Algebraic Applications of the Theory of Violator Spaces

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Definition (Gärtner et al, 2008)

A **violator space** is a pair \((H, V)\), where \(H\) is a finite set and \(V: 2^H \rightarrow 2^H\) is a mapping such that:

1. For all \(G \subseteq H\), \(G \cap V(G) = \emptyset\) (consistency)
2. For all \(F \subseteq G \subseteq H\), such that \(G \cap V(F) = \emptyset\), \(V(G) = V(F)\) (locality)

- The mapping \(V\) associates to every subset \(G \subseteq H\) the set of things in \(H\) that “violate” \(G\)
- Think of \(H\) as a set of constraints
- Get to choose what “violates” means for your particular problem
- Examples: LP-type problems, geometric optimization problems, smallest enclosing ball problem
Smallest enclosing ball in $\mathbb{R}^2$: 

**Problem**: Given a set of points in $\mathbb{R}^2$, find the smallest circle containing them.

**Setup**:

- $H$, a set of points $\mathbb{R}^2$
- $\mathcal{V}$: For $G \subset H$, a point $p$ outside of $G$ violates $G$ if adding $p$ to $G$ increases the size of the smallest circle containing $G$. 

![Diagram of points in $\mathbb{R}^2$]
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- $G =$ blue points
- Red point violates $G$
- Green point does not violate $G$
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Key observation:
At most 3 points of $H$ determine the unique smallest circle containing $H$. 
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Definition (Gärtner et al, 2008)

A **basis** of a violator space $(H, V)$ is a subset $B \subseteq H$ such that $B \cap V(F) \neq \emptyset$ holds for all proper subsets $F \subsetneq B$. The **combinatorial dimension** is the size of the largest basis for $(H, V)$.
Key idea:

Violator spaces provide an abstract framework for formulating many types of optimization problems which is useful for designing efficient algorithms.

Clarkson’s algorithm (Clarkson, 1995):

A randomized algorithm that performs biased sampling to find a basis.

Input: \((H, V)\); \(\delta\), the combinatorial dimension

Output: \(B\), a basis for \(H\)

Given a violator space \((H, V)\), some subset \(G \subseteq H\), and some elements \(h \in H \setminus G\), the primitive test decides whether \(h \in V(G)\).

Theorem (Clarkson, 1995; Škovroň, 2007)

Clarkson’s algorithm finds a basis \(B\) for \((H, V)\) in an expected \(O(\delta |H| + \delta^O(\delta))\) calls to the primitive.
Goal:
Take problems from computational algebra and fit them into the framework of violator spaces.

Each potential application requires three ingredients:
- The right notion of “violates”
- A bound on $\delta$, the combinatorial dimension
- A primitive test
Overdetermined systems of polynomials

Problem

Suppose that \( \{f_1, \ldots, f_s\} \) \((s \gg 0)\) is a collection of polynomials in \(n\) variables and we are interested in solving the system \(f_1 = \cdots = f_s = 0\).

The ingredients (De Loera-Petrović-Stasi, 2015):

- **Violator**: \( H = \{f_1, \ldots, f_s\} \); if \( G \subset H \), \( f_i \) violates \( G \) if \( f_i \) does not vanish on the variety \( V(G) \).

- **Combinatorial dimension**: rank of coefficient matrix:

\[
\delta = \text{rank} \begin{pmatrix}
\text{monomials} \\
\begin{pmatrix}
f_1 \\
\vdots \\
f_s
\end{pmatrix} \\
\text{coefficients}
\end{pmatrix}
\]

- **Primitive test**: GB calculation
Overdetermined systems of polynomials

Problem

Suppose that \( \{f_1, \ldots, f_s\} \ (s \gg 0) \) is a collection of polynomials in \( n \) variables and we are interested in solving the system \( f_1 = \cdots = f_s = 0 \).

Example (Mayr-Meyer Ideal:)

The Mayr-Meyer ideal \( J(n,d) \) is an ideal in \( 10n + d \) variables where the minimal generators have degree \( d + 2 \). It is a pathological example that is not to achieve the doubly exponential bound in \( n \) for GB computations. In the case \( n = d = 2 \), we added two polynomials to the 24 minimal generators of \( J(2,2) \) to make the system infeasible. Using a prototype for \( V_{\text{solve}} \) in Macaulay2, we found a basis of size 2 an average of 8 seconds. The Gröbner computation on the same machine lasted 18+ hours without terminating.
Small generating sets and semi-algebraic sets

Theorem (De Loera-Petrović-Stasi, 2015)

There exists a violator $V_{\text{SmallGen}}$ for finding small generating sets of homogenous ideals in a polynomials ring.

Theorem (De Loera-Petrović-Stasi-W., 2016+)

There exists a violator $V_{\text{SemiAlg}}$ for finding minimal representations of elementary semi-algebraic sets.
Current work

What’s happening next:

- Showing that the violators $V_{\text{Solve}}$, $V_{\text{SmallGen}}$, and $V_{\text{SemiAlg}}$ satisfy addition properties in the violator framework
- Extending to other problems in computation algebra
- Finding nice applications

