Invariants of non-unique factorization

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Joint with Roberto Pelayo

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Definition

An integral domain $R$ is factorial if for each non-unit $r \in R$,

1. there is a factorization $r = u_1 \cdots u_k$ as a product of irreducibles, and
2. this factorization is unique (up to reordering and unit multiple).
Factorial domains

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Example

$\mathbb{Z}$ is factorial: each $z = p_1 \cdots p_k$ for primes $p_1 \cdots p_k$. 
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An integral domain $R$ is \textit{atomic} if for each non-unit $r \in R$,

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If $R = \mathbb{Z}[\sqrt{-5}]$, then $6 \in R$ has two distinct factorizations:

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$
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To prove: define a valuation $a + b\sqrt{-5} \mapsto a^2 + 5b^2$. 
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6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})
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To prove: define a *valuation* \( a + b\sqrt{-5} \mapsto a^2 + 5b^2 \).

The point: it’s nontrivial.
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1. $x^2$ and $x^3$ are irreducible.
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1. $x^2$ and $x^3$ are irreducible.
2. $x^6 = x^3 \cdot x^3 = x^2 \cdot x^2 \cdot x^2$. 

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  $$ (R, +, \cdot) \leadsto (R \setminus \{0\}, \cdot) $$
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  $$(R, +, \cdot) \leadsto (R \setminus \{0\}, \cdot)$$
  $$(\mathbb{C}[M], +, \cdot) \leadsto (M, \cdot)$$
Interesting monoids

**Definition**

An *arithmetical congruence monoid* is a **multiplicative** submonoid

\[ M_{a,b} = \{ n : n \equiv a \mod b \} \subset \mathbb{Z}_{>0} \]

for \( a, b > 0 \) with \( a^2 \equiv a \mod b \).
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The *Hilbert monoid* \( M_{1,4} = \{1, 5, 9, 13, 17, \ldots \} \).
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- \( 9, 21, 49 \in M_{1,4} \) are irreducible.
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An arithmetic congruence monoid is a multiplicative submonoid

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A *numerical monoid* \( S \) is an *additive* submonoid of \( \mathbb{N} \) with \( |\mathbb{N} \setminus S| < \infty \).
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Let $S = \langle 2, 3 \rangle = \{2, 3, 4, 5, \ldots\}$ under **addition**.
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$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$. 

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Example

$\text{McN} = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$. “McNugget Monoid”
Factorization invariants

Definition
Fix a commutative, cancellative monoid \((M, \cdot)\). For each non-unit \(m \in M\),
\[\mathbb{Z}(m) = \{\text{factorizations } m = \prod u_i\}\]
denotes the set of factorizations of \(m\).
The elasticity of \(m\) is
\[\rho(m) = \max \text{ length in } \mathbb{Z}(m) \text{ } \text{ min length in } \mathbb{Z}(m) .\]
The elasticity of \(M\) is
\[\rho(M) = \sup m \in M \rho(m) .\]
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- Every factorization of \(m \in M_{1,4}\) has the same length.
- This is (almost) the best we could hope for.
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Numerical monoids: \(McN = \langle 6, 9, 20 \rangle \subset \mathbb{N}\). \(\rho(McN) = 20/6\).
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- \(\rho(n) \leq 20/6\) for all \(n \in McN\).
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The Meyerson monoid: \(\rho(M_{4,6}) = 2\).

- \(\rho(m) < 2\) for all \(m \in M_{4,6}\)!
- Elasticity of \(M_{4,6}\) is not accepted.
Factorization lengths in numerical monoids

Let $S = \langle n_1, \ldots, n_k \rangle \subset (\mathbb{N}, +)$. For $n \in S$,

$$M(n) = \text{max length in } Z(n) \quad m(n) = \text{min length in } Z(n)$$
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**Observations**

Max length factorization: lots of small irreducibles

Min length factorization: lots of large irreducibles

Theorem (Barron–O–Pelayo, 2014)

Let $S = \langle n_1, \ldots, n_k \rangle$. For $n > n_k$ ($n_k - 1$),

$$M(n + n_1) = 1 + M(n) \quad m(n + n_k) = 1 + m(n)$$

Equivalently, $M(n), m(n)$ are eventually quasilinear:

$$M(n) = 1 + n_1 a_0(n) \quad m(n) = 1 + n_k b_0(n)$$

for periodic functions $a_0(n), b_0(n)$. 

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for periodic functions $a_0(n), b_0(n)$. 
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$$M(n + n_1) = 1 + M(n) \quad \text{and} \quad m(n + n_k) = 1 + m(n).$$

$S = \langle 6, 9, 20 \rangle$: 

$m(1000001) = 50002$ and $1000001 = 2(6) + 1(9) + 49993(20)$
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Factorization lengths in numerical monoids

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\[ S = \langle 6, 9, 20 \rangle: \]

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$S = \langle 5, 16, 17, 18, 19 \rangle$: 

Theorem (Barron–O–Pelayo, 2014)

Let \( S = \langle n_1, \ldots, n_k \rangle \). For \( n > n_k(n_{k-1} - 1) \),

\[
M(n + n_1) = 1 + M(n) \quad \text{and} \quad m(n + n_k) = 1 + m(n).
\]

\( S = \langle 5, 16, 17, 18, 19 \rangle \):

\[
m(n) : S \to \mathbb{N}
\]