Convex incidences, neuroscience, and ideals

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Biological Motivation

**Place cells:** Neurons which are active in a particular region of an animal’s environment. (Nobel Prize 2014, Physiology or Medicine, O’Keefe/Moser-Moser)

https://upload.wikimedia.org/wikipedia/commons/5/5e/Place_Cell_Spiking_Activity_Example.png
Biological Motivation

How is data on place cells collected?
Biological Motivation

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\[ C = \{000, 100, 001, 011, 110, 111\} \]
Mathematical Formulation

Neural codes capture an animal’s response to a stimulus.

We assume that the receptive fields for place cells are open convex sets in Euclidean space.
Mathematical Formulation

We associate collections of convex sets to binary codes.

**Definition (Curto et. al, 2013)**

Let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be a collection of convex open sets. The code of $\mathcal{U}$ is

$$\mathcal{C}(\mathcal{U}) := \left\{ v \in \{0, 1\}^n \left| \bigcap_{v_i=1} U_i \setminus \bigcup_{v_j=0} U_j \neq \emptyset \right. \right\}$$

Let $\mathcal{U} = \{U_1, U_2, U_3\}$

$\mathcal{C}(\mathcal{U}) = \{000, 100, 010, 001, 110, 011\}$
The Question

Let $C \subseteq \{0, 1\}^n$ be a code. If there exists a collection of convex open sets $U$ so that $C = C(U)$, we say that $C$ is convex. We call $U$ a convex realization of $C$.

Question: How can we detect whether a code $C$ is convex?
The Question

Definition
Let $C \subseteq \{0,1\}^n$ be a code. If there exists a collection of convex open sets $\mathcal{U}$ so that $C = C(\mathcal{U})$ we say that $C$ is convex. We call $\mathcal{U}$ a convex realization of $C$.

Question
How can we detect whether a code $C$ is convex?
Non-Example

Consider the code \( \mathcal{C} = \{000, 100, 010, 110, 011, 101\} \)
Consider the code $\mathcal{C} = \{000, 100, 010, 110, 011, 101\}$

$\mathcal{C}$ is not realizable!
Classifying Convex Codes

**Question**

*Can we find meaningful criteria that guarantee a code is convex?*

**Answer:** Yes!
Question

Can we find meaningful criteria that guarantee a code is convex?

Answer: Yes!

- Simplicial complex codes (Curto et. al, 2013)
- Codes with 11⋯1 in them (Curto et. al, 2016)
- Intersection complete codes (Kronholm et. al, 2015)
- Many more (results from several papers)
Other Ideas....

Use Ideals!
An Algebraic Approach

We will work in the polynomial ring $\mathbb{F}_2[x_1, \ldots, x_n]$. 

Definition (CIVCY2013)

Let $v \in \{0, 1\}^n$. The indicator pseudomonomial for $v$ is

$$\rho_v : = \prod_{i=1}^{n} x_i / \prod_{j \in v^c} (1 - x_j).$$

$\rho_{110} = x_1 x_2 (1 - x_3)$. Note that $\rho_v(u) = 1$ only if $u = v$.

Definition (CIVCY2013)

Let $C \subseteq \{0, 1\}^n$ be a code. The neural ideal $J_C$ of $C$ is the ideal $J_C : = \cup \rho_v / v \not\in C$.
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**Definition (CIVCY2013)**

Let $C \subseteq \{0, 1\}^n$ be a code. The *neural ideal* $J_C$ of $C$ is the ideal

$$J_C := \langle \rho_v \mid v \notin C \rangle.$$
Neural Ideal Example

Definition (CIVCY2013)

Let $C \subseteq \{0, 1\}^n$ be a code. The neural ideal $J_C$ of $C$ is the ideal

$$J_C := \langle \rho_v \mid v \notin C \rangle.$$ 

- $C = \{000, 100, 010, 001, 011\}$

$$J_C = \langle \rho_v \mid v \notin C \rangle = \langle x_1x_2(1 - x_3), x_1x_3(1 - x_2), x_1x_2x_3 \rangle$$

$$= \langle x_1x_2, x_1x_3(1 - x_2) \rangle$$
**Definition (CIVCY2013)**

Let $J_C$ be a neural ideal. The *canonical form* of $J_C$ is the set of minimal pseudomonomials in $J_C$ with respect to division. Equivalently:

$$CF(J_C) := \{ f \in J_C \mid f \text{ is a PM and no proper divisor of } f \text{ is in } J_C \}.$$
Consider the code $C = \{00000, 10000, 01000, 00100, 00001, 11000, 10001, 01100, 00110, 00101, 00011, 11100, 00111\}$. 
Consider the code \( C = \{00000, 10000, 01000, 00100, 00001, 11000, 10001, 
01100, 00110, 00101, 00011, 11100, 00111\} \).

\[
J_C = 
\langle x_4(1-x_1)(1-x_2)(1-x_3)(1-x_5), x_1x_3(1-x_2)(1-x_4)(1-x_5), x_1x_4(1-x_2)(1-x_3)(1-x_5), 
  x_2x_4(1-x_1)(1-x_3)(1-x_5), x_2x_5(1-x_1)(1-x_3)(1-x_4), x_1x_2x_4(1-x_3)(1-x_5), 
  x_1x_2x_5(1-x_3)(1-x_4), x_1x_3x_4(1-x_2)(1-x_5), x_1x_3x_5(1-x_2)(1-x_4), 
  x_1x_4x_5(1-x_2)(1-x_3), x_2x_3x_4(1-x_1)(1-x_5), x_2x_3x_5(1-x_1)(1-x_4), 
  x_2x_4x_5(1-x_1)(1-x_3), x_2x_3x_4x_5(1-x_1), x_1x_3x_4x_5(1-x_2), 
  x_1x_2x_4x_5(1-x_3), x_1x_2x_3x_5(1-x_4), x_1x_2x_3x_4(1-x_5), x_1x_2x_3x_4x_5 \rangle
\]

Uggghhhhh!
Canonical Form and Constructing Codes

Canonical Form (Minimal description!)

\[ J_C = \langle x_1 x_3 x_5, x_4 (1 - x_3)(1 - x_5), x_1 x_4, x_1 x_3 (1 - x_2), x_2 x_4, x_2 x_5 \rangle \]
Canonical Form and Constructing Codes

\[ J_C = \langle x_1 x_3 x_5, x_4 (1 - x_3) (1 - x_5), x_1 x_4, x_1 x_3 (1 - x_2), x_2 x_4, x_2 x_5 \rangle \]

- \( x_1 x_3 x_5 \)
Canonical Form and Constructing Codes

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\[ x_1 x_3 x_5 \Rightarrow U_1 \cap U_3 \cap U_5 = \emptyset, \]
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\( x_1x_3x_5 \Rightarrow U_1 \cap U_3 \cap U_5 = \emptyset, \)
\( U_1 \cap U_3 \neq \emptyset, U_1 \cap U_5 \neq \emptyset, U_3 \cap U_5 \neq \emptyset. \)
Canonical Form and Constructing Codes

The picture so far:
$J_C = \langle x_1 x_3 x_5, x_4 (1 - x_3)(1 - x_5), x_1 x_4, x_1 x_3 (1 - x_2), x_2 x_4, x_2 x_5 \rangle$

$\iff x_1 x_3 x_5 \implies U_1 \cap U_3 \cap U_5 = \emptyset, \ U_1 \cap U_3 \neq \emptyset, \ U_1 \cap U_5 \neq \emptyset, \ U_3 \cap U_5 \neq \emptyset.$
Canonical Form and Constructing Codes

\[ J_C = \langle x_1 x_3 x_5, x_4 (1 - x_3)(1 - x_5), x_1 x_4, x_1 x_3 (1 - x_2), x_2 x_4, x_2 x_5 \rangle \]

- \[ x_1 x_3 x_5 \Rightarrow U_1 \cap U_3 \cap U_5 = \emptyset, \ U_1 \cap U_3 \neq \emptyset, \ U_1 \cap U_5 \neq \emptyset, \ U_3 \cap U_5 \neq \emptyset. \]
- \[ x_4 (1 - x_3)(1 - x_5) \]
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\[ J_C = \langle x_1 x_3 x_5, x_4 (1 - x_3) (1 - x_5), x_1 x_4, x_1 x_3 (1 - x_2), x_2 x_4, x_2 x_5 \rangle \]

- \( x_1 x_3 x_5 \Rightarrow U_1 \cap U_3 \cap U_5 = \emptyset, U_1 \cap U_3 \neq \emptyset, U_1 \cap U_5 \neq \emptyset, U_3 \cap U_5 \neq \emptyset. \)
- \( x_4 (1 - x_3) (1 - x_5) \Rightarrow U_4 \subseteq U_3 \cup U_5, \)
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\[ J_C = \langle x_1 x_3 x_5, x_4 (1 - x_3)(1 - x_5), x_1 x_4, x_1 x_3 (1 - x_2), x_2 x_4, x_2 x_5 \rangle \]

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- \( x_2 x_4 \Rightarrow U_2 \cap U_4 = \emptyset \),
- \( x_2 x_5 \Rightarrow U_2 \cap U_5 = \emptyset \).
Final picture:
The Neural Ideal in Summary

\[ C \rightarrow J_C \rightarrow CF(J_C) \]

We associate codes to neural ideals, and use the canonical form to compactly present the neural ideal and encode information about the code and its realizations.
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We hope to understand convex codes by examining neural ideals and their canonical forms.
Definition

We say a homomorphism \( \phi : \mathbb{F}_2^n \to \mathbb{F}_2^m \) respects neural ideals if for every \( C \subseteq \{0, 1\}^n \) there exists \( D \subseteq \{0, 1\}^n \) so that

\[
\phi(J_C) = J_D.
\]

That is, if \( \phi \) maps neural ideals to neural ideals.

Can we classify all such homomorphisms? Do they have geometric meaning?
**Homomorphisms Respecting Neural Ideals**

**Restriction:** Mapping $x_i \mapsto 1$ or $x_i \mapsto 0$ for some $i$.

- $x_i \mapsto 1$ corresponds with replacing each $U_j$ by $U_j \cap U_i$.
- $x_i \mapsto 0$ corresponds with replacing each $U_j$ by $U_j \setminus U_i$. 
Homomorphisms Respecting Neural Ideals

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Graphical representation:

- $U_4$ interacts with $U_3$.
- $U_2$ interacts with $C$.
- $U_3$ interacts with $U_2$.
- $CF(J_c)$ transforms into $CF(J_{c'})$.

Diagram showing set intersections and mappings.
Homomorphisms Respecting Neural Ideals

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**Bit Flipping:** Mapping $x_i \mapsto 1 - x_i$ for some $i$.
- Corresponds to taking the complement of $U_i$. 

![Diagram](image-url)
**Homomorphisms Respecting Neural Ideals**

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**Bit Flipping:** Mapping \( x_i \mapsto 1 - x_i \) for some \( i \).
- Corresponds to taking the complement of \( U_i \).

**Permutation:** Permuting labels on the variables in \( \mathbb{F}_2[n] \).
- Corresponds to permuting labels on the sets in a realization.
Theorem (Jeffs, O.)

Let $\phi : \mathbb{F}_2^n \to \mathbb{F}_2^m$ be a homomorphism respecting neural ideals. Then $\phi$ is the composition of the three types of maps previously described:

- Permutation
- Restriction
- Bit flipping

Moreover, there is an algorithm to present $\phi$ as such a composition.
Homomorphisms Respecting Neural Ideals: Proof Idea

1. If $\phi : \mathbb{F}_2[n] \to \mathbb{F}_2[m]$ respects neural ideals if and only if $\phi$ is
   - surjective, and
   - sends pseudonomials to pseudomonomials or 0
2. $\phi(x_i) \in \{x_j, 1 - x_j, 0, 1\}$, and for every $j \in [m]$ there is a unique $i \in [n]$ so that $\phi(x_i) \in \{x_j, 1 - x_j\}$.
3. (Carefully) piece things together variable by variable.
Conclusion

In This Talk:

- We associated polynomial ideals to codes.
- We used these ideals to understand codes and their realizations.
- We described a class of homomorphisms which play nicely with these ideals. These homomorphisms can be used to understand convex codes, and also computationally construct them.
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In This Talk:

- We associated polynomial ideals to codes.
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What’s Next?

- How do maps respecting neural ideals affect canonical forms?
- What other algebraic techniques can be leveraged?
Thank You!