Decompositions of Binomials Ideals

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Polynomial Ideals

\[ R = \mathbb{k}[x_1, \ldots, x_n] \] the polynomial ring over a field \( \mathbb{k} \).

A **monomial** is a polynomial with one term, a **binomial** is a polynomial with at most two terms.

**Monomial ideals** are generated by monomials, **binomial ideals** are generated by binomials.

**Monomial ideals:**
- Algebra, Combinatorics, Topology.

**Toric Ideals:**
- Prime binomial ideals.
- Algebra, Combinatorics, Geometry.
Theorem (Eisenbud and Sturmfels, 1994)

$I \subseteq R$ a binomial ideal, $\mathbb{k}$ algebraically closed.

- **Geometric Statement:**
  $\text{Var}(I)$ is a union of toric varieties.

- **Algebraic Statement:**
  The associated primes and primary components of $I$ can be chosen binomial.
Why are Noetherian rings called Noetherian?

A commutative ring with 1, Noetherian (ascending chains of ideals stabilize).

A proper ideal \( I \subset R \) is **prime** if \( xy \in I \) implies \( x \in I \) or \( y \in I \).

\( I \) is **primary** if \( xy \in I \) and \( x^n \notin I \ \forall n \in \mathbb{N} \), implies \( y \in I \).

**Theorem (Lasker 1905 (special cases), Noether 1921)**

*Every proper ideal \( I \subsetneq R \) has a decomposition as a finite intersection of primary ideals.*

The radicals of the primary ideals appearing in the decomposition are the **associated primes** of \( I \).
Binomial Ideals

Theorem (Eisenbud and Sturmfels, 1994)

$I \subset R$ a binomial ideal, $\mathbb{k}$ algebraically closed.

- **Geometric Statement:**
  \( \text{Var}(I) \) is a union of toric varieties.

- **Algebraic Statement:**
  The associated primes and primary components of $I$ can be chosen binomial.

- **Combinatorial Statement:**
  The subject of this talk.

Need $\mathbb{k}$ algebraically closed; $\text{char}(\mathbb{k})$ makes a difference.

Example: In $\mathbb{k}[y]$, consider $I = \langle y^p - 1 \rangle$.

No hope of nice combinatorics for trinomial ideals.
There is combinatorics! (Slide of joy)

\[ I = \langle x^2 - y^3, x^3 - y^4 \rangle \]

\[ = \langle x - 1, y - 1 \rangle \cap (I + \langle x^4, x^3 y, x^2 y^2, xy^4, y^5 \rangle) \]
There is combinatorics! (Slide of joy)

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Works for binomial ideals over \( k = \overline{k} \) with \( \text{char}(k) = 0 \).
But how to make sure we have all bounded components?
Switch gears: Lattice Ideals

If $L \subseteq \mathbb{Z}^n$ is a lattice, and $\rho : L \to \mathbb{k}^*$ is a group homomorphism,

$$I(\rho) = \langle x^u - \rho(u-v)x^v \mid u, v \in \mathbb{N}^n, u - v \in L \rangle \subset \mathbb{k}[x_1, \ldots, x_n]$$

is a lattice ideal.

Theorem (Eisenbud–Sturmfels)

A binomial ideal $I$ is a lattice ideal iff $mb \in I$ for $m$ monomial, $b$ binomial $\Rightarrow b \in I$.

If $\mathbb{k}$ is algebraically closed, the primary decomposition of $I(\rho)$ can be explicitly determined in terms of extensions of $\rho$ to $\text{Sat}(L) = (\mathbb{Q} \otimes_{\mathbb{Z}} L) \cap \mathbb{Z}^n$. 
Lattice Ideals are easy to decompose

Example

\[ L = \text{span}_\mathbb{Z}\{(\begin{pmatrix} -1 \\ 0 \\ 3 \\ 2 \end{pmatrix}, (\begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}) \} \subset \mathbb{Z}^4. \]

\[ \rho : \mathbb{Z}^4 \to \mathbb{k}^* \text{ the trivial character.} \]

\[ I(\rho) = \langle xw^2 - z^3, x^2w - y^3 \rangle. \]

Sat(\(L\)) = \text{span}_\mathbb{Z}\{((1, -2, 1, 0), (0, 1, -2, 1))\} \quad \text{and} \quad |\text{Sat}(L)/L| = 3

If \(\text{char}(\mathbb{k}) \neq 3\), then \(I = I_1 \cap I_2 \cap I_3\), where

\[ I_j = \langle yz - \omega^j xw, xz - \omega^j y^2, z^2 - \omega^{2j} yw \rangle, \quad \omega^3 = 1, \quad \omega \neq 1. \]

If \(\text{char}(\mathbb{k}) = 3\), \(I\) is primary.
What next

The good:
Relevant combinatorics: monoid congruences.
Laura, don’t forget to explain what congruences are.

The not so good:
Field assumptions, computability issues.

Take a deep breath: Stop decomposing at the level of lattice ideals.

The choices:
- Finest possible
  → Mesoprimary Decomposition [Kahle-Miller]
- Coarsest possible
  → Unmixed Decomposition [Eisenbud-Sturmfels], [Ojeda-Piedra], [Eser-M]
Too many definitions

Colon ideal and saturation:
\[(I : x) = \{ f \mid xf \in I \} \quad \text{and} \quad (I : x^\infty) = \{ f \mid \exists l > 0, x^lf \in I \}\]

I binomial ideal, m monomial \(\Rightarrow (I : m), (I : m^\infty)\) binomial.

Let \(\sigma \subseteq \{1, \ldots, n\}\). \(I \subseteq \mathbb{k}[x_1, \ldots, x_n]\) is \(\sigma\)-cellular if \(\forall i \in \sigma\), \((I : x_i) = I\), and \(\forall j \notin \sigma\), \(\exists l_j > 0\) such that \(x_j^{l_j} \in I\).

\(I\) a \(\sigma\)-cellular binomial ideal.

- \(I\) is mesoprime if \(I = \langle I_{\text{lat}} \rangle + \langle x_j \mid j \notin \sigma \rangle\) for some lattice ideal \(I_{\text{lat}} = I_{\text{lat}} \subset \mathbb{k}[x_i \mid i \in \sigma]\).
- \(I\) is mesopprimary if \(b \in \mathbb{k}[x_i \mid i \in \sigma]\) binomial, \(m\) monomial and \(bm \in I \Rightarrow m \in I\) or \(b \in I_{\text{lat}} = I \cap \mathbb{k}[x_i \mid i \in \sigma]\).
- \(I\) is unmixed if \(\text{Ass}(I) = \text{Ass}(\langle I_{\text{lat}} \rangle + \langle x_j \mid x_j \notin \sigma \rangle)\), where \(I_{\text{lat}} = I \cap \mathbb{k}[x_i \mid x_i \in \sigma]\).
Cellular, Mesoprimary, Unmixed

$I$ a $\sigma$-cellular binomial ideal, mesoprime.

- $I$ is mesoprime if $I = \langle I_{\text{lat}} \rangle + \langle x_j \mid j \notin \sigma \rangle$ for some lattice ideal $I_{\text{lat}} \subset \mathbb{k}[x_i \mid i \in \sigma]$.

- $I$ is mesoprimary if $b \in \mathbb{k}[x_i \mid i \in \sigma]$ binomial, $m$ monomial and $bm \in I \Rightarrow m \in I$ or $b \in I_{\text{lat}} = I \cap \mathbb{k}[x_i \mid i \in \sigma]$.

- $I$ is unmixed if $\text{Ass}(I) = \text{Ass}(\langle I_{\text{lat}} \rangle + \langle x_j \mid x_j \notin \sigma \rangle)$, where $I_{\text{lat}} = I \cap \mathbb{k}[x_i \mid x_i \in \sigma]$.

Example

$I = \langle x^3 - 1, y(x - 1), y^2 \rangle$

cellular, unmixed, not mesoprimary, with decomposition

$I = \langle x^3 - 1, y \rangle \cap \langle x - 1, y^2 \rangle$.

If $\text{char}(\mathbb{k}) = 3$, $I$ is primary.
If $\text{char}(\mathbb{k}) \neq 3$, the primary decomposition is

$I = \langle x - \omega, y \rangle \cap \langle x - \omega^2, y \rangle \cap \langle x - 1, y^2 \rangle; \omega^3 = 1, \omega \neq 1.$
Cellular, Mesoprimary, Unmixed

$I$ a $\sigma$-cellular binomial ideal, mesoprime.

- $I$ is **mesoprime** if $I = \langle I_{\text{lat}} \rangle + \langle x_j \mid j \notin \sigma \rangle$ for some lattice ideal $I_{\text{lat}} \subset k[x_i \mid i \in \sigma]$.
- $I$ is **mesoprimary** if $b \in k[x_i \mid i \in \sigma]$ binomial, $m$ monomial and $bm \in I \Rightarrow m \in I$ or $b \in I_{\text{lat}} = I \cap k[x_i \mid i \in \sigma]$.
- $I$ is **unmixed** if $\text{Ass}(I) = \text{Ass}(\langle I_{\text{lat}} \rangle + \langle x_j \mid x_j \notin \sigma \rangle)$, where $I_{\text{lat}} = I \cap k[x_i \mid x_i \in \sigma]$.

**Example**

$I = \langle I_{\text{lat}} \rangle + \langle I_{\text{art}} \rangle$ is always mesoprimary but converse is not true. For instance

$$\langle x^2y^2 - 1, xz - yw, z^2, w^2 \rangle$$

is mesoprimary.
At last

**Theorem**

*Decompositions of binomial ideals into*

- mesoprimary binomial ideals \([\text{Kahle-Miller}]\)
- unmixed cellular binomial ideals \([\text{Eisenbud-Sturmfels}]\) \([\text{Ojeda-Piedra}]\) \([\text{Eser-M}]\)

exist over any field.

**The punchline:**
Now primary decomposition is easy!
But how to do it? (Handwavy slide, we are all tired)

The easy case: $I$ is $\sigma$-cellular.

For $m$ monomial in $k[x_j \mid j \notin \sigma]$, $J_m = (I : m) \cap k[x_i \mid i \in \sigma]$ is a lattice ideal.

The **unmixed/mesoprimary** components of $I$ are of the form

$$((I + J_m) : \prod_{i \in \sigma} x_i^\infty) + "combinatorial" \ monomial \ ideal$$

**Mesoprimary decomposition:** largest possible monomial ideal.

**Unmixed decomposition:** smallest possible monomial ideal.

It is easy to produce mesoprimary/unmixed decompositions. **Controlling the decompositions is hard.**
Binomial ideals do not in general have irreducible binomial decompositions [Kahle-Miller-O’Neill].

$I$ a binomial ideal.

- When is $\mathbb{k}[x]/I$ Cohen–Macaulay?
- Gorenstein?
- What are the Betti numbers of $\mathbb{k}[x]/I$?
- Can a (minimal) free resolution be constructed?
- Is there something like the Ishida complex?
- Ask any interesting question here...

I do not know.

The optimistic ending: An emerging area, with lots of interesting open problems!
THANK YOU!
Proof of Noether’s theorem (slide of the second wind)

$I \subsetneq R$ is reducible if $I = J_1 \cap J_2$ with $J_1, J_2 \nsubseteq I$.

1. Every proper ideal has an irreducible decomposition.

   If $I$ does not have an irreducible decomposition, can produce a non-stabilizing ascending chain of ideals.

2. Irreducible ideals are primary.

   $I$ is primary iff every $x \in R$ is either nilpotent or a nonzerodivisor modulo $I$.

Suppose $x \in R$ is neither nilpotent nor a nonzerodivisor modulo $I$.
Then: \[(I : x) \subset (I : x^2) \subset (I : x^3) \subset \cdots \]
so $\exists N: \quad (I : x^N) = (I : x^{N+1}) = \cdots$
Claim.
\[I = (I + \langle x^N \rangle) \cap (I : x^N)\]