set of agents who receive the item for report $t$</td>
</tr>
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</table>

Table 3: Summary of Notation

**Proof of Theorem 4.** (UBM) can be decomposed by treating all possible exclusivity margin profiles separately, because none of the constraints involve variables that are related to
different exclusivity margins, furthermore the probabilities are independent among agents and among the dimensions of their type. Therefore it is sufficient to solve the subproblems separately for each fixed $m$ and acquire the expected revenue as the expectation over the optimal objective values of the subproblems. From Proposition 6 in Deng and Pekeč, 2013 it follows that for fixed $m$ the allocation rule $x$ of the optimal DSIC and EPIR mechanism given type profile $t = (s, m)$, is

$$x^i(t) = 1 \text{ for } \begin{cases} i \in a_s(t) & \text{when } \sum_{j \in a_s(t)} \phi_F(s^j) \geq \phi_F(s_{ae}(t)) + m_{ae}(t) \\ i = a_e(t) & \text{when } \sum_{j \in a_s(t)} \phi_F(s^j) < \phi_F(s_{ae}(t)) + m_{ae}(t) \end{cases}$$

and $x^i(t) = 0$ otherwise. The expected revenue under this allocation rule is equal to

$$E_s \left[ \sum_i p^i(s, m) \right] = E_s \left[ \sum_i \left( \phi_F(s^i) + m^i \prod_{j \neq i} (1 - x^j(s, m)) \right) x^i(s, m) \right].$$

Since 13 holds for any fixed $m$, it concludes 10. Finally, 11 follows from taking expectation over 14 with respect to the exclusivity margins.

Note that Deng and Pekeč, 2013 originally studies ex-post incentive compatible (EPIC) mechanisms, because their exclusivity margin is assumed to be a linear combination of the shared valuations of the other agents. In our case exclusive margins do not depend on the valuation of the other agents, hence the notions DSIC and EPIC coincide (for formal definition of EPIC see Definition 9).

Proof of Proposition 1.

We start with a lemma that uses (MON) to give insight in how an incentive compatible, ex-post individual rational mechanism $(x, p)$ for (DSA) assigns allocations and prices to types.

Lemma 1. Let $(x, p)$ be a feasible mechanism for (DSA). Let $i \in N$, $t^{-i} \in T^{-i}$, and assume
that $p^{i}(0,0,t^{-i})=0$. Then there exist $s^{*} \in \mathbb{R} \cup \{ \infty \}$ and $m^{*} \in \mathbb{R} \cup \{ \infty \}$, depending on $i$ and $t^{-i}$, such that

(1) For all $(s^{i},m^{i})$ such that $s^{i} < s^{*}$ and $s^{i} + m^{i} < s^{*} + m^{*}$, $x$ does not allocate the item to $i$, that is $x^{i}(s^{i},m^{i},t^{-i}) = 0$.

(2) For all $(s^{i},m^{i})$ such that $s^{i} > s^{*}$ and $m^{i} < m^{*}$, $x$ allocates the item shared to $i$, that is $x^{i}(s^{i},m^{i},t^{-i}) = 1$ and $|a(s^{i},m^{i},t^{-i})| > 1$.

(3) For all $(s^{i},m^{i})$ such that $s^{i} + m^{i} > s^{*} + m^{*}$ and $m^{i} > m^{*}$, $x$ allocates the item exclusive to $i$, that is $x^{i}(s^{i},m^{i},t^{-i}) = 1$ and $|a(s^{i},m^{i},t^{-i})| = 1$.

(4) If $x$ allocates no item to $i$ for some type $(s^{i},m^{i})$, then $p(s^{i},m^{i},t^{-i}) = 0$.

(5) If $x$ allocates the item shared to $i$ for some type $(s^{i},m^{i})$, then $p(s^{i},m^{i},t^{-i}) = s^{*}$.

(6) If $x$ allocates the item exclusive to $i$ for some type $(s^{i},m^{i})$, then $p(s^{i},m^{i},t^{-i}) = s^{*} + m^{*}$.

Proof. Fix agent $i$ and type profile $t^{-i}$. We begin by defining $(s^{*},m^{*})$ for four possible cases:

(a) If there exists no $(s^{i},m^{i})$ for which $x$ allocates either shared or exclusive to $i$, then let $m^{*} = s^{*} = \infty$.

(b) If there exists a $(s^{i},m^{i})$ for which $x$ allocates exclusively, but there is no type such that $x$ allocates shared including $i$, then

$$m^{*} = \inf\{m^{i} \mid \exists s^{i} \text{ such that } x \text{ allocates exclusive at } (s^{i},m^{i})\}$$

and

$$s^{*} = \inf\{s^{i} + m^{i} \mid \text{ } x \text{ allocates exclusive at } (s^{i},m^{i})\}.$$
(c) If there is no \((s^i, m^i)\) for which \(x\) allocates exclusive, but there is a type for which \(x\) allocates shared including \(i\), then \(m^* = \infty\) and

\[
s^* = \inf \{s^i | \exists m^i \text{ such that } x \text{ allocates shared at } (s^i, m^i) \}.
\]

(d) If there are types for both exclusive and shared allocations for \(i\), then

\[
m^* = \inf \{m^i | \exists s^i \text{ such that } x \text{ allocates exclusive at } (s^i, m^i) \}
\]

and

\[
s^* = \inf \{s^i | \exists m^i \text{ such that } x \text{ allocates shared at } (s^i, m^i) \}.
\]

As a visual aid, Figure 9 depicts the different allocation situations and the corresponding \((s^*, m^*)\) tuples.

![Figure 9: Allocation maps and the defined \((s^*, m^*)\) tuples as a function of \((s^i, m^i)\) for fixed \(i\) and \(t^{-i}\).](image-url)
Recall that (MON) says that for any two types \((s^i, m^i), (\hat{s}^i, \hat{m}^i)\) we have

\[
v^i(x(s^i, m^i, t^{-i}),(s^i, m^i)) - v^i(x(\hat{s}^i, \hat{m}^i, t^{-i}),(s^i, m^i)) + v^i(x(\hat{s}^i, \hat{m}^i, t^{-i}),(\hat{s}^i, \hat{m}^i)) - v^i(x(s^i, m^i, t^{-i}),(\hat{s}^i, \hat{m}^i)) \geq 0
\]

We prove first (1) to (3) for different cases. We start with the case where \(0 < m^* < \infty\) and \(0 < s^*\).

To see (1), observe that \(x\) cannot allocate shared at \((s^i, m^i)\) due to the definition of \(s^*\). If it allocates exclusive, we must have \(m^i \geq m^*\) by the definition of \(m^*\). Let \((\hat{s}^i, \hat{m}^i)\) be a type such that

\[
\hat{s}^i + \hat{m}^i > s^i + m^i,
\]

\[
\hat{m}^i < m^*
\]

\[
\hat{s}^i < s^*.
\]

By the definition of \(s^*\) and \(m^*\), \(x\) does not allocate an item to \(i\) at \((\hat{s}^i, \hat{m}^i)\). By (MON) for the two type \((s^i, m^i)\) and \((\hat{s}^i, \hat{m}^i)\), we get

\[
s^i + m^i - (\hat{s}^i + \hat{m}^i) \geq 0,
\]

which yields a contradiction. Therefore \(x^i(s^i, m^i, t^{-i}) = 0\)

To see (2), observe that \(x\) cannot allocate exclusive an item to \(i\) at \((s^i, m^i)\) by the definition of \(m^*\). Furthermore, the existence of a type as in (2) implies \(s^* < \infty\). Suppose \(x\) does not allocate any item at \((s^i, m^i)\). By the definition of \(s^*\) there exists \((\hat{s}^i, \hat{m}^i)\) such that \(\hat{s}^i < s^i\) and \(x\) allocates shared at \((\hat{s}^i, \hat{m}^i)\). Using again (MON) we get

\[
-s^i + \hat{s}^i \geq 0,
\]

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a contradiction. Therefore, \( x \) must allocate shared to \( i \) at \((s^i, m^i)\).

To see (3), observe first that the existence of such a type implies \( s^* < \infty \) and the assumption that \( m^* < \infty \) implies that there is a type for which \( x \) allocates exclusive to \( i \). As a next step, we show that for any \((s^i, m^i)\) with \( m^i > m^* \), \( x \) cannot allocate shared at \((s^i, m^i)\). Suppose it does, then let \((\tilde{s}^i, \tilde{m}^i)\) be a type such that \( \tilde{m}^i < m^i \) and \( x \) allocates exclusive at \((\tilde{s}^i, \tilde{m}^i)\). Using (MON), we get

\[
s^i - (s^i + m^i) + \tilde{s}^i + \tilde{m}^i - \tilde{s}^i \geq 0,
\]
yielding \( \tilde{m}^i \geq m^i \), a contradiction.

Now let \((s^i, m^i)\) as in (3). It remains to show that \( x \) cannot allocate nothing to \( i \) at this type. Suppose it does. Let \((\hat{s}^i, \hat{m}^i)\) be a type such that

\[
\hat{s}^i + \hat{m}^i < s^i + m^i
\]

\[
\hat{m}^i > m^*
\]

\[
\hat{s}^i > s^*
\]

By what we yet showed, \( x \) cannot allocate shared at this type. By using the same construction as in the proof of part (2), \( x \) can also not allocate nothing at this type. Therefore \( x \) must allocate exclusive at \((\hat{s}^i, \hat{m}^i)\). Invoking once more (MON) we get

\[
-(s^i + m^i) + \hat{s}^i + \hat{m}^i \geq 0,
\]
yielding a contradiction. Therefore \( x \) must allocate exclusive to \( i \) at \((s^i, m^i)\).

Next we consider the case where \( 0 < m^* < \infty \) and \( s^* = 0 \). For this case there is no type that satisfies (1), and we can focus on (2) and (3). We start with (2). Suppose \( m^i < m^* \), by definition of \( m^* \), \( x \) cannot allocate exclusive at \((s^i, m^i)\). If \( x \) allocates nothing at \((s^i, m^i)\),
let \((\hat{s}^i, \hat{m}^i)\) be a type with \(\hat{s}^i < s^i\) at which \(x\) allocates shared to \(i\). As in the proof of (2) before, we can show that this contradicts (MON). Next, consider (3). If \(x\) allocates shared at \((s^i, m^i)\), we can construct a contradiction as in the proof of (3) before. If \(x\) allocates nothing to \(i\) at \((s^i, m^i)\), we can use the same construction as in the second part of the proof of (3) before to yield a contradiction.

Next consider the case \(m^* = 0\). For this case, there is no type that satisfies (2), and we can focus on (1) and (3). First observe that \(x\) cannot allocate shared at any type \((s^i, m^i)\) with \(m^i > 0\). This follows from a similar construction as in the proof of (3) before.

To see (1), note that \(x\) cannot allocate exclusive by the definition of \(s^*\). Next suppose it allocates shared, then \(m^i = 0\). Let \((\hat{s}^i, \hat{m}^i)\) be a type such that

\[
\hat{s}^i + \hat{m}^i < s^* + m^*
\]

\[
\hat{m}^i > 0
\]

\[
\hat{s}^i > s^i
\]

Note that \(x\) cannot allocate anything at \((\hat{s}^i, \hat{m}^i)\). Thus (MON) yields

\[
s^i - \hat{s}^i \geq 0,
\]
a contradiction.

To see (3), note that if \(x\) allocates nothing at \((s^i, m^i)\) we can find a type \((\hat{s}^i, \hat{m}^i)\) for which \(\hat{s}^i + \hat{m}^i < s^i + m^i\) at which \(x\) allocates exclusive, yielding a contradiction to (MON). As \(m^i > 0\), a shared allocation is not an option either.

It remains the case \(m^* = \infty\), meaning that \(x\) does not allocate exclusive to \(i\), for any type \((s^i, m^i)\), given \(t^{-i}\). If in addition \(s^* = \infty\), no type exists satisfying (2) or (3), and no item is allocated for any type, showing that (1) holds trivially. Thus assume that \(s^* < \infty\).
We can focus on (1) and (2). Both follow from very similar arguments as in the previous cases, utilizing (MON). This finishes the proof of (1) to (3).

It remains to show (4), (5) and (6). First recall that for any two types that yield the same allocation the prices must be equal, otherwise the type with the higher price would have an incentive to report the type that yields a lower price. Therefore we will denote \( p_0 \), \( p_s \), and \( p_e \) the price that is charged for a non-, shared and exclusive allocation respectively.

As for (4), if the price \( i \) has to pay was strictly positive when getting no item, (EPIR) would be violated, if the price was strictly negative at such a type, a bidder of type \((0,0)\) would have an incentive to misreport this type.

As for (5), note that \( s^* < \infty \) as otherwise there is no type for which \( x \) allocates shared. Observe that by (EPIR) the price \( p_s \) cannot be strictly larger than \( s^* \). If \( s^* = 0 \), we are done as the price cannot be negative. If \( s^* > 0 \), and \( p_s < s^* \), consider a type \((s^i,0)\) with \( p_s < s^i < s^* \). By (2), \( x \) does not allocate an item at this type, and therefore this type would have an incentive to pretend to be of a type at which \( x \) allocates shared.

As for (6), note that \( m^* < \infty \) as otherwise there is no type at which \( x \) allocates exclusive. By (3) it follows that \( s^* + m^* = \inf \{ s^i + m^i \mid x \text{ allocates exclusive at } (s^i,m^i) \} \). Therefore, \( p_e \leq s^* + m^* \). If \( s^* + m^* = 0 \), we are done as the price cannot be negative. If \( s^* + m^* > 0 \) consider first the case \( s^* > 0 \). If \( p_e < s^* + m^* \), by (3), there is a type \((s^i,m^i)\) such that \( s^i + m^i > p_e \) at which \( x \) allocates nothing. This type would have an incentive to pretend to be of a type that allocates exclusively. Finally, consider the case \( s^* = 0 \). If \( p_e < m^* \), there is a type \((s^i,m^i)\) such that \( m^i > p_e \) at which \( x \) allocates shared. This type would have an incentive to pretend to be of a type that allocates exclusively, as it gets utility \( s^i \) if being truthful and \( s^i + m^i - p_e > s^i \) if not being truthful.

Now we are ready to prove Proposition 1. Let \((x,p)\) feasible for (DSA). First we may assume that \((x,p)\) satisfies the conditions of Lemma 2, as the payment of an agent \( i \) with type \((0,0)\), given any type of the other agents, cannot be larger than 0 by (EPIR), and if it
was smaller than 0, we can change the payment of this agent by a positive amount, thereby not decreasing the revenue of the mechanism. Given this assumption, we construct a truthful \((\overline{x}, \overline{p})\). Take any type \(t\) and any agent \(i\).

Suppose there exists no type \((s^i, m^i)\) such that \(a(s^i, m^i, t^{-i}) = \{i\}\), that is, \(m^* = \infty\). Then assign an item to \(i_s\) if and only if \(x\) assigns an item to \(i\). Assign no item to \(i_e\). Set the payment for \(i_s\) equal to \(s^*\) if \(i_s\) gets an item, 0 otherwise. Set the payment for \(i_e\) equal to 0.

Suppose there exists \((s^i, m^i)\) such that \(a(s^i, m^i, t^{-i}) = \{i\}\). Let \(m^*\) be as defined in Lemma 2. If \(m^i \neq m^*\), assign an item to \(i_s\) if and only if \(x\) assigns shared to \(i\) and assign an item to \(i_e\) if and only if \(x\) assigns exclusive to \(i\). If \(x\) assigns nothing to \(i\), then neither assign an item to \(i_s\) nor to \(i_e\). Set the price for \(i_s\) equal to \(s^*\) if \(i_s\) is assigned an item, 0 otherwise. For \(i_e\), let the price be \(s_i + \inf\{m_i \mid x\) assigns exclusive at \((s^i, m^i, t^{-i})\}\) if \(i_e\) gets an item, and 0 otherwise. If \(m^i = m^*\), assign neither an item to \(i_e\) nor to \(i_s\), and set the price for both agents equal to 0. Thereby note that by Lemma 2, for all \(s^i\) the price that \(i_e\) pays when receiving the item is larger than or equal to \(s^* + m^*\).

Note that \(p^i(t) \leq \overline{p}^i(s^i) + \overline{p}^e(s^i)\) except for \(t^i\) such that \(m^i = m^*\). Therefore the expected revenue from agent \(i\) satisfies

\[
\mathbb{E}_{t^{-i}} \left[ \mathbb{E}_{t^i} \left[ p(t^i, t^{-i}) \right] \right] \leq \mathbb{E}_{t^{-i}} \left[ \mathbb{E}_{(s^i, m^i)} \left[ \overline{p}^i(s^i, m^i, t^{-i}) + \overline{p}^e(s^i, m^i, t^{-i}) \right] \right]
\]

Indeed, the argument of the inner expected value on the left hand side is, except for a null set, smaller than or equal to the corresponding argument on the right hand side.

By Lemma 2, it is immediate that, given any type of the other other agents, the payment for \(i_s\) and \(i_e\), respectively, if winning the item equals the infimum over all valuations of that agent, such that she wins the item. It is well known, but can also easily be verified along the lines of the proof of Lemma 2, that such a a payment rule satisfies (EPIR) and (EPIC).

FROM HERE IS THE OLD PROOF OF GREG. DO NOT TOUCH UNTIL WE ARE
CONVINCED OF THE NEW ONE.

Before proceeding to the proof we recall a well-known lemma on the pricing scheme of any DSIC mechanism, here stated in the context of DSA∗. The lemma says that for the same allocation an agent has to be charged the same amount regardless of his bid, if the bids of the others kept unchanged. Together with ex-post individual rationality this implies an intuitive, but non-trivial upper bound for each payment.

Lemma 2. Let \((x, p)\) be a feasible mechanism for (DSA). For given \(i, t^{-i}, t^{i}\) and \(\hat{t}^{i}\), having \(x(t^{i}, t^{-i}) = x(\hat{t}^{i}, t^{-i})\) implies that \(p^{i}(t^{i}, t^{-i}) = p^{i}(\hat{t}^{i}, t^{-i})\). Moreover, for given \(i, t^{-i}, t^{i}\) we have that
\[
p^{i}(t^{i}, t^{-i}) \leq \inf_{\hat{t}^{i}} \{v^{i}(x(\hat{t}^{i}, t^{-i}), \hat{t}^{i}) \mid x(\hat{t}^{i}, t^{-i}) = x(t^{i}, t^{-i})\}.
\]

Proof. Follows directly from (DSIC) and ex-post individual rationality.

Regarding the allocation rule the following lemma states that the higher your valuation is, the more chance you have to receive the item. The statements are grouped based on which dimension of the type is changed. Fix agent \(i\), the type of the others \(t^{-i}\) and the allocation rule \(x\). Let \(R_{e}(x, t^{-i}) \subseteq T^{i}\), \(R_{s}(x, t^{-i}) \subseteq T^{i}\) be the set of types such that the allocation rule assigns the item to agent \(i\) exclusively and shared respectively. Similarly, \(R_{0}(x, t^{-i}) \subseteq T^{i}\) is the set of types such that agent \(i\) gets nothing.

Lemma 3. Let \(x\) be a monotone allocation rule and fix agent \(i\) and the report of the others \(t^{-i}\). Then the following two statements hold:

I/ For all \(s^{i}, m^{i}, \hat{m}^{i}\) such that \(\hat{m}^{i} > m^{i}\), we have that if \((s^{i}, m^{i}) \in R_{e}(x, t^{-i})\), then \((s^{i}, \hat{m}^{i}) \in R_{e}(x, t^{-i})\).

II/ There is an \(m^{*} \in M^{i} \cup \{\infty\}\) such that for all \(s^{i}, m^{i}, \hat{s}^{i}\), where \(\hat{s}^{i} > s^{i}\) and \(m^{i} \neq m^{*}\), we have that if \((s^{i}, m^{i}) \in R_{s}(x, t^{-i})\), then \((\hat{s}^{i}, m^{i}) \in R_{s}(x, t^{-i})\).
Proof. Assume that the first statement is not true, that is, \((s^i, \hat{m}^i) \in R_0(x, t^{-i})\) or \((s^i, \hat{m}^i) \in R_\bar{s}(x, t^{-i})\). Applying (MON) for both cases with \(t^i = (s^i, m^i)\) and \(\hat{t}^i = (s^i, \hat{m}^i)\) results in

\[
0 \geq \hat{m}^i - m^i.
\]

This is a contradiction as \(\hat{m}^i > m^i\) by assumption.

For the second statement define \(m^*\) as the infimum of exclusivity margins such that agent \(i\) is allocated exclusively, i.e.,

\[
m^* = \inf\{\bar{m}^i \mid (\bar{s}^i, \bar{m}^i) \in R_e(x, t^{-i})\}.
\] (15)

Now, assume the contrary of the second statement, that is, there are \(s^i, m^i, \check{s}^i\) such that \(\check{s}^i > s^i\), \(m^i \neq m^*\), \((s^i, m^i) \in R_\bar{s}(x, t^{-i})\) and we have that \((\check{s}^i, m^i) \in R_0(x, t^{-i})\) or \((\check{s}^i, m^i) \in R_\bar{e}(x, t^{-i})\). First tackle the case when \((\check{s}^i, m^i) \in R_0(x, t^{-i})\). Applying (MON) with \(t^i = (s^i, m^i)\) and \(\hat{t}^i = (\check{s}^i, m^i)\) results in

\[
0 \geq \check{s}^i - s^i.
\]

This is a contradiction as \(\check{s}^i > s^i\) by assumption. Finally, assume that \((\check{s}^i, m^i) \in R_\bar{e}(x, t^{-i})\). Note that due to the definition of \(m^*\) this assumption implies that \(m^i > m^*\) and that there exists a \((\bar{s}^i, \bar{m}^i)\) such that \(\bar{m}^i < m^i\) and \((\bar{s}^i, \bar{m}^i) \in R_\bar{e}(x, t^{-i})\). Applying (MON) with \(t^i = (s^i, m^i)\) and \(\hat{t}^i = (\check{s}^i, \check{m}^i)\) results in

\[
\check{s}^i + \check{m}^i - s^i - m^i \geq \check{s}^i - s^i.
\]

This is a contradiction as \(m^i > \check{m}^i\) by assumption.

Now, we are ready to prove directly Proposition 1. By construction we have that for every \(i \in N\) there is a pair \((i_s, i_e) \in \overline{N}\), and for every \(t \in T\) there is a \(\overline{t} \in \overline{T}\) such that \(t^i = (s^i, m^i) = \ldots\)

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As in (15). Now, set $t^{i} = (s^{i}, m^{i})$. Define $m^{*}$ as in (15). Let $t = (t^{i}, t^{-i})$, where $t^{i} = (s^{i}, m^{i})$. Fix agent $i$ and for agent $i$ to agent $t$ if and only if $t^{i} \in R_{x}(x, t^{-i})$, and set $\pi^{i}(t)$ = 1 if and only if $t^{i} \in R_{x}(x, t^{-i})$. In words, whenever $x$ allocates shared to agent $i$, then $\pi$ gives the item to representative $i_{s}$ provided that $m^{i} \neq m^{*}$. Furthermore, if $x$ allocates exclusively to agent $i$, then $\pi$ gives the item to representative $i_{e}$. It is easy to see that $\pi$ satisfies (FeasA) and (FeasB).

Next, let us define the payment rule $p$ for agent $i_{s} \in N_{s}$ as

$$p^{i_{s}}(\hat{t}^{i_{s}}, \hat{t}^{-i_{s}}) = \begin{cases} \inf \{ \hat{t}^{i_{s}} \mid \pi^{i_{s}}(\hat{t}^{i_{s}}, \hat{t}^{-i_{s}}) = 1 \} & \text{if } \pi^{i_{s}}(\hat{t}^{i_{s}}, \hat{t}^{-i_{s}}) = 1 \\ 0 & \text{otherwise.} \end{cases}$$ (16)

and for agent $i_{e} \in N_{e}$ as

$$p^{i_{e}}(\hat{t}^{i_{e}}, \hat{t}^{i_{s}}, \hat{t}^{-(i_{s}, i_{e})}) = \begin{cases} \inf \{ \hat{t}^{i_{e}} \mid \pi^{i_{e}}(\hat{t}^{i_{e}}, \hat{t}^{i_{s}}, \hat{t}^{-(i_{s}, i_{e})}) = 1 \} + \hat{t}^{i_{e}} & \text{if } \pi^{i_{e}}(\hat{t}^{i_{e}}, \hat{t}^{-(i_{s}, i_{e})}) = 1 \\ 0 & \text{otherwise.} \end{cases}$$ (17)

To see incentive compatibility, consider agent $i_{e}$ and fix $(\hat{t}^{i_{s}}, \hat{t}^{-(i_{s}, i_{e})})$. If there is no $\hat{t}^{i_{e}}$ such that $\pi^{i_{e}}(\hat{t}^{i_{e}}, \hat{t}^{i_{s}}, \hat{t}^{-(i_{s}, i_{e})}) = 1$, then (EPIC) and (EPIR2) trivially hold. Otherwise let $\underline{m} = \inf \{ \hat{t}^{i_{e}} \mid \pi^{i_{e}}(\hat{t}^{i_{e}}, \hat{t}^{i_{s}}, \hat{t}^{-(i_{s}, i_{e})}) = 1 \}$. Note that Lemma 3 ensures that $\pi^{i_{e}}(\hat{t}^{i_{e}}, \hat{t}^{i_{e}}, \hat{t}^{-(i_{s}, i_{e})})$ is non-decreasing in $\hat{t}^{i_{e}}$ for all $\hat{t}^{i_{e}}$. This implies that $\pi^{i_{e}}(\hat{t}^{i_{e}}, \hat{t}^{i_{e}}, \hat{t}^{-(i_{s}, i_{e})}) = 1$ for all $\hat{t}^{i_{e}} > \underline{m}$, and $\pi^{i_{e}}(\hat{t}^{i_{e}}, \hat{t}^{i_{e}}, \hat{t}^{-(i_{s}, i_{e})}) = 0$ for all $\hat{t}^{i_{e}} < \underline{m}$. It is easy to see that this allocation rule together with payment $p^{i_{e}}(\hat{t}^{i_{e}}, \hat{t}^{i_{s}}, \hat{t}^{-(i_{s}, i_{e})}) = \underline{m} + \hat{t}^{i_{e}}$ satisfy (EPIC) and (EPIR2). With respect to the comparison of the payments we can observe the following. When $x(t)$ allocates exclusively
to agent \(i\), then

\[
p^i(s^i, m^i, t^{-i}) \leq \inf \{ v^i(x(s^i, \hat{m}^i, t^{-i}), (\hat{s}^i, \hat{m}^i)) \mid x(s^i, \hat{m}^i, t^{-i}) = x(s^i, m^i, t^{-i}) \}
\]

\[
\leq \inf \{ \bar{t}^{i_s} + \hat{t}^{i_e} \mid \bar{x}^{i_s}(\hat{t}^{i_s}, \bar{t}^{i_e}, \bar{t}^{-(i_s,i_e)}) = 1, \bar{t}^{i_s} = s^i \}
\]

\[
= \bar{p}^{i_s}(\bar{t}^{i_s}, \bar{t}^{i_e}, \bar{t}^{-(i_s,i_e)}).
\]

The inequalities follow from Lemma 2 and from the fact that the second infimum is taken over a smaller set. This means that in case of exclusive allocation \(\bar{p}^{i_s}(\bar{t}) + \bar{p}^{i_e}(\bar{t}) \geq p^i(t)\).

Finally, consider agent \(i\) and fix \((\bar{t}^{i_s}, \bar{t}^{i_e}, \bar{t}^{-(i_s,i_e)})\). If \(\bar{t}^{i_e} = m^*\) or there is no \(\bar{t}^{i_s}\) such that \(\bar{x}^{i_s}(\bar{t}^{i_s}, \bar{t}^{i_e}, \bar{t}^{-(i_s,i_e)}) = 1\), then \(\bar{x}^{i_s}(\bar{t}^{i_s}, \bar{t}^{i_e}, \bar{t}^{-(i_s,i_e)}) = 0\) for all \(\bar{t}^{i_s}\), therefore (EPIC) and (EPIR2) trivially hold. Otherwise assume that \(\bar{t}^{i_e} \neq m^*\) and let \(s = \inf \{ \tilde{t}^{i_s} \mid \bar{x}^{i_s}(\tilde{t}^{i_s}, \bar{t}^{i_e}, \bar{t}^{-(i_s,i_e)}) = 1 \}\). Note that Lemma 3 ensures that \(\bar{x}^{i_s}(\bar{t}^{i_s}, \bar{t}^{i_e}, \bar{t}^{-(i_s,i_e)})\) is non-decreasing in \(\bar{t}^{i_s}\) for all \((\bar{t}^{i_s}, \bar{t}^{i_e}, \bar{t}^{-(i_s,i_e)})\).

As a result we have that \(\bar{x}^{i_s}(\bar{t}^{i_s}, \bar{t}^{i_e}, \bar{t}^{-(i_s,i_e)}) = 1\) for all \(\bar{t}^{i_s} > s\) and \(\bar{x}^{i_s}(\bar{t}^{i_s}, \bar{t}^{i_e}, \bar{t}^{-(i_s,i_e)}) = 0\) for all \(\bar{t}^{i_s} < s\). It is easy to see that this allocation rule together with payment \(\bar{p}^{i_s}(\bar{t}^{i_s}, \bar{t}^{i_e}, \bar{t}^{-(i_s,i_e)}) = s\) satisfy (EPIC) and (EPIR2). With respect to the comparison of the payments we can observe the following. When \(x(t)\) allocates shared to agent \(i\) and \(\bar{t}^{i_e} \neq m^*\), then

\[
p^i(s^i, m^i, t^{-i}) \leq \inf \{ v^i(x(s^i, \hat{m}^i, t^{-i}), (\hat{s}^i, \hat{m}^i)) \mid x(s^i, \hat{m}^i, t^{-i}) = x(s^i, m^i, t^{-i}) \}
\]

\[
\leq \inf \{ \tilde{t}^{i_s} \mid \bar{x}^{i_s}(\tilde{t}^{i_s}, \bar{t}^{i_e}, \bar{t}^{-(i_s,i_e)}) = 1, \tilde{t}^{i_s} = m^i \}
\]

\[
= \bar{p}^{i_s}(\tilde{t}^{i_s}, \bar{t}^{i_e}, \bar{t}^{-(i_s,i_e)}).
\]

The inequalities follow from Lemma 2 and from the fact that the second infimum is taken over a smaller set. This means that in case of shared allocation \(\bar{p}^{i_s}(\bar{t}) + \bar{p}^{i_e}(\bar{t}) \geq p^i(t)\).

Combining the results of all subcases implies that for any feasible solution \((x, p)\) for
(DSA) there is a feasible \((\overline{x}, \overline{p})\) for (UBR) such that

\[
\mathbb{E}_{t \in T} \left[ \sum_{i \in N} p^j(t) \right] = \sum_{i \in N} \mathbb{E}_{t \in T} \left[ \sum_{(s^i, m^i) \in T^i} \left[ p^j(s^i, m^i, t^{-i}) \right] \right]
\]

\[
= \sum_{i \in N} \mathbb{E}_{t \in T} \left[ \sum_{(s^i, m^i) \in T^i, m^i \neq m^*} \left[ p^j(s^i, m^i, t^{-i}) \right] \right]
\]

\[
\leq \sum_{i \in N} \mathbb{E}_{t \in T} \left[ \sum_{(s^i, m^i) \in T^i, t^{-i} = m^*} \left[ p^j(s^i, m^i, t^{-i}) \right] \right]
\]

\[
= \sum_{i \in N} \mathbb{E}_{t \in T} \left[ \sum_{(s^i, t^{-i}) \in T^i, t^{-i} = m^*} \left[ p^j(s^i, t^{-i}) \right] \right]
\]

The second and third equalities hold because type distributions are continuous, hence the measure of the type set where \(t^{-i} = m^*\) is zero. The inequality follows from the results of the previous subcases. This concludes the proof of Proposition 1.

**Proof of Theorem 5.** Roughgarden and Talgam-Cohen, 2013 extends the results of Myerson, 1981 to settings with informational externalities and correlated type distributions for mechanism that are ex-post incentive compatible and ex-post individual rational. In particular, it is shown that the expected revenue of any EPIC and EPIR mechanism equals to expected sum of virtual valuations provided that the payments are maximal, i.e., the utility of an agent with zero type is zero. Moreover, the revenue maximizing mechanism allocates such that the sum of virtual valuations is maximal for each type profile, given that the resulting allocation is monotone non-decreasing for each agent in their own type. Note that the virtual valuation of agent \(i_e\) having type \(\overline{t}^{-i_e}\) is \(\overline{t}^{-i_e} + \phi_G(\overline{t}^{-i_e})\) due to the informational externality. Having all these in mind the only thing left to show is that all agent’s probability of receiving the item is monotone non-decreasing in their reported types.

As \(F\) has monotone hazard rate, we have that \(\phi_F(\overline{t}^{-i_e})\) is monotone non-decreasing in \(\overline{t}^{-i_e}\)
for all $i_s \in N_s$. Therefore if $i_s \in \overline{a}_s(t^{i_s}, t^{-i_s})$, then $i_s \in \overline{a}_s(\hat{t}^{i_s}, t^{-i_s})$ for any $\hat{t}^{i_s} > t^{i_s}$. Similarly, as $G$ has monotone hazard rate, we have that $\phi_G(\overline{t}^{i_e})$ is also non-decreasing in $\overline{t}^{i_e}$ for all $i_e \in N_e$. This ensures that once $i_e \in \overline{a}_e(t^{i_e}, t^{-i_e})$, then $i_e \in \overline{a}_e(\hat{t}^{i_e}, t^{-i_e})$ also for any $\hat{t}^{i_e} > t^{i_e}$.

The last critical point is that when any agent $i_s \in N_s$ increases $t^{i_s}$, because then both $\phi_F(t^{i_s})$ and $t^{i_s} + \phi_G(t^{i_e})$ increase at the same time. To see why it is not a problem observe that the MHR assumption ensures that $\phi_F(t^{i_s})$ increases at least as fast as $t^{i_s} + \phi_G(t^{i_e})$ in $t^{i_s}$.

**Proof of Theorem 6.**

In order to prove this Theorem we need to elaborate first in more detail on the VCG auction and prove a couple of results. Let $SW_N(a \mid t) = \sum_{i \in N} v^i(a, t^i)$ denote the welfare of agents in $N$ for allocation $a$ and type profile $t$.

**Definition 10 (VCG mechanism).** Let $A_N$ be the set of feasible allocations for agents in $N$ and let $a(t) \in A_N$ be an allocation for each type profile $t$ such that

$$a(t) \in \arg \max_{a \in A_N} SW_N(a \mid t).$$

Similarly, let $A_{N \setminus \{i\}}$ be the set of feasible allocations for agents in $N \setminus \{i\}$ and let $a(\{t\}) \in A_{N \setminus \{i\}}$ be an allocation for each type profile $\{t\}$ such that

$$a(\{t\}) \in \arg \max_{a \in A_{N \setminus \{i\}}} SW_{N \setminus \{i\}}(a \mid \{t\}).$$

Then for each $t$ VCG chooses $a(t)$ as allocation and elicits payment

$$p^i(t) = \sum_{j \neq i} v^j(a(\{t\}), t^j) - \sum_{j \neq i} v^j(a(t), t^j). \quad (18)$$

VCG allocates efficiently, i.e., maximizes the total welfare pointwise, moreover, it is EPIR. The following Lemma helps computing its expected revenue.

**Lemma 4.** Let $N = \{1, \ldots, n\}$ be the set of agents with i.i.d. types, and let $Q = \{1, \ldots, n - 1\}$ denote the same set of agents with one less member. Let $T_N = \times_{i \in N} T^i$ denote the set of possible type profiles of agents in $N$ and, similarly let $T_Q = \times_{i \in Q} T^i$ be the set of possible
type profiles of $n − 1$ agents. Then

$$\text{Rev}(VCG) = n \mathbb{E}_{t \in T_Q} [SW_Q(a(t) \mid t)] - (n - 1) \mathbb{E}_{t \in T_N} [SW_N(a(t) \mid t)] \quad (19)$$

**Proof.** According to (18) the expected value of the VCG payments can be written as

$$\text{Rev}(VCG) = \mathbb{E}_{t \in T_N} \left[ \sum_{i \in N} \left( \sum_{j \in N \setminus \{i\}} v^j(a(t^{-i}), t^j) - \sum_{j \in N \setminus \{i\}} v^j(a(t), t^j) \right) \right]$$

$$= \mathbb{E}_{t \in T_N} \left[ \sum_{i \in N} \sum_{j \in N \setminus \{i\}} v^j(a(t^{-i}), t^j) \right] - \mathbb{E}_{t \in T_N} \left[ \sum_{i \in N} \sum_{j \in N \setminus \{i\}} v^j(a(t), t^j) \right]$$

$$= \sum_{i \in N} \mathbb{E}_{t^{-i} \in T_N} \left[ \mathbb{E}_{t \in T_N} \left[ \sum_{j \in N \setminus \{i\}} v^j(a(t^{-i}), t^j) \right] \right] - (n - 1) \mathbb{E}_{t \in T_N} [SW_N(a(t) \mid t)]$$

$$= \sum_{i \in N} \mathbb{E}_{t^{-i} \in T_N} \left[ \sum_{j \in Q} v^j(a(t^{-i}), t^j) \right] - (n - 1) \mathbb{E}_{t \in T_N} [SW_N(a(t) \mid t)]$$

$$= n \mathbb{E}_{t \in T_Q} \left[ \sum_{j \in Q} v^j(a(t), t^j) \right] - (n - 1) \mathbb{E}_{t \in T_N} [SW_N(a(t) \mid t)]$$

$$= n \mathbb{E}_{t \in T_Q} [SW_Q(a(t) \mid t)] - (n - 1) \mathbb{E}_{t \in T_N} [SW_N(a(t) \mid t)].$$

The equalities are direct consequences of the assumption that types are i.i.d.. \hfill \Box

**Corollary 1.** If

$$\frac{\mathbb{E}_{t \in T_Q} [SW_Q(a(t) \mid t)]}{\mathbb{E}_{t \in T_N} [SW_N(a(t) \mid t)]} \geq 1 - \rho,$$

then the expected revenue of VCG is at least $1 - n \rho$ times the expected optimal welfare.

The message of Corollary 1 is that if one wants to compare $\text{Rev}(VCG)$ to the optimal expected welfare, then it is sufficient to know the added value of an extra agent to the welfare. Let $H_i = \sum_{j=1}^i 1/j$ represent the $i^{th}$ Harmonic number and set $H_0 = 0$. The next lemma is useful for providing lower bounds on the revenue-welfare ratio.
Lemma 5 (Lemma 3 from Roughgarden and Sundararajan, 2007). Draw independently \( n \) times from an MHR distribution. Then the expected value of the \( l^{th} \)-largest value of \( n \) samples is at least \((H_n - H_{l-1})/(H_{n+j} - H_{l-1})\) times that of the \( l^{th} \)-largest value of \( n + j \) samples.

Theorem 7. Consider the single-item auction problem with \( n \geq 2 \) agents who have unit demand and single-dimensional valuations i.i.d. according to an MHR distribution. Then VCG extracts at least \( 1 - 1/H_n \) fraction of the optimal welfare in terms of expected revenue.

Proof. Let \( N = \{1, \ldots, n\} \) be the set of agents with i.i.d. types, and let \( Q = \{1, \ldots, n-1\} \) denote the same set of agents with one less member. Furthermore, denote the \( l^{th} \) largest value from \( n \) samples by \( v_{l:n} \). Lemma 5 implies that

\[
\mathbb{E} [v_{l:n-1}] \geq (H_{n-1} - H_{l-1})/(H_n - H_{l-1}) \mathbb{E} [v_{l:n}] .
\]

Therefore we have that

\[
\frac{\mathbb{E}_{t \in T_Q} [SW_Q(x(t) \mid t)]}{\mathbb{E}_{t \in T_N} [SW_N(x(t) \mid t)]} = \frac{\mathbb{E} [v_{1:n-1}]}{\mathbb{E} [v_{1:n}]} \geq \frac{(H_{n-1}/H_n) \mathbb{E} [v_{l:n}]}{\mathbb{E} [v_{l:n}]} = 1 - \frac{1}{nH_n}.
\]

The proof is concluded by invoking Corollary 1 and letting \( \varrho = \frac{1}{nH_n} \).

We note that according to Theorem 4 of Roughgarden and Sundararajan, 2007 the ratio of the VCG revenue to the optimal welfare is at least \( 1 - 1/n \) for monotone hazard rate distributions. Their result is apparently not precise as our bound is tight for exponential distributions and \( 1 - 1/H_n \) is generally lower than \( 1 - 1/n \).

Now we are ready to proof Theorem 6
According to Theorem 4 $\text{Rev}(\text{UBM}^*)$ is an upper bound on $\text{Rev}(\text{DSA}^*)$, therefore it is sufficient to show that $\text{Rev}(\text{MAXSIMPLE})$ approximates $\text{Rev}(\text{UBM}^*)$. Let $x_{\text{UBM}^*}$ represent the allocation rule of $\text{UBM}^*$. Define $T_{\text{UBM}^*}^e = \{ t \in T \mid \exists i : x_{\text{UBM}^*}^i(t) = 1, \forall j \neq i : x_{\text{UBM}^*}^j(t) = 0 \}$ and $T_{\text{UBM}^*}^s = \{ t \in T \mid \exists i, j \neq i : x_{\text{UBM}^*}^i(t) = x_{\text{UBM}^*}^j(t) = 1 \}$. Then according to Theorem 4 the revenue of $\text{UBM}^*$ can be split such that $\text{Rev}(\text{UBM}^*) = \text{Rev}(\text{UBM}^*)_s + \text{Rev}(\text{UBM}^*)_e$, where

$$
\text{Rev}(\text{UBM}^*)_s = \mathbb{E}_{t \in T_{\text{UBM}^*}^s} \left[ \sum_i \phi_F(s^i) x_{\text{UBM}^*}^i \right] \tag{20}
$$

and

$$
\text{Rev}(\text{UBM}^*)_e = \mathbb{E}_{t \in T_{\text{UBM}^*}^e} \left[ \sum_i (\phi_F(s^i) + m^i) x_{\text{UBM}^*}^i \right]. \tag{21}
$$

The idea of the proof is to bound the two terms separately. We start with $\text{Rev}(\text{UBM}^*)_e$.

As $\phi_F(s^i) + m^i \leq s^i + m^i$ for all $i$, we have that $\text{Rev}(\text{UBM}^*)_e$ is less than or equal to the optimal welfare of a single-item auction. Note that OE achieves at least as much expected revenue as VCG does for the single-item auction. Therefore due to Theorem 7 $\text{Rev}(\text{OE})$ is at least $1 - 1/H_n$ times the optimal welfare of a single-item auction. This leads to the conclusion that $\text{Rev}(\text{OE}) \geq (1 - 1/H_n) \text{Rev}(\text{UBM}^*)_e$.

To bound $\text{Rev}(\text{UBM}^*)_s$ observe that whenever $\text{UBM}^*$ allocates shared for type report $t$, then $x_{\text{UBM}^*}(t) = x_{\text{OS}}(t)$. This is due to the fact $a_s(t)$ is defined the same way for both
mechanism. Using this observation together with Lemma 3 we can write

\[
\text{Rev}(\text{OS}) = \mathbb{E}_t \left[ \sum_i \phi_F(s^i)x_{\text{OS}}^i(t) \right] = \mathbb{E}_{t \in \mathbb{T}_{\text{UBM}}} \left[ \sum_i \phi_F(s^i)x_{\text{OS}}^i(t) \right] + \mathbb{E}_{t \notin \mathbb{T}_{\text{UBM}}} \left[ \sum_i \phi_F(s^i)x_{\text{OS}}^i(t) \right]
\]

\[
= \text{Rev}(\text{UBM}^*)_s + \mathbb{E}_{t \notin \mathbb{T}_{\text{UBM}}} \left[ \sum_i \phi_F(s^i)x_{\text{OS}}^i(t) \right] \geq \text{Rev}(\text{UBM}^*)_s.
\]

The last inequality holds because under \text{OS} allocation only occur when the sum of virtual valuations is non-negative. Putting together the two bounds results in

\[
\frac{1}{1 - 1/H_n} \text{Rev}(\text{OE}) + \text{Rev}(\text{OS}) \geq \text{Rev}(\text{UBM}^*)_s + \text{Rev}(\text{UBM}^*)_e = \text{Rev}(\text{UBM}^*) \geq \text{Rev}(\text{DSA}^*).
\]

To finish the proof note that \( \text{Rev}(\text{MAXSIMPLE}) = \max\{\text{Rev}(\text{OS}), \text{Rev}(\text{OE})\} \), hence

\[
\left( \frac{1}{1 - 1/H_n} + 1 \right) \text{Rev}(\text{MAXSIMPLE}) \geq \text{Rev}(\text{DSA}^*).
\]

Finally, note that we can replace DSIC to Bayes-Nash incentive compatibility in (DSA) and relax the resulting mathematical program by letting the exclusivity margin public information as in (UBM). As we arrive at a single-dimensional setting it is folklore knowledge that the optimal Bayes-Nash mechanism is DSIC. This means that \text{UBM}^* is optimal even among Bayes-Nash implementable mechanisms, therefore it is also an upper bound for the Bayesian relaxation of (DSA). \qed