Irreducible decomposition of binomial ideals

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Joint with Thomas Kahle and Ezra Miller

January 18, 2014
An ideal \( I \subset \mathbb{k}[x_1, \ldots, x_n] \) is a *binomial ideal* if it is generated by polynomials with at most two terms.
Definition

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Example

The following are binomial ideals in $\mathbb{k}[x, y]$:

\[
\langle x^2 - xy, xy - y^2 \rangle, \quad \langle x^2 y - xy^2, x^3, y^3 \rangle, \quad \langle x^2 - y, x^2 + y \rangle = \langle x^2, y \rangle, \quad x^2 - xy, x^3 - x^2, x^4 y^2 + xy^2 \in \langle x^2, y^2, xy \rangle.
\]
The Question

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- \( \langle x^2 - xy, xy - y^2 \rangle \),
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Definition

An ideal $I \subset S$ is *irreducible* if whenever $I = J_1 \cap J_2$ for ideals $J_1, J_2 \subset S$, either $I = J_1$ or $I = J_2$. 

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Fact
Every ideal \( I \subset \mathbb{k}[x_1, \ldots, x_n] \) can be written as a finite intersection

\[
I = \bigcap_{i=1}^{r} J_i
\]

of irreducible ideals \( J_1, \ldots, J_r \) (an irreducible decomposition).
The Question

Question (Eisenbud-Sturmfels, 1996)

Assume $\mathbb{k}$ is algebraically closed. Does every binomial ideal $I$ have a binomial irreducible decomposition, that is, an expression $I = \bigcap_i J_i$ where each $J_i$ is irreducible and binomial?
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**Example**

If \( k = \mathbb{Q} \), then \( \langle x^4 + 4 \rangle = \langle x^2 - 2x + 2 \rangle \cap \langle x^2 + 2x + 2 \rangle \).
The Question

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Answer (Kahle-Miller-O., 2014)

No.
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Answer (Kahle-Miller-O., 2014)

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Example

\( I = \langle x^2 y - xy^2, x^3, y^3 \rangle \subset k[x, y] \).
The Question

State of affairs:
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- Question: easy to state
The Question

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- Counter example: small

So, why was this problem open for almost 20 years?

Answer: Needed to know where to look.
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Answer: Needed to know where to look.
Today:
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- Review primary decomposition
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- Irreducible decomposition of binomial ideals
- Examine the counterexample, with proof (time permitting).
**Definition**

An ideal $I$ is *primary* if $ab \in I$ implies $a^\ell \in I$ or $b^\ell \in I$ for some $\ell \geq 1$. If $I$ is primary, then $p = \sqrt{I}$ is prime, and we say $I$ is $p$-primary.

**Fact**

Any ideal in a Noetherian ring is a finite intersection of primary ideals (that is, admits a primary decomposition).

**Example**

Primary ideals in $\mathbb{Z}$ are of the form $\langle p^r \rangle$ for $p$ prime, and $\sqrt{\langle p^r \rangle} = \langle p \rangle$.

For $a = p^{r_1} \cdots p^{r_\ell} \in \mathbb{Z}$, $\langle a \rangle = \bigcap_i \langle p^{r_i} \rangle$. 

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**Primary Decomposition**

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For $a = p_1^{r_1} \cdots p_\ell^{r_\ell} \in \mathbb{Z}$, $\langle a \rangle = \bigcap_i \langle p_i^{r_i} \rangle$. 
Fact

Irreducible ideals are primary.
Irreducible Ideals

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Definition

Given a \( p \)-primary ideal \( I \subset \mathbb{k}[x_1, \ldots, x_n] \), the socle of \( I \) is the ideal

\[
\text{soc}_p(I) = \{ f : pf \subset I \} \supset I
\]

We say \( I \) has simple socle if \( \dim_{\mathbb{k}} \text{soc}_p(I)/I = 1 \).
Irreducible Ideals

Fact

Irreducible ideals are primary.

Definition

Given a $p$-primary ideal $I \subseteq \mathbb{k}[x_1, \ldots, x_n]$, the socle of $I$ is the ideal

$$\text{soc}_p(I) = \{ f : pf \subseteq I \} \supset I$$

We say $I$ has simple socle if $\dim_{\mathbb{k}} \text{soc}_p(I)/I = 1$.

Fact

A $p$-primary ideal $I$ is irreducible if and only if it has simple socle.
Let $I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \subset \mathbb{k}[x, y]$, and let $p = \langle x, y \rangle$. 
Let \( I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \subset k[x, y], \) and let \( p = \langle x, y \rangle. \)

\[ x - y \in \text{soc}_p(I) \]
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\]

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\]

so \( \text{dim}_\mathbb{k}(\text{soc}_p(I)/I) = 2 \).
Long long ago, in an algebraic setting not far away...
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Monomial Ideals
Monomial Ideals

\[ I = \langle x^4, x^3y, x^2y^2, y^4 \rangle \]
Monomial Ideals

\[ I = \left\langle x^4, x^3y, x^2y^2, y^4 \right\rangle \]

\[ x^a = x_1^{a_1} \cdots x_n^{a_n} \in k[x_1, \ldots, x_n] \]
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\[ x^a = x_1^{a_1} \cdots x_n^{a_n} \in \mathbb{k}[x_1, \ldots, x_n] \]

\[ \longleftrightarrow a = (a_1, \ldots, a_n) \in \mathbb{N}^n \]
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Connect all monomials \( x^a \in I \)
Monomial Ideals

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Connect all monomials \( x^a \in I \)

Staircase Diagram
\[ I = \langle x^4, x^3y, x^2y^2, y^4 \rangle \]

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Connect all monomials \( x^a \in I \)

Generators of \( I \) are
“Inward-pointing corners”
Monomial Ideals

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Staircase Diagram
Fact

If a monomial ideal \( I \) is \( \mathfrak{p} \)-primary, then \( \mathfrak{p} \) is a monomial ideal.
Monomial Ideals

Fact
If a monomial ideal $I$ is $p$-primary, then $p$ is a monomial ideal.

Fact
Any monomial ideal $I$ admits a monomial irreducible decomposition, that is, an expression of the form

$$I = \bigcap_{i=1}^{r} J_i$$

for irreducible monomial ideals $J_1, \ldots, J_r$. 
Monomial Ideals

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Monomial Ideals

$I = \langle x^4, x^3y, x^2y^2, y^4 \rangle$

$I$ is $p$-primary, $p = \langle x, y \rangle$
Monomial Ideals

\[ I = \langle x^4, x^3y, x^2y^2, y^4 \rangle \]

\[ I \text{ is } p\text{-primary, } p = \langle x, y \rangle \]

\[ \text{soc}_p(I)/I = k\{x^3, x^2y, xy^3\} \]
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“Outward-pointing corners”
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"Outward-pointing corners"

Irreducible decomposition:
\[ I = J_1 \cap J_2 \cap J_3 \]
Monomial Ideals

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Fix an *irredundant* irreducible decomposition

\[ I = \bigcap_{i=1}^{r} J_i \]

for a \( p \)-primary ideal \( I \).
Irreducible Decomposition

Facts

Fix an *irredundant* irreducible decomposition

\[ I = \bigcap_{i=1}^{r} J_i \]

for a \( p \)-primary ideal \( I \).

- \( r = \dim_{\mathbb{k}} \text{soc}_p(I)/I \).
Fix an \textit{irredundant} irreducible decomposition

\[ I = \bigcap_{i=1}^{r} J_i \]

for a \( p \)-primary ideal \( I \).

- \( r = \dim_{\mathbb{k}} \soc_p(I)/I \).
- For each \( i \), the map \( R/I \to R/J_i \) induces a nonzero map on socles.
Irreducible Decomposition

Facts

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- \( r = \dim_{\mathbb{k}} \soc_p(I)/I \).
- For each \( i \), the map \( R/I \rightarrow R/J_i \) induces a nonzero map on socles.
- More generally, \( \soc_p(I)/I \cong \bigoplus_{i=1}^{r} \soc_p(J_i)/J_i \).
Facts

Fix an *irredundant* irreducible decomposition

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- \( r = \dim_{\mathbb{k}} \text{soc}_p(I)/I \).
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- More generally, \( \text{soc}_p(I)/I \cong \bigoplus_{i=1}^{r} \text{soc}_p(J_i)/J_i \).
- If \( I \) is monomial ideal, then \( \text{soc}_p(I) \) is monomial.
And now, back to our original programming...
And now, back to our original programming...

Binomial ideals
Theorem (Eisenbud-Sturmfels, 1996)

If $k = \overline{k}$, every binomial ideal admits a binomial primary decomposition.
Theorem (Eisenbud-Sturmfels, 1996)

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Question (Eisenbud-Sturmfels, 1996)

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Question (Eisenbud-Sturmfels, 1996)

Does the same hold for irreducible decomposition?

- In 2008, Dickenstein, Matuschevich and Miller investigate the combinatorics of binomial primary decomposition.
- In 2013, Kahle and Miller give a combinatorial method of explicitly constructing binomial primary decomposition.
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]
\[ \mathcal{I} = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ x^a \in k[x_1, \ldots, x_n] \leftrightarrow a \in \mathbb{N}^n \]
Binomial Ideals

\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ x^a \in k[x_1, \ldots, x_n] \leftrightarrow a \in \mathbb{N}^n \]

Define relation \( \sim_I \) on \( \mathbb{N}^n \):
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ x^a \in \mathbb{k}[x_1, \ldots, x_n] \iff a \in \mathbb{N}^n \]

Define relation \( \sim_I \) on \( \mathbb{N}^n \):

\[ a \sim_I b \in \mathbb{N}^n \iff x^a - \lambda x^b \in I \]

for some nonzero \( \lambda \in \mathbb{k} \)
$$I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle$$

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Define relation $\sim_I$ on $\mathbb{N}^n$:

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$$x^2 - xy \in I,$$
\[ l = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ x^a \in \mathbb{k}[x_1, \ldots, x_n] \leftrightarrow a \in \mathbb{N}^n \]

Define relation \( \sim_I \) on \( \mathbb{N}^n \):

\[ a \sim_I b \in \mathbb{N}^n \leftrightarrow x^a - \lambda x^b \in l \]

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\[ x^2 - xy \in l, \ xy - y^2 \in l, \]
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ \mathbf{x}^a \in k[x_1, \ldots, x_n] \longleftrightarrow a \in \mathbb{N}^n \]

Define relation \( \sim_I \) on \( \mathbb{N}^n \):

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\[ x^2 - xy \in I, \ xy - y^2 \in I, \ x(x^2 - xy) = x^3 - x^2 y \in I, \]
Binomial Ideals

\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

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Define relation \( \sim_I \) on \( \mathbb{N}^n \):

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\[ x^2 - xy \in \mathcal{I}, \ xy - y^2 \in \mathcal{I}, \]

\[ x(x^2 - xy) = x^3 - x^2y \in \mathcal{I}, \ldots \]

\[ x^a, x^b \in \mathcal{I} \Rightarrow x^a - x^b \in \mathcal{I} \]
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

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Binomial Ideals

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Define relation \( \sim_I \) on \( \mathbb{N}^n \):

\[ a \sim_I b \in \mathbb{N}^n \iff x^a - \lambda x^b \in I \]

for some nonzero \( \lambda \in \mathbb{k} \)

\[ x^2 - xy \in I, \, xy - y^2 \in I, \]

\[ x(x^2 - xy) = x^3 - x^2 y \in I, \ldots \]

\[ x^a, x^b \in I \Rightarrow x^a - x^b \in I \]

\[ (x^2 = xy \text{ in } \mathbb{k}[x, y]/I) \]
Fix a binomial ideal $I \subset \mathbb{k}[x_1, \ldots, x_n]$. 
Fix a binomial ideal $I \subset \mathbb{k}[x_1, \ldots, x_n]$.

- The equivalence relation $\sim_I$ induced by $I$ on $\mathbb{N}^n$ is a congruence:

  $a \sim_I b$ implies $a + c \sim_I b + c$

  for $a, b, c \in \mathbb{N}^n$. In particular, $(\mathbb{N}^n/\sim_I, +)$ is well defined.
Fix a binomial ideal \( I \subset k[x_1, \ldots, x_n] \).

- The equivalence relation \( \sim_I \) induced by \( I \) on \( \mathbb{N}^n \) is a congruence:
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- The monomials in \( I \) form a single class \( \infty \in \mathbb{N}^n/\sim_I \), called the nil.
Fix a binomial ideal $I \subset \mathbb{k}[x_1, \ldots, x_n]$.

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- The monomials in $I$ form a single class $\infty \in \mathbb{N}^n/\sim_I$, called the nil.

- The nil $\infty$ corresponds to 0 in the quotient $\mathbb{k}[x_1, \ldots, x_n]/I$. 
Fix a binomial ideal \( I \subset k[x_1, \ldots, x_n] \).

- The equivalence relation \( \sim_I \) induced by \( I \) on \( \mathbb{N}^n \) is a congruence:

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a \sim_I b \implies a + c \sim_I b + c
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- The monomials in \( I \) form a single class \( \infty \in \mathbb{N}^n/\sim_I \), called the nil.

- The nil \( \infty \) corresponds to 0 in the quotient \( k[x_1, \ldots, x_n]/I \).

- Each non-nil \( \bar{a} \in \mathbb{N}^n/\sim_I \) represents a distinct monomial modulo \( I \).
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]
Binomial Ideals

\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

Monoid \( \mathbb{N}^2 / \sim_I \)
Theorem (Kahle-Miller, 2013)

For $k = \overline{k}$, every binomial ideal has an expression of the form

$$I = \bigcap_{i=1}^{r} J_i$$

where each $J_i$ is binomial, primary, and has a unique monomial in its socle.
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- $I$ is primary to the maximal ideal $\mathfrak{m}$,
Theorem (Kahle-Miller, 2013)

For $k = \overline{k}$, every binomial ideal has an expression of the form

$$I = \bigcap_{i=1}^{r} J_i$$

where each $J_i$ is binomial, primary, and has a unique monomial in its socle.

To construct a binomial irreducible decomposition for $I$, we can assume

- $I$ is primary to the maximal ideal $m$,
- $\text{soc}_m(I)/I$ has a unique monomial.
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]
Binomial Ideals

\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ \mathbb{N}^n / \sim_I \leftrightarrow \text{monomials mod } I \]
Binomial Ideals

\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ \mathbb{N}^n / \sim_I \leftrightarrow \text{monomials mod } I \]

\[ \text{soc}_m(I) / I = \mathbb{k}\{x^3, x - y\} \]
$I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle$

$\mathbb{N}^n / \sim_I \leftrightarrow \text{monomials mod } I$

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\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

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\mathbb{N}^n/\sim_I \longleftrightarrow \text{monomials mod } I

\text{soc}_m(I)/I = \mathbb{k}\{x^3, x - y\}

\text{witnesses: monomials that merge with something in each direction}
\begin{align*}
I &= \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \\
\mathbb{N}^n / \sim_I &\longleftrightarrow \text{monomials mod } I \\
\text{soc}_m(I)/I &= \mathbb{k}\{x^3, x - y\} \\
\text{witnesses: monomials that merge with something in each direction} \\
I\text{-witnesses: } x^3, x, y
\end{align*}
A monomial $x^a$ is a *witness* for $I$ if for each $x^p \in p$,

$$p + a \sim_I p + a'$$

for some $a' \not\sim_I a$,

that is, $x^a$ merges with another monomial modulo $I$ when multiplied by any monomial in $p$. 

Theorem (Kahle-Miller, 2013)

For any $p$-primary binomial ideal $I$, any $f \in \text{soc}_p(I)/I$ is a sum of witnesses.
Definition

A monomial $x^a$ is a *witness* for $I$ if for each $x^p \in p$,

$$p + a \sim_I p + a'$$

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Theorem (Kahle-Miller, 2013)

*For any $p$-primary binomial ideal $I$, any $f \in \text{soc}_p(I)/I$ is a sum of witnesses.*
Binomial Ideals

\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ \text{soc}_m(I)/I = \mathbb{K}\{x^3, x - y\} \]
$I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle$

$\text{soc}_m(I)/I = \mathbb{k}\{x^3, x - y\}$

**soccularize** $I$: “Force simple socle”
Binomial Ideals

\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ \text{soc}_m(I)/I = \mathbb{k}\{x^3, x - y\} \]

**soccularize** \( I \): “Force simple socle”
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ \text{soc}_m(I)/I = \mathbb{k}\{x^3, x - y\} \]

**soccularize** \( I \): “Force simple socle”

\[ J = \langle x - y, x^4, y^4 \rangle \]
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

\[ \text{soc}_m(I)/I = \mathbb{k}\{x^3, x - y\} \]

**soccularize** \( I \): “Force simple socle”

\[ J = \langle x - y, x^4, y^4 \rangle \]

\[ \text{soc}_m(J)/J = \mathbb{k}\{x^3\} \]
Plan of attack:
Plan of attack:

- One irreducible component per witness monomial.
Soccular Decomposition

Plan of attack:

- One irreducible component per witness monomial.
- For each component, force chosen witness to be maximal.
Plan of attack:

- One irreducible component per witness monomial.
- For each component, force chosen witness to be maximal.
- Soccularize to remove other socle elements.
I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle
Soccular Decomposition

\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x, y \)
Soccular Decomposition

\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x, y \)
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x, y \)
Soccular Decomposition

\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x, y \)

\[ J_1 = \langle x - y, x^4, y^4 \rangle, \]
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x, y \)

\[ J_1 = \langle x - y, x^4, y^4 \rangle, \]
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x, y \)

\[ J_1 = \langle x - y, x^4, y^4 \rangle, \]

\[ J_2 = \langle x^2, y \rangle, \]
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

Witnesses: \( x^3 \), \( x \), \( y \)

\[ J_1 = \langle x - y, x^4, y^4 \rangle, \]

\[ J_2 = \langle x^2, y \rangle, \]
Soccer Decomposition

\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x, y \)

\[ J_1 = \langle x - y, x^4, y^4 \rangle, \]
\[ J_2 = \langle x^2, y \rangle, J_3 = \langle x, y^2 \rangle \]
\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x, y \)

\[ J_1 = \langle x - y, x^4, y^4 \rangle, \]
\[ J_2 = \langle x^2, y \rangle, \quad J_3 = \langle x, y^2 \rangle \]

\[ I = J_1 \cap J_2 \cap J_3 \]
Soccular Decomposition

\[ I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x, y \)

\[ J_1 = \langle x - y, x^4, y^4 \rangle, \]
\[ J_2 = \langle x^2, y \rangle, J_3 = \langle x, y^2 \rangle \]

\[ I = J_1 \cap J_2 \cap J_3 = J_1 \cap J_2 \]
Soccular Decomposition

\[ I = \langle x^2 - xy, xy + y^2, x^4, y^4 \rangle \]
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Witnesses: \( x^3, x, y \)
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Witnesses: \( x^3, x, y \)

\[ \text{soc}_m(I)/I = \mathbb{k}\{x^3\} \]
Soccular Decomposition

\[ I = \langle x^2 - xy, xy + y^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x, y \)

\[ \text{soc}_m(I)/I = \mathbb{k}\{x^3\} \]

\[ I = I \cap \langle x^2, y \rangle \cap \langle x, y^2 \rangle \]
Soccular Decomposition

\[ I = \langle xy - y^2, x^3 - xy^2, x^4, y^4 \rangle \]
Soccular Decomposition

\[ I = \langle xy - y^2, x^3 - xy^2, x^4, y^4 \rangle \]
Soccular Decomposition

\[ I = \langle xy - y^2, x^3 - xy^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x^2, xy \)
Soccular Decomposition

\[ I = \langle xy - y^2, x^3 - xy^2, x^4, y^4 \rangle \]

 Witnesses: \( x^3, x^2, xy \)

\[ \text{soc}_m(I)/I = \mathbb{k}\{x^3, x^2 - xy\} \]
Soccular Decomposition

\[ I = \langle xy - y^2, x^3 - xy^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x^2, xy \)

\[ \text{soc}_m(I)/I = \mathbb{k}\{x^3, x^2 - xy\} \]

Soccularize:

\[ \text{soc}_m(I)/I = \mathbb{k}\{x^3, x^2 - xy\} \]
I = \langle xy - y^2, x^3 - xy^2, x^4, y^4 \rangle

Witnesses: x^3, x^2, xy

soc_m(I)/I = \mathbb{k}\{x^3, x^2 - xy\}

Soccularize:
$I = \langle xy - y^2, x^3 - xy^2, x^4, y^4 \rangle$

Witnesses: $x^3, x^2, xy$

$soc_m(I)/I = \mathbb{k}\{x^3, x^2 - xy\}$

Soccularize:
Soccular Decomposition

\[ I = \langle xy - y^2, x^3 - xy^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x^2, xy \)

\[ \text{soc}_m(I)/I = \mathbb{k}\{x^3, x^2 - xy\} \]

Soccularize: New witnesses!
\[ I = \langle xy - y^2, x^3 - xy^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x^2, xy \)

\[ \text{soc}_m(I)/I = \mathbb{k}\{x^3, x^2 - xy\} \]

Soccularize: New witnesses!

*Protected* witnesses: \( x, y \)
Soccular Decomposition

\[ l = \langle xy - y^2, x^3 - xy^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x^2, xy \)

\[ \text{soc}_m(l)/l = \mathbb{k}\{x^3, x^2 - xy\} \]

Soccularize: New witnesses!

*Protected* witnesses: \( x, y \)
Soccular Decomposition

\[ I = \langle xy - y^2, x^3 - xy^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x^2, xy \)

\[ \text{soc}_m(I)/I = \mathbb{k}\{x^3, x^2 - xy\} \]

Soccularize: New witnesses!

*Protected* witnesses: \( x, y \)

\[ J_1 = \langle x - y, x^4, y^4 \rangle \]
\[ I = \langle xy - y^2, x^3 - xy^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x^2, xy \)

\[ \text{soc}_m(I)/I = \mathbb{k}\{x^3, x^2 - xy\} \]

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Soccularize: New witnesses!

*Protected* witnesses: \( x, y \)

\[ J_1 = \langle x - y, x^4, y^4 \rangle \]

\[ J_2 = \langle x^3, y \rangle \]
Soccular Decomposition

\[ I = \langle xy - y^2, x^3 - xy^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x^2, xy \)

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\[ J_1 = \langle x - y, x^4, y^4 \rangle \]
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Witnesses: \( x^3, x^2, xy \)

\[ \text{soc}_m(I)/I = \mathbb{k}\{x^3, x^2 - xy\} \]

Soccularize: New witnesses!

Protected witnesses: \( x, y \)

\[ J_1 = \langle x - y, x^4, y^4 \rangle \]
\[ J_2 = \langle x^3, y \rangle \]
\[ J_3 = \langle xy - y^2, x^2, y^3 \rangle \]
Soccular Decomposition

\[ I = \langle xy - y^2, x^3 - xy^2, x^4, y^4 \rangle \]

Witnesses: \( x^3, x^2, xy \)

\[ \text{soc}_m(I)/I = \mathbb{k}\{x^3, x^2 - xy\} \]

Soccularize: New witnesses!

*Protected* witnesses: \( x, y \)

\[ J_1 = \langle x - y, x^4, y^4 \rangle \]
\[ J_2 = \langle x^3, y \rangle \]
\[ J_3 = \langle xy - y^2, x^2, y^3 \rangle \]

\[ I = J_1 \cap J_2 \cap J_3 \]
Algorithm for decomposing a binomial ideal $I$:

1. **Soccular Decomposition**
   - One component for each $I$-witness.
   - For the component at a witness $w$:
     - Add monomials not below $w$, so $w$ is a unique monomial socle element.
     - "Soccularize" by merging witness pairs below $w$.
   - Repeat with protected witnesses until no new witness pairs are created.

**Theorem (Kahle-Miller-O., 2014)**
For $k = k$, any binomial ideal $I$ can be written as

$$I = \bigcap_{i=1}^{r} J_i,$$

where each $J_i$ is binomial and $p_i$-primary, and the socle $soc_{p_i}(J_i) / J_i$ contains a unique monomial and no other binomials.
Algorithm for decomposing a binomial ideal $I$:

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Algorithm for decomposing a binomial ideal $I$:

- One component for each $I$-witness.
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  - "Soccularize" by merging witness pairs below $w$.
  - Repeat with protected witnesses until no new witness pairs are created.

Theorem (Kahle-Miller-O., 2014)

For $k = k$, any binomial ideal $I$ can be written as $I = \bigcap_{i=1}^{r} J_i$, where each $J_i$ is binomial and $p_i$-primary, and the socle $soc_{p_i}(J_i)/J_i$ contains a unique monomial and no other binomials.
Algorithm for decomposing a binomial ideal $I$:

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The Counterexample

\[ I = \langle x^2y - xy^2, x^3, y^3 \rangle \]
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\[ I = \langle x^2 + y^2 - xy, x^3, y^3 \rangle \cap \langle x^3, y \rangle \]
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Theorem (Kahle-Miller-O., 2014)

\( I = \langle x^2 y - xy^2, x^3, y^3 \rangle \) admits no binomial irreducible decomposition.
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Proof.

Fix an irredundant irreducible decomposition \( I = \bigcap_{i=1}^{r} J_i \).
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We have \( r = \dim_k (\text{soc}_m(I)/I) = 2 \), so \( I = J_1 \cap J_2 \).
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Write \( \alpha = x^2 + y^2 - xy \), \( \beta = x^2 y \), so \( \text{soc}_m(I)/I = k\{\alpha, \beta\} \).
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We know
\[ \text{soc}_m(I)/I \cong \text{soc}_m(J_1)/J_1 \oplus \text{soc}_m(J_2)/J_2, \]
so we have \( \alpha + \lambda \beta \in \text{soc}_m(J_i)/J_i \) for some \( i \), say \( i = 1 \).
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This means \( I + \langle \alpha + \lambda \beta \rangle \subset J_1 \).
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so we have $\alpha + \lambda \beta \in \text{soc}_m(J_i)/J_i$ for some $i$, say $i = 1$.

This means $I + \langle \alpha + \lambda \beta \rangle \subset J_1$.

But $I + \langle \alpha + \lambda \beta \rangle$ already has simple socle, so $J_1 = I + \langle \alpha + \lambda \beta \rangle$. 

Christopher O'Neill (Duke University)  Irreducible decomposition of binomial ideals  January 18, 2014  33 / 36
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Thanks!
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