My research lies in combinatorial commutative algebra of rings and monoids. I am also interested in discrete geometry and its connections to computation, algorithms, and the development and use of software in mathematical research.

**Research overview.** Non-unique factorization theory aims to classify integral domains and commutative cancellative monoids where uniqueness of factorization of non-unit elements as products of irreducibles fails. Since its inception in the study of algebraic number fields, non-unique factorization theory has remained closely connected to additive group theory and combinatorial number theory. Combinatorial factorization invariants play a major role in the study of non-unique factorization, as they give concrete methods of quantifying the abundance and variety of factorizations.

Finitely generated monoids have played an increasingly important role in the study of non-unique factorization theory. Many landmark results in factorization theory center around demonstrating extremal behavior of certain factorization invariants, and the setting of finitely generated monoids is broad enough to encompass a large variety of such behavior [ACHP07, CK15]. At the same time, finitely generated monoids benefit from algorithms and implementations for explicit computation of invariant values, which have been instrumental in their study [DGM15, Gar15]. As such, finitely generated monoids provide a wealth of explicit examples, and obtaining a more thorough understanding of their factorization structure has implications on the general setting.

My recent work involving finitely generated monoids has produced

(i) a definitive link with combinatorial commutative algebra, yielding a new framework through which to examine factorization structure [One15], and

(ii) more efficient algorithms for computing invariant values [BOP14b, DGM15].

In Sections 2-5, several factorization invariants from the literature are discussed and methods for future work concern the effective computation of factorization invariants for finitely generated monoids are presented, following an overview of the my recent contributions in Section 1.

**Undergraduate research.** Suitably chosen research problems on factorization theory in monoids are ideal for undergraduate projects, as they offer a high probability of producing publishable output while remaining accessible to undergraduate students. While significant contributions to open problems in commutative algebra often require extensive background in the subject, factorization theory can place algebraic objects in combinatorial settings, making them more concrete to work with explicitly. In fact, several of the motivating results presented below are products of undergraduate research projects I have coadvised, either in the REU setting [CCMMP14, HKMOP14, KOP15, OPTW15] or independently [BOP14a, BOP14b].

Many of the problems presented below are excellent starting places for undergraduate research projects, and I intend to include undergraduates on several such projects. As an example, initial investigations for Project 1 will involve utilizing mathematics software packages to compute concrete examples, providing a natural starting place for students to acquire familiarity and build intuition. Project 4 will also benefit from computation in its early stages, and a first semester in abstract algebra is a sufficient prerequisite to begin work with block monoids. Additionally, the survey article [OPe14] contains several open problems accessible to undergraduates.
1. Invariants of non-unique factorization

A factorization of an element \( m \) in an integral domain \( R \) is an expression of \( m \) as a product of irreducible elements; \( R \) is factorial if every nonzero nonunit admits a unique factorization (up to multiplication by a unit). First examined in the context of rings of integers of algebraic number fields [Car60], non-unique factorization theory aims to classify and quantify the non-uniqueness of factorizations that arise in integral domains that fail to be factorial.

Since the factorial condition on a domain \( R \) does not depend on the additive structure of \( R \), one can study factorizations in \( R \) by focusing on the structure of \( R^* = R \setminus \{0\} \), its multiplicative (cancellative, commutative) monoid. Often, this is done by either considering \( R^* \) itself, or by restricting to a suitable submonoid of \( R^* \). For instance, when \( R \) is the monoid algebra over a cancellative commutative monoid \( M \), the factorization properties of \( R \) are often studied by examining the monoid \( M \). Since the emergence of this idea [And97], non-unique factorization theory is frequently studied in monoids in lieu of integral domains [GH06], resulting in applications to combinatorial number theory [Hal91, Man73], algebraic geometry [Kun70, Ros99], and elsewhere.

The non-uniqueness of factorizations in a given monoid \( M \) can be quantified using combinatorially-flavored factorization invariants, which assign to each element of \( M \) (or to \( M \) as a whole) a value measuring its failure to admit unique factorization. For example, the catenary degree invariant assigns to each monoid element \( m \in M \) a non-negative integer \( c(m) \) measuring the distance between its factorizations, computed using a discrete graph whose vertices are factorizations [CCMMP14, CGL09]. The catenary degree \( c(M) \) of the monoid is then defined as the supremum of the catenary degrees achieved by its elements [BGG11], providing a global (monoid-wide) measure of the factorization structure. Many other standard invariants, such as delta sets [BCKR06, CGP14], elasticity [BOP14a, CHM06, CGGR01], and \( \omega \)-primality [AC10, OPe14], have been well studied for several classes of monoids and will each be formally introduced in later sections.

Let \( \mathbb{Z}_M(m) \) denote the set of factorizations of the monoid element \( m \in M \). The set

\[
\mathcal{Z}(M) = \{ \mathbb{Z}_M(m) : m \in M \},
\]

of factorizations of the monoid \( M \) is an example of a perfect factorization invariant since it encapsulates enough information to uniquely determine \( M \) up to isomorphism [OPe15]. However, such complete information comes at a cost: extracting information from it (or even simply writing it down) is a nontrivial task. In fact, the set of factorizations of a sufficiently large finitely generated monoid element alone contains enough information to recover the monoid structure of \( M \). Many invariants derived from the set of factorizations, such as the elasticity and catenary degree invariants, are more manageable and easier to work with, but necessitate a loss of information.

Further traction is often gained by examining factorization invariants for particular classes of monoids arising in the literature. The factorization structure of numerical monoids [GR09] (cofinite additive submonoids of \( \mathbb{Z}_{\geq 0} \)) has been of much recent interest [BCKR06, CK15, OPe15], due in part to its role in related areas of mathematics. For example, factorizations of a given numerical monoid element coincide with non-negative integer solutions to a linear equation, which is central to discrete optimization and integer programming [CGLR06, DeL05, DDK13]. Additionally, numerical monoids arise naturally in the Frobenius coin-exchange problem, which asks for the largest non-negative integer value that cannot be evenly changed using a collection of relatively prime coin values \( n_1, \ldots, n_k \) [BR07, Chapter 1]. In this context, each coin value represents an irreducible element of the numerical monoid \( M = \langle n_1, \ldots, n_k \rangle \subset \mathbb{Z}_{\geq 0} \), and factorizations of a monoid element \( m \in M \) correspond to distinct ways of making change for \( m \).

Recent work examining the factorization structure of numerical monoids has uncovered explicit characterizations of the eventual behavior of numerous factorization invariants [OPe15]. For example, the catenary degree invariant \( c : M \to \mathbb{Z}_{\geq 0} \) (defined in Section 5) for any numerical monoid \( M = \langle n_1, \ldots, n_k \rangle \) satisfies \( c(m + n_1 \cdots n_k) = c(m) \) for \( m \gg 0 \) (that is, the function \( c \) is eventually
Figure 1. A plot showing the catenary degree (left) and \(\omega\)-primality (right) invariants for the numerical monoid \(\langle 7, 13, 17 \rangle \subset \mathbb{Z}_{\geq 0}\), produced using SAGE [SAGE].

periodic), and the \(\omega\)-primality invariant \(\omega : M \to \mathbb{Z}_{>0}\) (defined in Section 4) satisfies
\[
\omega(m) = \frac{1}{n_1} m + a_0(m)
\]

for some \(n_1\)-periodic function \(a_0\) and \(m \gg 0\) [CCMMP14, OPe13]. The above function is an example of a quasipolynomial, that is, a polynomial with periodic coefficients. Figure 1 depicts the eventual behavior of both the catenary degree and \(\omega\)-primality functions for a particular numerical monoid.

Many of the aforementioned eventual behavior results rely on the fact that numerical monoids (i) possess only finitely many irreducible elements and (ii) can be totally ordered. Recently, I have successfully generalized many of these results from numerical monoids to affine monoids (finitely generated submonoids of \(\mathbb{Z}^k_{\geq 0}\)), using techniques from combinatorial commutative algebra [One15]. As affine monoids need not possess a total ordering, these extensions are stated using an appropriate multivariate quasipolynomial analog.

**Definition 1.1.** A function \(f : \mathbb{Z}^k_{\geq 0} \to \mathbb{R}\) is eventually quasipolynomial if (i) \(f\) restricts to a polynomial on each of finitely many cones (that is, sets of the form
\[
C(\beta; \alpha_1, \ldots, \alpha_r) = \beta + \alpha_1 \mathbb{Z}_{\geq 0} + \cdots + \alpha_r \mathbb{Z}_{\geq 0}
\]

for \(\beta\) and linearly independent \(\alpha_1, \ldots, \alpha_r \in \mathbb{Z}_{\geq 0}^k\), and (ii) the union of these cones equals \(\mathbb{Z}^k_{\geq 0}\).

Hilbert’s theorem, a cornerstone of commutative algebra, states that the Hilbert function of any finitely generated, positively multigraded module is eventually quasipolynomial [Sta96, Fie00]. Hilbert’s theorem is often applied to problems in enumerative combinatorics by constructing a graded module whose Hilbert function coincides with a counting function of interest [Ehr62, Mac71, DLMO09]. For example, constructions of this type yield an alternative proof that the number of proper \(k\)-colorings of a finite undirected graph \(G\) is a polynomial in \(k\) whose degree is the number of vertices of \(G\) [Sta12].

My recent work characterizing the eventual behavior of factorization invariants for affine monoids applies Hilbert’s theorem in this manner [One15]. More specifically, given \(M \subset \mathbb{Z}^k_{\geq 0}\) affine and one of several factorization invariants defined on the level of monoid elements, a family of multigraded modules is constructed whose Hilbert functions determine the invariant value of every element of \(M\). Applying Hilbert’s theorem to the modules in this family yields a description of the factorization invariant in terms of multivariate eventually quasipolynomial functions. When \(M\) is a numerical monoid (that is, when \(k = 1\)), this specializes to a known result for each invariant.

As an example, the aforementioned \(\omega\)-primality invariant \(\omega : M \to \mathbb{Z}_{>0}\) over any affine monoid \(M\) is shown to coincide with an eventual quasipolynomial in which the degree of each polynomial restriction is at most 1. If \(M = \langle n_1, \ldots, n_k \rangle\) is a numerical monoid, then more can be said. In particular, the polynomial restrictions occur on \(n_1\) 1-dimensional cones, each generated by \(n_1\),
whose union has finite complement in \(M\). The restriction of the \(\omega\)-function to each 1-dimensional cone is linear, and the leading term of each linear restriction is \(\frac{1}{n_1}\). The remaining cones are all 0-dimensional, one for each element of \(M\) outside of the 1-dimensional cones.

In addition to generalizing several core numerical monoid results to affine monoids, my work has produced a definitive link between combinatorial commutative algebra and the factorization invariants of interest in this setting. The result is a new framework through which to examine the factorization structure of affine monoids, which should help to ease the transition from the recent focus on numerical monoids to this more general setting. Projects proposed below are motivated both by this newfound connection, and by fundamental questions arising in factorization theory.

2. Factorization length and elasticity

The length of a factorization \(f\) of a monoid element \(m \in M\) is the number of irreducibles in \(f\); \(L(m)\) denotes the set of factorization lengths of \(m\) (its length set). Many factorization invariants are derived from factorization lengths, the simplest and most natural of which are maximum and minimum factorization length, which assign to each monoid element the values \(M(m) = \max L(m)\) and \(m(m) = \min L(m)\), respectively.

My work discussed in Section 1 characterizes the eventual behavior of maximum and minimum factorization length for affine monoids [One15], generalizing an earlier result for numerical monoids from joint work with Barron and Pelayo [BOP14a] and specializing foundational asymptotic results [GH92, CGGR02]. As a consequence of Theorem 2.1 below, the maximum and minimum factorization length functions on any affine monoid \(M\) are uniquely determined by their values on a bounded region, and an explicit region is given when \(M\) is a numerical monoid.

**Theorem 2.1** ([One15]). The max and min factorization length functions \(M, m : M \rightarrow \mathbb{Z}_{\geq 0}\) are eventually quasilinear for any affine monoid \(M \subset \mathbb{Z}^k_{\geq 0}\). Moreover, if \(M = \langle n_1, \ldots, n_k \rangle \subset \mathbb{Z}_{\geq 0}\) is a numerical monoid, then

\[
M(m) = \frac{1}{n_1}m + a_0(m) \quad \text{and} \quad m(m) = \frac{1}{n_k}m + b_0(m)
\]

for \(m \geq n_{k-1}n_k\), where \(a_0\) and \(b_0\) are \(n_1\)-periodic and \(n_k\)-periodic functions, respectively.

Maximum and minimum factorization length arise in discrete optimization problems, where factorizations in a given affine monoid coincide with integer solutions to a given set of linear equations [CGLR06, DeL05, DDK13]. In this setting, a monoid element’s factorizations that achieve the maximum (resp. minimum) factorization length are precisely those solutions with maximal (resp. minimal) \(\ell_1\)-norm. Theorem 2.1 implies that the extremal \(\ell_1\)-norms of the factorizations of sufficiently large affine monoid elements \(m \in M\) are parametrized by the extremal \(\ell_1\)-norms of their divisors. Moreover, the proof of Theorem 2.1 for numerical monoids characterizes which factorizations achieve such values, yielding an explicit parametrization ideal for direct application to discrete optimization problems.

**Project 1.** Characterize the cones arising in the quasipolynomials in Theorem 2.1.

The connection to Hilbert functions discussed in Section 1 brings additional tools from combinatorial commutative algebra to the table for investigating Project 1. For instance, recent developments involving the so-called multigraded Castelnuovo-Mumford regularity [MS04, MS05] bound the translation factors of cones arising in multigraded Hilbert function restrictions.

3. Delta sets

One of the first invariants studied in factorization theory [CS90, CS92], the delta set of a monoid element \(m \in M\) is defined as the set of successive differences of elements of its length set \(L(m)\) and provides a coarse measure of the “gaps” between the factorization lengths of \(m\). More precisely, if we write the length set \(L(m) = \{l_1 < \cdots < l_r\}\) of \(m\) in increasing order and without repetition, we define the delta set of \(m\) to be the set \(\Delta_M(m) = \{l_i - l_{i-1} : 2 \leq i \leq r\}\). This invariant detects several different kinds of non-unique factorization behavior. For instance, if \(\Delta(m) = \emptyset\), then every
factorization of \( m \) has the same length, and if \( \Delta(m) = \{1\} \), then the factorization lengths of \( m \) are consecutive. Generally speaking, the delta set \( \Delta(m) \) measures how sparse the length set of \( m \) is, and its larger elements indicate “missing factorizations” in \( M \).

Delta sets in a given monoid \( M \) are often studied using the union \( \Delta(M) = \bigcup_{m \in M} \Delta_M(m) \) of the delta sets of its elements, a set which globally measures how sparse the factorization lengths of \( M \) are. For instance, \( \Delta(M) = \emptyset \) if and only if \( M \) is half-factorial, meaning any two factorizations of the same element have identical length.

It is known that \( \gcd(\Delta(M)) = \min(\Delta(M)) \) for any cancellative, commutative monoid \( M \) with nonempty delta set [GH06]. Additionally, under certain conditions (such as if \( M \) is finitely generated), the delta set \( \Delta(M) \) is known to be finite. However, no further restrictions are known regarding which finite sets occur. In particular, it is not known if there is a finite set \( D \) satisfying \( \gcd(D) = \min(D) \) which does not occur as the delta set of some monoid. This motivates one of the most persistent open questions concerning delta sets, namely the delta set realization problem:

**Problem 3.1.** Given a finite set \( D \subset \mathbb{Z}_{\geq 0} \) satisfying \( \gcd(D) = \min(D) \), does there exist a monoid \( M \) satisfying \( \Delta(M) = D \)?

Further restrictions are known for certain classes of monoids, but no such restrictions are known for numerical monoids. In fact, recent results of Colton and Kaplan provide strong evidence that the conditions stated in Problem 3.1 may in fact be the only conditions governing numerical monoid delta sets [CK15]. In particular, they construct an infinite family of numerical monoids whose delta sets have arbitrarily large gaps, taking the form \( \{d, td\} \) for any given \( d \) and \( t \).

The difficulty of Problem 3.1 lies in proving that a particular value does not appear in the delta set of a given monoid \( M \), as doing so requires a high level of control over the relations between factorizations throughout \( M \). For example, the numerical monoid \( M = \langle 17, 33, 53, 71 \rangle \subset \mathbb{Z}_{\geq 0} \) has delta set \( \Delta(M) = \{2, 4, 6\} \), but the value 6 only occurs in the delta sets of the monoid elements 266, 283 and 300; see Figure 2. Such examples demonstrate the subtlety in producing monoids like those constructed by Colton and Kaplan in [CK15].

Numerical monoids are a natural setting in which to investigate Problem 3.1. Much of the recent work involving factorization invariants in numerical monoids benefits from established computer software packages [DGM15, Gar15, SAGE], and the delta set invariant is no exception. In particular, two recently developed algorithms for numerical monoid delta set computation include counterexamples to several conjectures, each claiming that some particular finite set does not occur as the delta set of any numerical monoid [CK15]. The first algorithm (joint work with García-Sánchez) utilizes techniques from combinatorial commutative algebra to compute the delta set algorithm of any affine monoid. The key ingredient is Theorem 3.2 below, which generalizes the eventual periodicity of numerical monoid delta sets [CHK09].
Theorem 3.2 ([One15]). Given an affine monoid $M \subset \mathbb{Z}^k_{\geq 0}$ and fixing $d \in \Delta(M)$, the function $M \to \{0,1\}$, defined by sending $m \mapsto 1$ whenever $d \in \Delta(m)$, is eventually quasiconstant. As a consequence, if $M = (n_1, \ldots, n_k) \subset \mathbb{Z}_{\geq 0}$ is a numerical monoid, then for all $m \gg 0$ we have
\[ \Delta(m + n_1 n_k) = \Delta(m). \]

As with the proof of Theorem 2.1, the quasiconstant function appearing in Theorem 3.2 is constructed using Hilbert’s theorem for multigraded modules. More specifically, given an affine monoid $M$, a sequence of multigraded polynomial ideals $I_0 \subset I_1 \subset I_2 \subset \cdots$ is constructed with the property that the quotient $I_d/I_{d-1}$ has a nonzero element of graded degree $m \in M$ if and only if $d \in \Delta(m)$. Since each quotient $I_d/I_{d-1}$ is graded, this implies that $I_{d-1} \subseteq I_d$ and only if $j \in \Delta(M)$, a membership criterion for $\Delta(M)$ that avoids searching individual monoid elements.

The minimal monoid elements $m \in M$ for which $d \in \Delta(m)$ can be recovered directly from the elements of the reduced Gröbner basis of each ideal $I_d$. A deeper understanding of these minimal monoid elements is critical for progress in Problem 3.1; indeed, a recent preprint from García Sánchez et al. [GLM15] provides a solution to Project 2 below for numerical monoids with three minimal generators, resulting in a complete answer to Problem 3.1 in this special case. Additionally, an answer to Project 2 would improve the efficiency of Algorithm 3.3 by further restricting the set of monoid elements whose delta sets must be computed.

**Project 2.** For affine $M \subset \mathbb{Z}^k_{\geq 0}$, characterize the minimal elements of $M$ whose delta sets contain each element of $\Delta(M)$.

A second algorithm for computing numerical monoid delta sets, joint with Barron and Pelayo [BOP14b], utilizes dynamic programming to compute length sets in finitely generated monoids without the need to first compute factorizations. This allows the delta sets of large monoid elements to be quickly computed, since the number of distinct factorization lengths in finitely generated monoids has linear growth rate [FH06, GH92]. Algorithm 3.3 below, together with the eventual periodicity of the delta set for numerical monoids, produces an algorithm to compute the delta set of any numerical monoid.

**Algorithm 3.3 ([BOP14b, DGM15]).** The length set of any affine monoid element $m \in M$ can be dynamically computed without computing any monoid element factorizations.

In contrast to numerical monoids, the monoid $\mathcal{B}(G)$ of zero-sum sequences over any finite Abelian group $G$ (called a block monoid) is known to have delta set $\Delta(\mathcal{B}(G)) = \{1, \ldots, \max \Delta(\mathcal{B}(G))\}$. Although $\max \Delta(\mathcal{B}(G))$ has been found for some Abelian groups $G$, it remains an open problem in general [GZ15]. The use of computer software in computing $\max \Delta(\mathcal{B}(G))$ has been infeasible until recently, due to the large (exponential in $|G|$) number of generators that block monoids possess [Pon04]. Indeed, the set of factorizations grows too quickly to be effectively computed for large elements, and computing the ideals used in Theorem 3.2 is infeasible since the underlying ring has one variable per monoid generator. Algorithm 3.3 avoids this issue, since the size of the length set grows independently of the number of generators. Additionally, for block monoids, it is known that $\max \Delta(\mathcal{B}(G))$ occurs in the delta set of some element whose length set contains 2, yielding a bounded set of elements whose delta sets must be computed in order to obtain $\Delta(\mathcal{B}(G))$ [GGS11].

**Project 3.** Optimize Algorithm 3.3 to compute $\max \Delta(\mathcal{B}(G))$ for finite Abelian groups $G$.

4. Measuring primality

Developed by Geroldinger [Ger97], the $\omega$-primality invariant assigns to each non-unit element $r$ in an integral domain $R$ a value $\omega(r) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, defined as follows. Recall that $r$ is prime if whenever $r$ divides a product $u_1 \cdots u_n$ of $n$ elements, then $r$ must divide one of the $u_i$. In this case, we set $\omega(r) = 1$. If $\omega(r) = 3$, then whenever $r$ divides a product $u_1 \cdots u_n$ of $n \geq 3$ elements, there exist a collection of 3 elements $u_i$, $u_j$, and $u_k$ appearing in the product such that $r$ divides
In this way, \( \omega(r) = 1 \) if and only if \( r \) is prime in \( R \), and the value \( \omega(r) \) measures how far the element \( r \) is from being prime.

Every prime element in an integral domain (or more generally, a cancellative commutative monoid) is irreducible, but the converse need not hold. In fact, the existence of non-prime irreducible elements in a monoid \( M \) coincides with the existence of non-unique factorizations. As such, the behavior of the \( \omega \)-function over the whole monoid \( M \) measures how far from factorial \( M \) is. Much of the initial work studying \( \omega \)-primality focused on \( \omega \)-values of irreducible elements [Ger97, GH08], and more recently, examining \( \omega \)-values of arbitrary non-unit elements has produced algorithms and closed formulas in several settings [AC10, CPS14, OPe14].

Theorem 4.1 below describes the eventual behavior of \( \omega \)-primality for affine monoids [One15], generalizing the existing result for numerical monoids [OPe13, GMV14a]. The proof uses Hilbert’s theorem in a similar manner to that used in Theorem 2.1, except that the final step in recovering the \( \omega \)-function requires taking the element-wise maximum of finitely many (eventually quasilinear) Hilbert functions. Although this still implies the \( \omega \)-function is eventually quasilinear, the resulting characterization leaves the most room for refinement among those appearing in [One15].

**Theorem 4.1 ([One15]).** The \( \omega \)-primality function \( \omega : M \to \mathbb{Z}_{>0} \) is eventually quasilinear for any affine monoid \( M \subset \mathbb{Z}^k_{\geq 0} \). Moreover, if \( M = \langle n_1, \ldots, n_k \rangle \subset \mathbb{Z}_{\geq 0} \) is numerical, then

\[
\omega(m) = \frac{1}{n_1} m + a_0(m)
\]

for some \( n_1 \)-periodic function \( a_0 \) and \( m \gg 0 \).

The \( \omega \)-values of irreducible elements in block monoids admit a particularly concise characterization [Ger97, GK10]. However, little is known about the \( \omega \)-values of the remaining block monoid elements. As noted earlier, block monoids are affine and satisfy a certain saturation condition, making them a natural setting in which to refine Theorem 4.1. Although the inductive step of Algorithm 4.2 immediately generalizes to affine monoids [One15], the base step may require infinitely many computations. In practice, only finitely many preliminary computations are actually required, but it is difficult to pin down a finite set which is sufficient to begin the inductive process. Preliminary computations indicate that the saturation condition characterizing block monoids may yield a finite set suitable for initiating the dynamic algorithm. In addition to providing the first step generalizing Algorithm 4.2 to affine monoids, an answer to Project 5 below would allow the use of computer data on Project 4.
Figure 3. A SAGE plot [SAGE] depicting the catenary degrees of elements of the numerical monoid \langle 17, 41, 43, 59, 61 \rangle.

Project 5. Develop a dynamic algorithm to compute \( \omega \)-values for block monoids.

Joint work on Project 5 is already in its early stages with the other authors of [BOP14b].

5. The Catenary Degree

The catenary degree invariant, described briefly in Section 1, assigns to each element \( m \in M \) a nonnegative integer \( c(m) \) derived from combinatorial properties of the set of factorizations of \( m \). Generally speaking, the value \( c(m) \) measures the distance between the factorizations of \( m \), analogous to how delta sets measure distance between factorization lengths.

The catenary degree of a monoid element \( m \) is formally defined in two steps. First, the distance \( d(f, f') \) between any two factorizations \( f \) and \( f' \) of \( m \) is defined in such a way that two factorizations are closer together if they have more irreducibles in common. Second, the catenary degree of \( m \) is defined as the minimum non-negative integer \( c(m) = r \) so that in the complete graph whose vertices are the factorizations of \( m \) and whose edges are labeled by the distance function, any two vertices are connected by a path whose edges are labeled at most \( c(m) \). Informally, an element \( m \) has a larger catenary degree if its factorizations are “further apart”.

Like the delta set invariant, the catenary degree function has been shown to be eventually periodic in numerical monoids [CCMMP14], and my recent work has generalized this result to Theorem 5.1 below for affine monoids [One15]. Figure 3 plots the catenary degree function \( c : M \to \mathbb{Z}_{\geq 0} \) for the numerical monoid \( M = \langle 17, 41, 43, 59, 61 \rangle \), which satisfies \( c(m + 17) = c(m) \) for all \( m > 250 \).

Theorem 5.1 ([One15]). The catenary degree function \( c : M \to \mathbb{Z}_{\geq 0} \) is eventually quasiconstant for any affine monoid \( M \subset \mathbb{Z}^k_{\geq 0} \). As a consequence, if \( M = \langle n_1, \ldots, n_k \rangle \subset \mathbb{Z}_{\geq 0} \) is a numerical monoid, then for all \( m \gg 0 \) we have

\[
  c(m + n_1 \cdots n_k) = c(m).
\]

The proof of Theorem 5.1 for numerical monoids appearing in [CCMMP14] is purely existential and relies on the fact that a nonincreasing sequence of positive integers is eventually constant. As such, no concrete bound on the start of catenary degree periodicity is known. Additionally, the eventual period is only known to divide \( n_1 \cdots n_k \), but in all computed examples the actual period is much smaller, usually dividing \( n_1 n_k \). This lack of detail is in stark contrast to the analogous results for the delta set and \( \omega \)-primality invariants.

A dynamic algorithm for computing the catenary degree in numerical monoids, similar to Algorithms 3.3 and 4.2 for computing delta sets and \( \omega \)-primality, would benefit furthering investigations into this periodic behavior. The iterative nature of the proof of Theorem 5.1 for numerical monoids given in [CCMMP14] provides strong evidence that such an algorithm exists.
**Project 6.** Develop a dynamic algorithm in the spirit of Algorithms 3.3 and 4.2 to compute catenary degrees of numerical monoid elements.

The catenary degree $c(M)$ of a monoid $M$ is defined as the supremum of the catenary degrees achieved by elements of $M$ [GH06]. For finitely generated monoids, this supremum is known to be finite, and always occurs on a finite (computable) set of monoid elements (called Betti elements) [CGLPR06, BGG11]. However, this value need not accurately reflect the factorization structure of $M$. For some numerical monoids, the maximum catenary degree is the only nonzero catenary degree achieved, yet for others, every non-negative integer between 2 and the maximum catenary degree occurs as the catenary degree of some monoid element.

The catenary degree $c(m)$ of a monoid element $m \in M$ is closely related to the delta set $\Delta(m)$, although neither completely determines the other. On the other hand, the monoid-wide factorization invariants $\Delta(M)$ and $c(M)$, both of which are standard in the literature, describe the factorization structure of $M$ in drastically different levels of detail. Indeed, the former maintains a comprehensive set of values, while the latter recalls only a single value. The set $C(M) = \{c(m) : m \in M\}$ of catenary degrees of $M$ is a closer analog of the delta set $\Delta(M)$, as it provides a comparably detailed description of the factorization structure of $M$. Recently, joint work with Ponomarenko, Tate, and Webb [OPTW15] demonstrates the following:

(i) the minimum nonzero value of $C(M)$ also occurs at a Betti element, and
(ii) some elements of $C(M)$ need not occur at Betti elements.

Currently, there is no known algorithm to compute the entire set of catenary degrees of a numerical monoid [Gar15]. However, the multigraded modules constructed in the proof of Theorem 5.1 are very similar to those used to prove Theorem 3.2 for delta sets. In particular, given an affine monoid $M$, an ascending chain of multigraded modules $N_0 \subset N_1 \subset N_2 \subset \cdots$ is constructed so that the Hilbert functions of each pair of sequential modules $(N_{j-1}, N_j)$ determine which monoid elements have catenary degree $j$. As such, there should be an algorithm that computes any affine monoid’s set of catenary degrees, analogous to the delta set algorithm resulting from Theorem 3.2.

**Project 7.** Develop an algorithm to compute the set of catenary degrees of any affine monoid by computing the multigraded modules arising in the proof of Theorem 5.1.
References


