Invariants of non-unique factorization

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November 21, 2014
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First half: Catenary degree (combinatorial)
Joint with Vadim Ponomarenko, Reuben Tate*, and Gautam Webb*

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First half: Catenary degree (combinatorial)
Joint with Vadim Ponomarenko, Reuben Tate*, and Gautam Webb*

Second half: \(\omega\)-primality (algebraic)
Joint with Thomas Barron* and Roberto Pelayo

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Factorial domains

Definition

An integral domain $R$ is factorial if for each non-unit $r \in R$,

1. there is a factorization $r = u_1 \cdots u_k$ as a product of irreducibles, and
2. this factorization is unique (up to reordering and unit multiple).
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Example
$\mathbb{Z}$ is factorial: each $z = p_1 \cdots p_k$ for primes $p_1 \cdots p_k$. 
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Definition
An integral domain $R$ is \textit{atomic} if for each non-unit $r \in R$,
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If $R = \mathbb{Z}[\sqrt{-5}]$, then $6 \in R$ has two distinct factorizations:

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$
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To prove: define a \textit{valuation} $a + b\sqrt{-5} \mapsto a^2 + 5b^2$. 
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The point: it’s complicated.
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1. \( x^2 \) and \( x^3 \) are irreducible.
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2. \( x^6 = x^3 \cdot x^3 = x^2 \cdot x^2 \cdot x^2 \).
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Observation
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- Where’s the addition?
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- Where’s the addition?
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- Where’s the addition?
- Factorization in (cancellative commutative) monoids:
  
  $$(R, +, \cdot) \sim (R \setminus \{0\}, \cdot)$$
  
  $$(\mathbb{C}[M], +, \cdot) \sim (M, \cdot)$$
A numerical monoid $S$ is an additive submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$. 
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Example

Let $S = \langle 2, 3 \rangle = \{2, 3, 4, 5, \ldots \}$ under addition.
Numerical monoids

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A *numerical monoid* $S$ is an *additive* submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

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Let $S = \langle 2, 3 \rangle = \{2, 3, 4, 5, \ldots\}$ under *addition*. $\mathbb{C}[S] = \mathbb{C}[x^2, x^3]$. 

'$\Rightarrow$' $60 = 7(6) + 2(9) = 3(20)$
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$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots \}$. 
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$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots \}$. “McNugget Monoid”
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$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$. “McNugget Monoid”

\[ 60 = 7(6) + 2(9) \quad \leadsto \quad (7, 2, 0) \]
\[ = 3(20) \quad \leadsto \quad (0, 0, 3) \]
Define a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$, $Z_S(n) = \{ (a_1, \ldots, a_k) \in \mathbb{N}^k : n = a_1 n_1 + \cdots + a_k n_k \}$ denotes the set of factorizations of $n$.

For $f, f' \in Z_S(n)$, $|f| =$ length of $f$ and $\gcd(f, f') = (\min(f_1, f'_1), \ldots, \min(f_k, f'_k))$.

$$d(f, f') = \max\{|f - \gcd(f, f')|, |f' - \gcd(f, f')|\}$$
Factorization invariants: towards the catenary degree

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Example

\[ S = \langle 4, 6, 7 \rangle \subset \mathbb{N}, \]
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\[ S = \langle 4, 6, 7 \rangle \subset \mathbb{N}, \quad f = (3, 1, 1), \quad f' = (1, 0, 3) \in \mathbb{Z}_S(25). \]
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- \( d(f, f') \)
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- $g = \gcd(f, f') = (1, 0, 1)$.
- $d(f, f') = \max \{|f - g|, |f' - g|\} = 3$. 

\[
\begin{array}{c}
4 \\
4 \\
4 \\
(3,1,1) \\
4 \\
7 \\
7 \\
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\]
Fix a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$, define the catenary degree $c(n)$ as follows:

1. Construct a complete graph $G$ with vertex set $\mathbb{Z}_S(n)$ where each edge $(f, f')$ has label $d(f, f')$ (catenary graph).
2. Locate the largest edge weight $e$ in $G$.
3. Remove all edges from $G$ with weight $e$.
4. If $G$ is disconnected, return $e$. Otherwise, return to step 2.

If $|\mathbb{Z}_S(n)| = 1$, define $c(n) = 0$. 
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If $|Z_S(n)| = 1$, define $c(n) = 0$. 
A Big Example

\[ S = \langle 11, 36, 39 \rangle, \ n = 450 \]
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A Big Example

$S = \langle 11, 36, 39 \rangle$, $n = 450$, $c(n) = 16$
A Big Example, Method 2

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A Big Example, Method 2

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\[ (0, 6, 6) \quad (27, 1, 3) \]

\[ (3, 4, 7) \quad (24, 3, 2) \]

\[ (6, 2, 8) \quad (21, 5, 1) \]

\[ (9, 0, 9) \quad (18, 7, 0) \]
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Betti elements

Definition
For an element $n \in S = \langle n_1, \ldots, n_k \rangle$, let $\nabla_n$ denote the subgraph of the catenary graph in which only edges $(f, f')$ with $\gcd(f, f') \neq 0$ are drawn.

Example
$S = \langle 10, 15, 17 \rangle$ has Betti elements 30 and 85.

$\nabla_{30}$:
\[(0, 2, 0)\]

$\nabla_{85}$:
\[(3, 0, 0)\]
\[(0, 2, 0)\]
\[(3, 0, 0)\]
\[(0, 0, 5)\]
\[(7, 1, 0)\]
\[(4, 3, 0)\]
\[(1, 5, 0)\]
Betti elements

**Definition**

For an element \( n \in S = \langle n_1, \ldots, n_k \rangle \), let \( \nabla_n \) denote the subgraph of the catenary graph in which only edges \((f, f')\) with \( \gcd(f, f') \neq 0 \) are drawn. We say \( n \) is a *Betti element* of \( S \) if \( \nabla_n \) is disconnected.

Example

\[ S = \langle 10, 15, 17 \rangle \] has Betti elements 30 and 85.

\[ \nabla_{30} : (0,2,0) \]

\[ \nabla_{85} : (3,0,0) \]

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Betti elements

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Example
$S = \langle 10, 15, 17 \rangle$ has Betti elements 30 and 85.

$\nabla_{30}$:

- $(3,0,0)$
- $(0,2,0)$

$\nabla_{85}$:

- $(0,0,5)$
- $(7,1,0)$
- $(4,3,0)$
- $(1,5,0)$

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Maximal catenary degree in $S$

**Theorem**

$$\max\{c(n) : n \in S\} = \max\{c(b) : b \text{ Betti element of } S\}.$$
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Key concept: Cover morphisms.
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Key concept: Cover morphisms.
Maximal catenary degree in $S$

**Theorem**

$max\{c(n) : n \in S\} = max\{c(b) : b \text{ Betti element of } S\}.$

Key concept: Cover morphisms.

$Z_S(n) \xrightarrow{f} Z_S(n + n_i) \xrightarrow{f + e_i}$
Maximal catenary degree in $S$

**Theorem**

\[ \max\{c(n) : n \in S\} = \max\{c(b) : b \text{ Betti element of } S\}. \]

Key concept: Cover morphisms.

\[ Z_S(n) \]

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Idea for proof: Certain edges (determined by Betti elements) connect the catenary graph of each $n \in S$. 
Maximal catenary degree in $S$

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\[ \text{Z}(n) \]
Maximal catenary degree in $S$

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Idea for proof: Certain edges (determined by Betti elements) connect the catenary graph of each $n \in S$. 

![Diagram showing connections between $Z(b_1)$ and $Z(n)$]
Maximal catenary degree in $S$

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Idea for proof: Certain edges (determined by Betti elements) connect the catenary graph of each $n \in S$.\[\begin{align*}
\text{Z}(b_1) & \quad \rightarrow \quad \text{Z}(n) \\
\text{Z}(b_2) & \quad \rightarrow \\
\end{align*}\]
Maximal catenary degree in $S$

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Idea for proof: Certain edges (determined by Betti elements) connect the catenary graph of each $n \in S$. 

\[\begin{array}{ccc}
Z(b_1) & \rightarrow & Z(n) \\
\downarrow & & \downarrow \\
Z(b_2) & \rightarrow & Z(b_3)
\end{array}\]
Conjecture

\[ \min \{ c(n) > 0 : n \in S \} = \min \{ c(b) : b \text{ Betti element of } S \} \].
**Conjecture Theorem (O., Ponomarenko, Tate, Webb)**

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\min\{c(n) > 0 : n \in S\} = \min\{c(b) : b \text{ Betti element of } S\}.
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\[B = \min\{c(b) : b \text{ Betti element of } S\}.\]
**Conjecture** Theorem (O., Ponomarenko, Tate, Webb)

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**Lemma**

If \( f, f' \in \mathbb{Z}_S(n) \)

\[ f \bullet \]

\[ f' \bullet \]
Minimal (nonzero) catenary degree in $S$

**Conjecture Theorem (O., Ponomarenko, Tate, Webb)**

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**Lemma**

*If* $f, f' \in \mathbb{Z}_S(n)$ *and* $d(f, f') < B$,
**Conjecture Theorem** (O., Ponomarenko, Tate, Webb)

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\min\{c(n) > 0 : n \in S\} = \min\{c(b) : b \text{ Betti element of } S\}.
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**Lemma**

If \( f, f' \in Z_S(n) \) and \( d(f, f') < B \), then there exists \( f'' \in Z_S(n) \)
Conjecture Theorem (O., Ponomarenko, Tate, Webb)

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**Lemma**

If \( f, f' \in Z_S(n) \) and \( d(f, f') < B \), then there exists \( f'' \in Z_S(n) \) with

\[
\max\{|f|, |f'|\} < |f''|.
\]

\[f\]

\[f'\]

\[f''\]
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**Conjecture Theorem (O., Ponomarenko, Tate, Webb)**

$$\min \{ c(n) > 0 : n \in S \} = \min \{ c(b) : b \text{ Betti element of } S \}.$$ 

**Proof of theorem:**

- Draw edges with weight $\langle B \rangle \in \mathbb{Z}^S(n)$ with $|f|_{\text{maximal}}$.
- $\Rightarrow |f''| > |f|$.
- Maximality of $|f| \Rightarrow f''$ has no edges!
Conjecture Theorem (O., Ponomarenko, Tate, Webb)

\[ \min\{c(n) > 0 : n \in S\} = \min\{c(b) : b \text{ Betti element of } S\}. \]

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Catenary graph of \( n \):
Minimal (nonzero) catenary degree in $S$

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- Fix $n \in S$
- Draw edges with weight $< B$

Catenary graph of $n$: 

![Catenary graph image]
Minimal (nonzero) catenary degree in $S$

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Proof of theorem:

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- Draw edges with weight $< B$
- $f \in Z_S(n)$ with $|f|$ maximal
Conjecture Theorem (O., Ponomarenko, Tate, Webb)

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Proof of theorem:

1. Fix \( n \in S \)
2. Draw edges with weight \(< B\)
3. \( f \in Z_S(n) \) with \(|f| \) maximal
4. \( f' \in Z_S(n) \) with \( d(f, f') < B \)

Catenary graph of \( n \):
Minimal (nonzero) catenary degree in $S$

**Conjecture Theorem (O., Ponomarenko, Tate, Webb)**

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Catenary graph of $n$:
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- Lemma $\Rightarrow |f''| > |f|$  
- Maximality of $|f| \Rightarrow f''$ has no edges!

Catenary graph of $n$: 

- Nodes representing elements of $S$
- Edges connecting elements with weight less than $B$
- Blue node indicating the maximality of $f'$. 

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Switching gears: $\omega$-primality

Definition ($\omega$-primality)

Fix a cancellative, commutative, atomic monoid $M$. For $x \in M$, $\omega(x)$ is the smallest positive integer $m$ such that whenever $x \mid \prod_{r > m} u_i$ for $r > m$, there exists a subset $T \subset \{1, \ldots, r\}$ with $|T| \leq m$ such that $x \mid \prod_{i \in T} u_i$. 
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**Fact**

\( \omega(x) = 1 \) if and only if \( x \) is prime (i.e. \( x \mid ab \) implies \( x \mid a \) or \( x \mid b \)).
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**Fact**

\( M \) is factorial if and only if every irreducible element \( u \in M \) is prime. Moreover, \( \omega(p_1 \cdots p_r) = r \) for any primes \( p_1, \ldots, p_r \in M \).
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**Example**

\[ x^2 \mid x^2, \quad x^2 \mid x^3 \cdot x^3 \text{ since } x^4 \in R, \quad x^2 \mid u_1 u_2 u_3 \] with each \( u_i = x^2 \) or \( x^3 \).

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$R = \mathbb{C}[x^2, x^3]$
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$R = \mathbb{C}[x^2, x^3]$ (think $S = \langle 2, 3 \rangle \subset \mathbb{N}$).
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- $x^2 \mid x^2$,
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Switching gears: \( \omega \)-primality

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Switching gears: $\omega$-primality

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$R = \mathbb{C}[x^2, x^3]$ (think $S = \langle 2, 3 \rangle \subset \mathbb{N}$). To compute $\omega(x^2)$:

- $x^2 | x^2$,
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- $\omega(x^2) = 2$. 
Switching gears: \( \omega \)-primality

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Switching gears: $\omega$-primality

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**Definition**

A *bullet* for $x \in M$ is a product $u_1 \cdots u_r$ of irreducible elements such that (i) $x$ divides $u_1 \cdots u_r$, and (ii) $x$ does not divide $u_1 \cdots u_r / u_i$ for each $i \leq r$. The set of bullets of $x$ is denoted $\text{bul}(x)$. 

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Switching gears: \(\omega\)-primality

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**Proposition**

\[\omega_M(x) = \max\{r : u_1 \cdots u_r \in \text{bul}(x)\}.\]
Algorithms to compute $\omega$-primality

$\omega$-primality in a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$:
ω-primality in a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$:

$$b_1 n_1 + \cdots + b_k n_k \in \text{bul}(n)$$
Algorithms to compute $\omega$-primality

$\omega$-primality in a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$:

$$b_1 n_1 + \cdots + b_k n_k \in \text{bul}(n) \quad \longleftrightarrow \quad \vec{b} = (b_1, \ldots, b_k) \in \mathbb{N}^k$$
ω-primality in a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$:

$$b_1 n_1 + \cdots + b_k n_k \in \text{bul}(n) \iff \vec{b} = (b_1, \ldots, b_k) \in \mathbb{N}^k$$

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Algorithms to compute $\omega$-primality

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$(b_1 n_1 + \cdots + b_k n_k - n \in S)$

**Example**

$S = \langle 6, 9, 20 \rangle$. 

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 Algorithms to compute $\omega$-primality

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Example

$S = \langle 6, 9, 20 \rangle$. \text{bul}(54) = \{(9, 0, 0), (6, 2, 0), (0, 6, 0), (3, 4, 0), (0, 0, 3)\}.$
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$S = \langle 6, 9, 20 \rangle$. $\text{bul}(54) = \{(9, 0, 0), (6, 2, 0), (0, 6, 0), (3, 4, 0), (0, 0, 3)\}$. Here, $(0, 0, 3) \in \text{bul}(54)$ since $3(20) - 54 \in S$. 

Remark

Several improvements on this algorithm exist.

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Algorithms to compute $\omega$-primality

$\omega$-primality in a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$:

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$$(b_1 n_1 + \cdots + b_k n_k - n \in S)$$

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$\omega$-primality in a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$:

$b_1 n_1 + \cdots + b_k n_k \in \text{bul}(n) \iff \vec{b} = (b_1, \ldots, b_k) \in \mathbb{N}^k$

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\[ S = \langle 3, 7 \rangle \]

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Quasilinearity for numerical monoids

*Dissonance point*: minimum $N_0$ such that $\omega(n)$ is quasilinear for $n > N_0$. 

Question (O.-Pelayo, 2013)
The upper bound for dissonance point is large. Can we do better?

Roadblock
Existing algorithms are slow for large $n$.

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Can we dynamically (inductively) compute several $\omega$-values at once?

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\begin{align*}
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Definition/Proposition (Cover morphisms)

Fix \( n \in S \) and \( i \leq k \). The \( i \)-th cover morphism for \( n \) is the map

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\vec{b} \mapsto \begin{cases} 
\vec{b} + \vec{e}_i & \sum_{j=1}^k b_j n_j - n - n_i \notin S \\
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Moreover, \( \text{bul}(n) = \bigcup_{i \leq k} \psi_i(\text{bul}(n - n_i)) \).**
Toward a dynamic algorithm... the base case

**Definition (ω-primality in numerical monoids)**

Fix a numerical monoid $S$ and $n \in S$. 
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$\omega_S(n)$ is the minimal $m$ such that whenever $(\sum_{i=1}^{r} n_{ji}) - n \in S$ for $r > m$, there exists $T \subset \{1, \ldots, r\}$ with $|T| \leq m$ and $(\sum_{i \in T} n_{ji}) - n \in S$. 

Remark: All properties of $\omega$ extend from $S$ to $\mathbb{Z}$. 

Proposition: For $n \in \mathbb{Z}$, the following are equivalent: 

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A dynamic algorithm!

Example

\[ McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots \} . \]
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\[ \begin{array}{c|c|c}
\leq -44 & 0 & \{0\} \\
-43 & 1 & \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \\
-42 & 0 & \{0\} \\
\vdots & \vdots & \vdots \\
-38 & 0 & \{0\} \\
-37 & 2 & \{2\vec{e}_1, \vec{e}_2, \vec{e}_3\} \\
-36 & 0 & \{0\} \\
-35 & 0 & \{0\} \\
-34 & 2 & \{\vec{e}_1, 2\vec{e}_2, \vec{e}_3\} \\
\end{array} \]
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A dynamic algorithm!

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<tr>
<td>150</td>
<td>25</td>
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A dynamic algorithm!

**Example**

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Runtime comparison


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<table>
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<tr>
<th>$S$</th>
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<th>$\omega_S(n)$</th>
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<th>Dynamic</th>
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<td>6ms</td>
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<td>15ms</td>
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<td>915</td>
<td>———</td>
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<tr>
<td>$\langle 15, 27, 32, 35 \rangle$</td>
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<td>3m 54.7s</td>
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20 / 23
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GAP Numerical Semigroups Package, available at
http://www.gap-system.org/Packages/numericalsgps.html.
Future directions: $\omega$-primality

What about more general (finitely generated) monoids $M$?
Future directions: \( \omega \)-primality

What about more general (finitely generated) monoids \( M \)?

- Characterization of \( \omega_M \) in terms of maximal length bullets?
Future directions: $\omega$-primality

What about more general (finitely generated) monoids $M$?

- Characterization of $\omega_M$ in terms of maximal length bullets? ✓

- Extension of $\omega_M$ to $q(M)$?

- Iterative construction of bullets from cover maps?

- Issue: the base case!

Problem: Find a dynamic algorithm to compute $\omega$-primality in $M$.

Problem: Characterize the eventual behavior of $\omega$-primality in $M$. 

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Future directions: $\omega$-primality

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Issue: the base case!

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Problem
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Future directions: catenary degree
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Find a (canonical) finite set on which every catenary degree is achieved.
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Thanks!