Conductor ideals of affine monoids and $K$-theory

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Outline

- Frobenius number of a numerical semigroup
- Affine monoid, normalization, seminormalization
- Conductor ideals & gaps in affine monoids
- Crash course in $K$-theory
- Affine monoid rings and their $K$-theory
- Nilpotence of higher $K$-theory of toric varieties
- Conjecture
Frobenius number of a numerical semigroup

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Computing the Frobenius number of $g(a_1, \ldots, a_n)$ of $\mathbb{Z}_{\geq 0}a_1 + \cdots + \mathbb{Z}_{\geq 0}a_n$ is **hard**. The only value of $n$ for which there is a formula is $n = 2$:

$$g(a_1, a_2) = a_1a_2 - a_1 - a_2$$
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Huge existing literature – *Postage Stamp Problem*, *Coin Problem*, *McNugget Problem (special case)*, *Arnold Conjecture (on asymptotics of \( g(a_1, \ldots, a_n) \))*, etc.
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A (positive) affine monoid $M$ is seminormal if

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The normalization of \( M \) is the smallest normal submonoid \( \widetilde{M} \subset \mathbb{Z}^d \) containing \( M \), i.e., \( \widetilde{M} = C(M) \cap \text{gp}(M) \) – ‘saturation’ of \( M \)
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The seminormalization of $M$ is the smallest seminormal submonoid $\text{sn}(M) \subset \mathbb{Z}^d$ containing $M$, i.e.,

$$\bar{M} = \{ x \in \mathbb{Z}^d \mid 2x, 3x \in M \}$$

– ‘saturation’ of $M$ along the rational rays inside the cone $C(M)$
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FACT. $\bar{M} \cap \text{int} C(M) = \text{sn}(M) \cap \text{int} C(M)$
Conductor ideals & gaps in affine monoids

The conductor ideal of an affine monoid $M$ is

$$c_{\tilde{M}/M} := \{ x \in \tilde{M} \mid x + \tilde{M} \subset M \} \subset M$$

It is an ideal of $M$ because $c_{\tilde{M}/M} + M \subset M$
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Proof. Let \( \bar{M} \) is module finite over \( M \). Let \( \{ x_1 - y_1, \ldots, x_n - y_n \} \subset \text{gp}(M) \) be a generating set \( x_i, y_i \in M \). Then \( y_1 + \cdots + y_n \in c_{\bar{M}/M} \). \( \square \)
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(Katthän, 2015)

$$\bar{M} \setminus M = \bigcup_{j=1}^{l} (q_j + \text{gp}(M \cap F)) \cap C(M),$$

where the $F_j$ are faces of the cone $C(M)$ and $q_j \in \bar{M}$
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Moreover, $c_{\bar{S}/S} = g(S) + \mathbb{Z}_{>0}$, where $g(S)$ is the Frobenius number of $S$ (Reid-Roberts, 2001). Let $\{v_1, \ldots, v_d, v_{d+1}\} \subset \mathbb{Z}_{\geq 0}^d$ be a circuit (no $d$ elements are linearly dependent) and $M = \mathbb{Z}_{\geq 0}v_1 + \cdots + \mathbb{Z}_{\geq 0}v_1$. 
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$$c_{\bar{M}/M} = g + (\text{int} \ C(M) \cap \text{gp}(M))$$
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$$c_{\bar{M}/M} = g + \left(\text{int } C(M) \cap \text{gp}(M)\right)$$

where

$$g = \left(\sum_{i=1}^{d+1} d_i v_i\right) / 2 - \sum_{i=1}^{d+1} v_i$$

$d_i$ being the order of $\mathbb{Z}^d$ modulo $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{d+1}$
Crash course in $K$-theory

Grothendieck’s group $K_0(R)$ of a ring $R$ measures how far the projective $R$-modules overall are from being free (actually, stably free, which is a certain functorial weakening of ‘free’).
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Informally, these groups are syzygies between elementary transformation of invertible matrices over $R$. Formally, they are higher homotopy groups of a certain $K$-theoretical space, associated to $R$ (Quillen, the 1970s)
$K$-theory of monoid rings

Let $R$ be a (commutative) regular ring and $M \subset \mathbb{Z}^d$ an affine monoid
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(G., 1988) \hspace{1cm} K_0(R) = K_0(R[M]) \hspace{1cm} \text{iff} \hspace{1cm} M = \text{sn}(M)
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(G., 1988) \quad K_0(R) = K_0(R[M]) \quad \text{iff} \quad M = \text{sn}(M)

Corollary: \quad K_0(R[M])/K_0(R) \cong R(\text{sn}(M) \setminus M) \quad \text{when} \quad \mathbb{Q} \subset R
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**Corollary:** \[ K_0(R[M])/K_0(R) \cong R(\text{sn}(M) \setminus M) \] when $\mathbb{Q} \subset R$

(G., 1992) \[ K_*(R) = K_*(R[M]) \text{ iff } M \cong \mathbb{Z}_0^r \text{ for some } r \geq 0 \]
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Corollary: \[ K_0(R[M])/K_0(R) \cong R(\text{sn}(M) \setminus M) \quad \text{when } Q \subset R \]

(G., 1992) \[ K_*(R) = K_*(R[M]) \iff M \cong \mathbb{Z}^r_{\geq 0} \quad \text{for some } r \geq 0 \]

(G., 2005) Assume $Q \subset R$ and $c \geq 2$. Then high iterations of the homothety $M \to cM$, defined by $m \mapsto cm$, kill $K_*(R[M])/K_*(R)$.
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\text{(G., 1988)} \quad K_0(R) &= K_0(R[M]) \text{ iff } M = \text{sn}(M) \\
\text{Corollary:} \quad K_0(R[M])/K_0(R) &\cong R(\text{sn}(M) \setminus M) \text{ when } Q \subset R \\
\text{(G., 1992)} \quad K_*(R) &= K_*(R[M]) \text{ iff } M \cong \mathbb{Z}_{\geq 0}^r \text{ for some } r \geq 0 \\
\text{(G., 2005)} \quad \text{Assume } Q \subset R \text{ and } c \geq 2. \text{ Then high iterations of the homothety } M \rightarrow cm, \text{ defined by } m \mapsto cm, \text{ kill } K_*(R[M])/K_*(R) \\
\text{(Cortiñas, Haesemayer, Walker, Weibel, announced in 2016)} \quad \text{The condition } Q \subset R \text{ in the statement above can be dropped}
\end{align*}
Conjecture. \( R \) a regular ring, containing \( \mathbb{Q} \) : for every finitely generated monomial algebra \( R[M] \) without nontrivial units we have the equality
Conjecture. Let $R$ be a regular ring, containing $\mathbb{Q}$: for every finitely generated monomial algebra $R[M]$ without nontrivial units we have the equality

$$K_i(R[M])/K_i(R) \cong (\text{a finitely generated } M\text{-graded thin } R[M]\text{-module})$$

and on this module the map $M \to M$, $m \mapsto cm$, acts by dilating the $M$-degrees by factor $c$. 
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Informally, the mentioned thinness means that every element of \( K_i(R[M])/K_i(R) \) is pushed by sufficiently high iterations of the map \( M \to M, \ m \mapsto cm \), to the \( M \) -graded zero zone. In particular, this conjecture implies the aforementioned nilpotence of \( K_i(R[M])/K_i(R) \).
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It is known that \( K_i(R[M]) / K_i(R) \) is an \( R \)-module; this follows from the Bloch-Stienstra action of the big Witt vectors.
Conjecture
REFERENCES


REFERENCES


Thank you