Discussion problems. The problems below should be completed in class.

(D1) Binomial Theorem. Recall the binomial theorem from Tuesday:

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

(a) Give a combinatorial proof of the following identity, valid whenever $k + m \leq n$:

$$\binom{n}{m} \binom{n-m}{k} = \binom{n}{k} \binom{n-k}{m}.$$

(b) Consider the following identity, valid whenever $n \geq 1$:

$$\sum_{k=1}^{n} k \binom{n}{k} = n 2^{n-1}.$$

(i) Give a combinatorial proof of this identity.

(ii) Give an algebraic proof of this identity. Hint: apply $\frac{d}{dx}$ to the binomial theorem.

(D2) Applications of Inclusion-Exclusion. Recall the Sieve formula from Tuesday:

$$|A_1 \cup \cdots \cup A_n| = \sum_{S \subseteq [n]} (-1)^{|S|} |\bigcap_{j \in S} A_j|.$$

(a) A derangement is a permutation that has no fixed points.

(i) Find the number $D_n$ of derangements $[n] \to [n]$ using the Sieve formula.

(ii) Give a combinatorial proof that the sequence $D_n$ satisfies

$$D_n = n! - \sum_{i=k}^{n} \binom{n}{k} D_{n-k}.$$

(iii) Give a combinatorial proof that the sequence $D_n$ satisfies

$$D_{n+1} = n(D_n + D_{n-1}).$$

(b) For $n,m \geq 1$, let $O_{n,m}$ denote the total number of surjective functions $[n] \to [m]$.

(i) For $n \geq m$, use the Sieve formula to show that

$$O_{n,m} = m^n - \sum_{k=0}^{m-1} (-1)^{m-k} \binom{m}{k} k^n.$$

(ii) Show that $O_{n,m}$ satisfies the relation

$$O_{n,m} = m^n - \sum_{k=0}^{m-1} \binom{m}{k} O_{n,k}.$$

Hint: use the fact that any function is surjective onto its image.
Required problems. As the name suggests, you must submit all required problem with this homework set in order to receive full credit.

(R1) Find the coefficient of $x^{11}y^7$ in the expansion of $(x + y)^{18} + x^3(x - y)^{15}$.

(R2) How many ways can we arrange the characters 1, 1, 2, 2, 3, 4, 5 so that no two consecutive digits are identical?

(R3) Use induction on $n$ to prove that for all $n \geq 1$,

$$\sum_{i=0}^{n} (-1)^k \binom{n}{k} = 0.$$  

Hint: use the identity $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$ in your inductive step.

(R4) Find $\phi(210)$, where $\phi(n)$ denote the number of elements of $[n]$ relatively prime to $n$.

Selection problems. You are required to submit all parts of one selection problem with this problem set to be eligible for full credit. However, on this assignment only, you may also submit additional selection problems, and your scores will count towards your overall homework score.

(S1) Find 4 infinite subsets of $\mathbb{N}$ such that the intersection of any 3 of them is infinite, but the intersection of all 4 is empty.

(S2) Let $\phi(n)$ denote the number of elements of $[n]$ relatively prime to $n$. Find $\phi(p_1^{a_1} \cdots p_k^{a_k})$ for $a_1, \ldots, a_k \geq 1$ and distinct primes $p_1, \ldots, p_k$. Hint: first prove this for the case $k = 1$.

(S3) Prove that for all $n \geq 1$,

$$\sum_{a_1+a_2+a_3=n} \binom{n}{a_1, a_2, a_3} = 3^n.$$  

(S4) Use the binomial theorem to prove that for all $n \geq 1$,

$$\sum_{k=0}^{n} \frac{(-1)^{k+1}}{k+1} \binom{n}{k} = 0.$$  

Why might it be hard to find a combinatorial proof of this identity?

(S5) Consider the following identity, valid for all $n \geq 1$:

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$  

(a) Give an algebraic proof of this identity.

(b) Give a combinatorial proof of this identity.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Give a combinatorial proof of the following identity:

$$\binom{n-1}{k-1} \binom{n}{k+1} \binom{n+1}{k} = \binom{n}{k-1} \binom{n-1}{k} \binom{n+1}{k+1}.$$