Irreducible decomposition of binomial ideals

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Joint with Thomas Kahle and Ezra Miller

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An ideal $I \subset k[x_1, \ldots, x_n]$ is a *binomial ideal* if it is generated by polynomials with at most two terms.
The Question

Definition
An ideal \( I \subset \mathbb{k}[x_1, \ldots, x_n] \) is a *binomial ideal* if it is generated by polynomials with at most two terms.

Example
\[ \langle x - y \rangle \subset \mathbb{k}[x, y], \langle x^2 - xy, xy - y^2 \rangle \subset \mathbb{k}[x, y]. \]
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Definition

An ideal \( I \subset \mathbb{k}[x_1, \ldots, x_n] \) is a \textit{binomial ideal} if it is generated by polynomials with at most two terms.

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\[ \langle x^2 - y, x^2 + y \rangle = \langle x^2, y \rangle \subset \mathbb{k}[x, y], \langle x^2y - xy^2, x^3, y^3 \rangle \subset \mathbb{k}[x, y]. \]
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\]

Example

\[
x^2 - xy, x^3 - x^2, x^4 y^2 + xy^2 \in \langle x^2, y^2, xy \rangle \subset \mathbb{k}[x, y].
\]
The Question

Definition

An ideal \( I \subseteq S \) is \textit{irreducible} if whenever \( I = J_1 \cap J_2 \) for ideals \( J_1, J_2 \subseteq S \), either \( I = J_1 \) or \( I = J_2 \).
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Definition
An ideal \( I \subset S \) is irreducible if whenever \( I = J_1 \cap J_2 \) for ideals \( J_1, J_2 \subset S \), either \( I = J_1 \) or \( I = J_2 \).

Fact
Every ideal \( I \subset \mathbb{k}[x_1, \ldots, x_n] \) can be written as a finite intersection

\[
I = \bigcap_{i=1}^{r} J_i
\]

of irreducible ideals \( J_1, \ldots, J_r \) (an irreducible decomposition).
The Question

Question (Eisenbud-Sturmfels, 1996)

Assume \( k \) is algebraically closed. Does every binomial ideal \( I \) have a \textit{binomial} irreducible decomposition, that is, an expression \( I = \bigcap_i J_i \) where each \( J_i \) is irreducible and binomial?

Example

If \( k = \mathbb{Q} \), then \( \langle x^4 + 4 \rangle = \langle x^2 - 2x + 2 \rangle \cap \langle x^2 + 2x + 2 \rangle \).

Answer (Kahle-Miller-O., 2014)

No.

Example

\( I = \langle x^2y - xy^2, x^3, y^3 \rangle \subset k[x, y] \).
The Question

**Question (Eisenbud-Sturmfels, 1996)**

Assume $\mathbb{k}$ is algebraically closed. Does every binomial ideal $I$ have a *binomial* irreducible decomposition, that is, an expression $I = \bigcap_i J_i$ where each $J_i$ is irreducible and binomial?

**Example**

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\( I = \langle x^2y - xy^2, x^3, y^3 \rangle \subset k[x, y] \).
State of affairs:
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So, why was this problem open for almost 20 years?

Answer: Needed to know where to look.
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Answer: Needed to know where to look.
Today:

Storytime!

Review primary decomposition
Irreducible decomposition of monomial ideals
Irreducible decomposition of binomial ideals
Examine the counterexample, with proof (time permitting).

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- Review primary decomposition
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Primary Decomposition

**Definition**

An ideal $I$ is *primary* if $ab \in I$ implies $a^\ell \in I$ or $b^\ell \in I$ for some $\ell \geq 1$. If $I$ is primary, then $p = \sqrt{I}$ is prime, and we say $I$ is $p$-primary.

**Fact**

Any ideal in a Noetherian ring is a finite intersection of primary ideals (that is, admits a primary decomposition).

**Example**

Primary ideals in $\mathbb{Z}$ are of the form $\langle p^r \rangle$ for $p$ prime, and $\sqrt{\langle p^r \rangle} = \langle p \rangle$. For $a = p^{r_1} \cdots p^{r_\ell} \in \mathbb{Z}$, $\langle a \rangle = \bigcap_i \langle p^{r_i} \rangle$. 
Primary Decomposition

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Primary ideals in \( \mathbb{Z} \) are of the form \( \langle p^r \rangle \) for \( p \) prime, and \( \sqrt{\langle p^r \rangle} = \langle p \rangle \).

For \( a = p_1^{r_1} \cdots p_\ell^{r_\ell} \in \mathbb{Z} \), \( \langle a \rangle = \bigcap_i \langle p_i^{r_i} \rangle \).
Fact

Irreducible ideals are primary.
Irreducible Ideals

Fact

*Irreducible ideals are primary.*

Definition

Given a \( p \)-primary ideal \( I \subset k[x_1, \ldots, x_n] \), the *socle* of \( I \) is the ideal

\[
\text{soc}_p(I) = \{ f : pf \subset I \} \subset I
\]

We say \( I \) has *simple socle* if \( \dim_k \text{soc}_p(I)/I = 1 \).
Irreducible Ideals

Fact
Irreducible ideals are primary.

Definition
Given a $p$-primary ideal $I \subset k[x_1, \ldots, x_n]$, the socle of $I$ is the ideal

$$\text{soc}_p(I) = \{ f : pf \subset I \} \subset I$$

We say $I$ has simple socle if $\dim_k \text{soc}_p(I)/I = 1$.

Fact
A $p$-primary ideal $I$ is irreducible if and only if it has simple socle.
Let \( I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \subset \mathbb{k}[x, y] \), and let \( p = \langle x, y \rangle \).
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so $\dim_{\mathbb{k}}(\text{soc}_p(I)/I) = 2$. 
Long long ago, in an algebraic setting not far away...
Monomial Ideals

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Monomial Ideals
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\[ x^a = x_1^{a_1} \cdots x_n^{a_n} \in \mathbb{k}[x_1, \ldots, x_n] \]
Monomial Ideals

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Connect all monomials \( x^a \in I \)
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Staircase Diagram
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Generators of \( I \) are

“Inward-pointing corners”

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Connect all monomials \( x^a \in I \)

Generators of \( I \) are "Inward-pointing corners"
Fact

If a monomial ideal $I$ is $\mathfrak{p}$-primary, then $\mathfrak{p}$ is a monomial ideal.
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If a monomial ideal $I$ is $\mathfrak{p}$-primary, then $\mathfrak{p}$ is a monomial ideal.

Fact

Any monomial ideal $I$ admits a monomial irreducible decomposition, that is, an expression of the form

$$I = \bigcap_{i=1}^{r} J_i$$

for irreducible monomial ideals $J_1, \ldots, J_r$. 
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\( I \) is \( p \)-primary, \( p = \langle x, y \rangle \)
\[ I = \langle x^4, x^3y, x^2y^2, y^4 \rangle \]

\[ I \text{ is } p\text{-primary, } p = \langle x, y \rangle \]

\[ \text{soc}_p(I)/I = \mathbb{k}\{x^3, x^2y, xy^3\} \]
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“Outward-pointing corners”
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“Outward-pointing corners”

Irreducible decomposition:

$I = J_1 \cap J_2 \cap J_3$
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Fix an *irredundant* irreducible decomposition

\[ I = \bigcap_{i=1}^{r} J_i \]

for a \( p \)-primary ideal \( I \).
Fix an irredundant irreducible decomposition

\[ I = \bigcap_{i=1}^{r} J_i \]

for a \( p \)-primary ideal \( I \).

- \( r = \dim_{\mathbb{k}} \text{soc}_p(I)/I \).
Fix an *irredundant* irreducible decomposition

\[ I = \bigcap_{i=1}^{r} J_i \]

for a \( p \)-primary ideal \( I \).

- \( r = \dim_k \text{soc}_p(I)/I \).
- For each \( i \), the map \( R/I \rightarrow R/J_i \) induces a nonzero map on socles.
Irreducible Decomposition

Facts

Fix an irredundant irreducible decomposition

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- For each \( i \), the map \( R/I \to R/J_i \) induces a nonzero map on socles.
- More generally, \( \text{soc}_p(I)/I \cong \bigoplus_{i=1}^{r} \text{soc}_p(J_i)/J_i \).
Fix an *irredundant* irreducible decomposition

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- For each \( i \), the map \( R/I \to R/J_i \) induces a nonzero map on socles.
- More generally, \( \text{soc}_p(I)/I \cong \bigoplus_{i=1}^{r} \text{soc}_p(J_i)/J_i \).
- If \( I \) is monomial ideal, then \( \text{soc}_p(I) \) is monomial.
And now, back to our original programming...
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Binomial ideals
Theorem (Eisenbud-Sturmfels, 1996)

If $k = \overline{k}$, every binomial ideal admits a binomial primary decomposition.
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Does the same hold for irreducible decomposition?
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Does the same hold for irreducible decomposition?

- In 2002, Dickenstein, Matusevich and Miller investigate the combinatorics of binomial primary decomposition.
Theorem (Eisenbud-Sturmfels, 1996)

If $k = \overline{k}$, every binomial ideal admits a binomial primary decomposition.

Question (Eisenbud-Sturmfels, 1996)

Does the same hold for irreducible decomposition?

- In 2002, Dickenstein, Matusevich and Miller investigate the combinatorics of binomial primary decomposition.
- In 2013, Kahle and Miller give a combinatorial method of explicitly constructing binomial primary decomposition.
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]
\[ l = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ x^a \in k[x_1, \ldots, x_n] \leftrightarrow a \in \mathbb{N}^n \]
Binomial Ideals

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Define relation \( \sim_I \) on \( \mathbb{N}^n \):
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Define relation \( \sim_I \) on \( \mathbb{N}^n \):

\[ a \sim_I b \in \mathbb{N}^n \longleftrightarrow x^a - \lambda x^b \in I \]

for some nonzero \( \lambda \in k \)
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ x^a \in \mathbb{k}[x_1, \ldots, x_n] \iff a \in \mathbb{N}^n \]

Define relation \( \sim_I \) on \( \mathbb{N}^n \):

\[ a \sim_I b \in \mathbb{N}^n \iff x^a - \lambda x^b \in I \quad \text{for some nonzero } \lambda \in \mathbb{k} \]

\[ x^2 - xy \in I, \]
Binomial Ideals

\[ l = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ x^a \in k[x_1, \ldots, x_n] \leftrightarrow a \in \mathbb{N}^n \]

Define relation \( \sim_I \) on \( \mathbb{N}^n \):

\[ a \sim_I b \in \mathbb{N}^n \leftrightarrow x^a - \lambda x^b \in l \]

for some nonzero \( \lambda \in k \)

\[ x^2 - xy \in l, \ xy - y^2 \in l, \]
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ x^a \in \mathbb{k}[x_1, \ldots, x_n] \leftrightarrow a \in \mathbb{N}^n \]

Define relation \( \sim_I \) on \( \mathbb{N}^n \):

\[ a \sim_I b \in \mathbb{N}^n \leftrightarrow x^a - \lambda x^b \in I \]

for some nonzero \( \lambda \in \mathbb{k} \)

\[ x^2 - xy \in I, \ xy - y^2 \in I, \ x(x^2 - xy) = x^3 - x^2y \in I, \]
$$I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle$$

$$x^a \in k[x_1, \ldots, x_n] \leftrightarrow a \in \mathbb{N}^n$$

Define relation $\sim_I$ on $\mathbb{N}^n$:

$$a \sim_I b \in \mathbb{N}^n \leftrightarrow x^a - \lambda x^b \in I$$

for some nonzero $\lambda \in k$

$$x^2 - xy \in I, \ xy - y^2 \in I,$$
$$x(x^2 - xy) = x^3 - x^2y \in I, \ldots$$
\[ l = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ x^a \in k[x_1, \ldots, x_n] \leftrightarrow a \in \mathbb{N}^n \]

Define relation \( \sim_l \) on \( \mathbb{N}^n \):

\[ a \sim_l b \in \mathbb{N}^n \leftrightarrow x^a - \lambda x^b \in l \]

for some nonzero \( \lambda \in k \)

\[ x^2 - xy \in l, \ xy - y^2 \in l, \ x(x^2 - xy) = x^3 - x^2y \in l, \ldots \]

\[ x^a, x^b \in l \Rightarrow x^a - x^b \in l \]
$I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle$

$x^a \in k[x_1, \ldots, x_n] \iff a \in \mathbb{N}^n$

Define relation $\sim_I$ on $\mathbb{N}^n$:

$a \sim_I b \in \mathbb{N}^n \iff x^a - \lambda x^b \in I$

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$x^2 - xy \in I$, $xy - y^2 \in I$,
$x(x^2 - xy) = x^3 - x^2y \in I$, ...

$x^a, x^b \in I \Rightarrow x^a - x^b \in I$
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ x^a \in \mathbb{k}[x_1, \ldots, x_n] \iff a \in \mathbb{N}^n \]

Define relation \( \sim_I \) on \( \mathbb{N}^n \):

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for some nonzero \( \lambda \in \mathbb{k} \)

\[ x^2 - xy \in I, \ xy - y^2 \in I, \ x(x^2 - xy) = x^3 - x^2y \in I, \ldots \]

\[ x^a, x^b \in I \implies x^a - x^b \in I \]

\[ (x^2 = xy \text{ in } \mathbb{k}[x, y]/I) \]
Fix a binomial ideal \( I \subset \mathbb{k}[x_1, \ldots, x_n] \).
Fix a binomial ideal $I \subset k[x_1, \ldots, x_n]$.

- The equivalence relation $\sim_I$ induced by $I$ on $\mathbb{N}^n$ is a congruence:

  $$a \sim_I b \implies a + c \sim_I b + c$$

  for $a, b, c \in \mathbb{N}^n$. In particular, $(\mathbb{N}^n/\sim_I, +)$ is well defined.
Fix a binomial ideal $I \subset \mathbb{k}[x_1, \ldots, x_n]$.

- The equivalence relation $\sim_I$ induced by $I$ on $\mathbb{N}^n$ is a congruence:

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- The monomials in $I$ form a single class $\infty \in \mathbb{N}^n/\sim_I$, called the nil.
- The nil $\infty$ corresponds to 0 in the quotient $k[x_1, \ldots, x_n]/I$.
- Each non-nil $\bar{a} \in \mathbb{N}^n/\sim_I$ represents a distinct monomial modulo $I$. 
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

Monoid \( \mathbb{N}^2/\sim_I \)
Theorem (Kahle-Miller, 2013)

For \( \overline{k} = \overline{k} \), every binomial ideal has an expression of the form

\[
I = \bigcap_{i=1}^{r} J_i
\]

where each \( J_i \) is binomial, primary, and has a unique monomial in its socle.
Theorem (Kahle-Miller, 2013)

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To construct a binomial irreducible decomposition for $I$, we can assume...
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For $k = \overline{k}$, every binomial ideal has an expression of the form

\[ I = \bigcap_{i=1}^{r} J_i \]

where each $J_i$ is binomial, primary, and has a unique monomial in its socle.

To construct a binomial irreducible decomposition for $I$, we can assume

- $I$ is primary to the maximal ideal $m$,
- $\text{soc}_m(I)/I$ has a unique monomial.
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ \mathbb{N}^n/\sim_I \longleftrightarrow \text{monomials mod } I \]
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ \mathbb{N}^n / \sim_I \leftrightarrow \text{monomials mod } I \]

\[ \text{soc}_m(I)/I = \mathbb{k}\{x^3, x - y\} \]
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ \mathbb{N}^n / \sim_I \leftrightarrow \text{monomials mod } I \]

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$I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle$

$\mathbb{N}^n/\sim_I \leftrightarrow$ monomials mod $I$

$\text{soc}_m(I)/I = \mathbb{k}\{x^3, x - y\}$
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

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Binomial Ideals

\( I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \)

\( \mathbb{N}^n / \sim_I \leftrightarrow \text{monomials mod } I \)

\( \text{soc}_m(I) / I = \mathbb{k}\{x^3, x - y\} \)

\text{witnesses}: \text{monomials that merge with something in each direction}
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ \mathbb{N}^n / \sim_I \leftrightarrow \text{monomials mod } I \]

\[ \text{soc}_m(I)/I = \mathbb{k}\{x^3, x - y\} \]

witnesses: monomials that merge with something in each direction

\[ I\text{-witnesses: } x^3, x, y \]
Binomial Ideals

**Definition**

A monomial $x^a$ is a *witness* for $I$ if for each $x^p \in p$,

$$p + a \sim_I p + a' \text{ for some } a' \not\sim_I a,$$

that is, $x^a$ merges with another monomial modulo $I$ when multiplied by any monomial in $p$. 

Theorem (Kahle-Miller, 2013)

For any $p$-primary binomial ideal $I$, any $f \in \text{soc}_p(I)/I$ is a sum of witnesses.
Definition

A monomial $x^a$ is a *witness* for $I$ if for each $x^p \in p$,

$$p + a \sim_I p + a'$$

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Theorem (Kahle-Miller, 2013)

*For any $p$-primary binomial ideal $I$, any $f \in \text{soc}_p(I)/I$ is a sum of witnesses.*
$$I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle$$
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ \text{soc}_m(I)/I = \mathbb{k}\{x^3, x - y\} \]
Binomial Ideals

\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ \text{soc}_m(I)/I = \mathbb{k}\{x^3, x - y\} \]

\textit{soccularize} \( I \): “Force simple socle”
$I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle$

$soc_m(I)/I = \mathbb{k}\{x^3, x - y\}$

_soccularize_ $I$: “Force simple socle”
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Binomial Ideals

\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ \text{soc}_m(I)/I = \mathbb{k}\{x^3, x - y\} \]

\textit{sockularize} \( I \): “Force simple socle”

\[ J = \langle x - y, x^4, y^4 \rangle \]

\[ \text{soc}_m(J)/J = \mathbb{k}\{x^3\} \]
Plan of attack:
Plan of attack:

- One irreducible component per witness monomial.
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- One irreducible component per witness monomial.
- For each component, force chosen witness to be maximal.
Plan of attack:

- One irreducible component per witness monomial.
- For each component, force chosen witness to be maximal.
- Soccularize to remove other socle elements.
Soccural Decomposition

\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]
Soccular Decomposition

\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x, y \)
\[ l = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x, y \)
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x, y \)
$I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle$

Witnesses: $x^3, x, y$

$J_1 = \langle x - y, x^4, y^4 \rangle$, 

\begin{center}
\begin{tikzpicture}
\end{tikzpicture}
\end{center}
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x, y \)

\[ J_1 = \langle x - y, x^4, y^4 \rangle, \]
Soccular Decomposition

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Witnesses: \( x^3, x, y \)

\[ J_1 = \langle x - y, x^4, y^4 \rangle, \]

\[ J_2 = \langle x^2, y \rangle, \]
Soccular Decomposition

\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x, y \)

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Witnesses: \( x^3, x, y \)

\[ J_1 = \langle x - y, x^4, y^4 \rangle, \]
\[ J_2 = \langle x^2, y \rangle, \]
\[ J_3 = \langle x, y^2 \rangle \]
$I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle$

Witnesses: $x^3, x, y$

$J_1 = \langle x - y, x^4, y^4 \rangle,$

$J_2 = \langle x^2, y \rangle, \ J_3 = \langle x, y^2 \rangle$

$I = J_1 \cap J_2 \cap J_3$
I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle

Witnesses: x^3, x, y

J_1 = \langle x - y, x^4, y^4 \rangle,

J_2 = \langle x^2, y \rangle, J_3 = \langle x, y^2 \rangle

I = J_1 \cap J_2 \cap J_3
   = J_1 \cap J_2
Soccular Decomposition

\[ I = \langle x^2 - xy, xy + y^2, x^4, y^4 \rangle \]
$I \ = \ \langle x^2 - xy, xy + y^2, x^4, y^4 \rangle$
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Witnesses: \( x^3, x, y \)
Soccular Decomposition

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Witnesses: \( x^3, x, y \)

\[ \text{soc}_m(I)/I = \mathbb{k}\{x^3\} \]
$I = \langle x^2 - xy, xy + y^2, x^4, y^4 \rangle$

Witnesses: $x^3, x, y$

$\text{soc}_m(I)/I = \mathbb{k}\{x^3\}$

$I = I \cap \langle x^2, y \rangle \cap \langle x, y^2 \rangle$
Soccular Decomposition

\[ I = \langle xy - y^2, x^3 - xy^2, x^4, y^4 \rangle \]
\[ I = \langle xy - y^2, x^3 - xy^2, x^4, y^4 \rangle \]
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Witnesses: \( x^3, x^2, xy \)
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Witnesses: \( x^3, x^2, xy \)

\[ \operatorname{soc}_m(I)/I = \mathbb{k}\{x^3, x^2 - xy\} \]
\[ I = \langle xy - y^2, x^3 - xy^2, x^4, y^4 \rangle \]

**Witnesses:** \( x^3, x^2, xy \)

\[ \text{soc}_m(I)/I = \mathbb{k}\{x^3, x^2 - xy\} \]

**Soccularize:**
\[ I = \langle xy - y^2, x^3 - xy^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x^2, xy \)

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Witnesses: \( x^3, x^2, xy \)

\[ \text{soc}_m(I)/I = \mathbb{K}\{x^3, x^2 - xy\} \]

Soccularize: New witnesses!
Soccular Decomposition

\[ I = \langle xy - y^2, x^3 - xy^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x^2, xy \)

\[ \text{soc}_m(I)/I = \mathbb{k}\{x^3, x^2 - xy\} \]

Soccularize: New witnesses!

*Protected* witnesses: \( x, y \)
Soccular Decomposition

\[ I = \langle xy - y^2, x^3 - xy^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x^2, xy \)

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Soccularize: New witnesses!

Protected witnesses: \( x, y \)

\[ J_1 = \langle x - y, x^4, y^4 \rangle \]
Soccular Decomposition

\[ I = \langle xy - y^2, x^3 - xy^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x^2, xy \)

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Soccularize: New witnesses!

Protected witnesses: \( x, y \)

\[ J_1 = \langle x - y, x^4, y^4 \rangle \]
\[ J_2 = \langle x^3, y \rangle \]
\[ \mathcal{I} = \langle xy - y^2, x^3 - xy^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x^2, xy \)

\[ \text{soc}_m(\mathcal{I})/\mathcal{I} = \mathbb{K}\{x^3, x^2 - xy\} \]

Soccularize: New witnesses!

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Soccularize: New witnesses!
Protected witnesses: $x, y$

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Soccularize: New witnesses!

*Protected* witnesses: \( x, y \)

\[
J_1 = \langle x - y, x^4, y^4 \rangle \\
J_2 = \langle x^3, y \rangle \\
J_3 = \langle xy - y^2, x^2, y^3 \rangle
\]

\[ I = J_1 \cap J_2 \cap J_3 \]
Soccular Decomposition

Algorithm for decompositing a binomial ideal $I$:

One component for each $I$-witness. For the component at a witness $w$:

Add monomials not below $w$, so $w$ is a unique monomial socle element.

"Soccularize" by merging witness pairs below $w$.

Repeat with protected witnesses until no new witness pairs are created.

Theorem (Kahle-Miller-O., 2014)

For $k = k$, any binomial ideal $I$ can be written as $I = \bigcap_{i=1}^{r} J_i$, where each $J_i$ is binomial and $p_i$-primary, and the socle $soc_{p_i}(J_i) / J_i$ contains a unique monomial and no other binomials.
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  - Repeat with protected witnesses until no new witness pairs are created

Theorem (Kahle-Miller-O., 2014)

For $k = k_1$, any binomial ideal $I$ can be written as $I = \bigcap_{i=1}^{r} J_i$, where each $J_i$ is binomial and $p_i$-primary, and the socle $\text{soc}_{p_i}(J_i)/J_i$ contains a unique monomial and no other binomials.
Algorithm for decompositing a binomial ideal $I$:

- One component for each $I$-witness.
- For the component at a witness $w$:
  - Add monomials not below $w$, so $w$ is a unique monomial socle element.
  - “Soccularize” by merging witness pairs below $w$.
  - Repeat with protected witnesses until no new witness pairs are created.

Theorem (Kahle-Miller-O., 2014)

For $k = \bar{k}$, any binomial ideal $I$ can be written as $I = \bigcap_{i=1}^{r} J_i$, where each $J_i$ is binomial and $\mathfrak{p}_i$-primary, and the socle $\text{soc}_{\mathfrak{p}_i}(J_i)/J_i$ contains a unique monomial and no other binomials.
The Counterexample

\[ I = \langle x^2y - xy^2, x^3, y^3 \rangle \]
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Witnesses: \( x^2y, x^2, xy, y^2 \)
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\[ I = \langle x^2 y - xy^2, x^3, y^3 \rangle \]

Witnesses: \( x^2 y, x^2, xy, y^2 \)

\[ \text{soc}_m(I)/I = \mathbb{k}\{x^2 y, x^2 + y^2 - xy\} \]
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\[ I = \langle x^2 + y^2 - xy, x^3, y^3 \rangle \cap \langle x^3, y \rangle \]
The Theorem (Kahle-Miller-O., 2014) states that:

\[ I = \langle x^2y - xy^2, x^3, y^3 \rangle \] admits no binomial irreducible decomposition.
Theorem (Kahle-Miller-O., 2014)

\[ I = \langle x^2y - xy^2, x^3, y^3 \rangle \text{ admits no binomial irreducible decomposition.} \]

Proof.

Fix an irredundant irreducible decomposition \( I = \bigcap_{i=1}^{r} J_i \).
The Counterexample

**Theorem (Kahle-Miller-O., 2014)**

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**Proof.**

Fix an irredundant irreducible decomposition \( I = \bigcap_{i=1}^{r} J_i \).

We have \( r = \dim_\mathbb{K}(\text{soc}_m(I)/I) = 2 \), so \( I = J_1 \cap J_2 \).
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Write \( \alpha = x^2 + y^2 - xy \), \( \beta = x^2y \), so \( \soc_m(I)/I = \mathbb{k}\{\alpha, \beta\} \).
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We know

\[ \text{soc}_m(I)/I \cong \text{soc}_m(J_1)/J_1 \oplus \text{soc}_m(J_2)/J_2, \]

so we have \( \alpha + \lambda \beta \in \text{soc}_m(J_i)/J_i \) for some \( i \), say \( i = 1 \).
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Theorem (Kahle-Miller-O., 2014)

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This means \( I + \langle \alpha + \lambda \beta \rangle \subset J_1 \).
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Write \( \alpha = x^2 + y^2 - xy \), \( \beta = x^2y \), so \( \text{soc}_m(I)/I = k\{\alpha, \beta\} \).
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This means \( I + \langle \alpha + \lambda \beta \rangle \subset J_1 \).
But \( I + \langle \alpha + \lambda \beta \rangle \) already has simple socle, so \( J_1 = I + \langle \alpha + \lambda \beta \rangle \). \( \square \)
References

David Eisenbud, Bernd Sturmfels (1996)
Binomial ideals.

Ezra Miller, Bernd Sturmfels (2005)
Combinatorial commutative algebra.

Thomas Kahle, Ezra Miller (2013)
Decompositions of commutative monoid congruences and binomial ideals.
arXiv:1107.4699 [math].

Thomas Kahle, Ezra Miller, Christopher O'Neill (2014)
Irreducible decompositions of binomial ideals.
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Thanks!
When do they exist?

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Witnesses: \( x^4, x^3, x^2 y, y^3 \)
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