Discussion problems. The problems below should be completed in class.

(D1) Graph automorphisms. An automorphism of a graph $G$ is an unlabeled graph isomorphism from $G$ to itself.

(a) Find all automorphisms of the complete graph $K_n$ for $n \geq 2$.
(b) Find all automorphisms of the circle graph $C_n$ for $n \geq 3$.
(c) Find all automorphisms of the star graph $S_n$ for $n \geq 2$.
(d) Find all automorphisms of the wheel graph $W_n$ for $n \geq 3$.
(e) Are any of the above graphs isomorphic?

Use the following graphs as a guide for the above problem.

![Graphs](image1.png)

Complete graph $K_5$  Circle graph $C_8$  Star graph $S_8$  Wheel graph $W_8$

(D2) Eulerian walks. Fix a graph $G = (V, E)$. A walk on $G$ is Eulerian if every edge in $G$ is used exactly once.

(a) Which of the following graphs have closed Eulerian walk? Justify your answers.

![Graphs](image2.png)

(b) Can you find a criterion to ensure a given simple graph $G$ has a closed Eulerian walk?
(c) Which of the above graphs has a (not necessarily closed) Eulerian walk?
(d) Under what conditions on $G$ can we find an Eulerian walk?
(e) Make each of the above graphs directed by manually adding a direction to each edge. Now, which graphs have an Eulerian walk?
(f) Suppose $G$ is a directed graph. What conditions must $G$ satisfy in order to apply your proof from part (b)?
**Required problems.** As the name suggests, you must submit all required problem with this homework set in order to receive full credit.

(R1) Prove that if there is a walk from a vertex $v_1$ to a vertex $v_2$ in a simple graph $G$, then there is a path from $v_1$ to $v_2$.

(R2) Is there a non-connected graph on 7 vertices such that every vertex has degree at least 3?

(R3) How many distinct Hamiltonian cycles does the complete graph $K_n$ have?

(R4) Suppose $G$ is a $k$-regular graph (that is, a simple graph in which every vertex has degree exactly $k$). Prove that $G$ has a cycle of length at least $k + 1$.

(R5) Find all non-isomorphic simple graphs on four vertices. Be sure to give an argument that you found all of them!

**Selection problems.** You are required to submit one selection problem with this problem set. You may also submit additional selection problems, but the total number of points awarded (excluding challenge problems) won’t exceed the total possible score on this problem set.

(S1) Find a simple graph $G$ that has no nontrivial automorphisms. What is the minimum number $n > 1$ of vertices a simple graph with this property can have?

(S2) Suppose $G$ is a simple graph with $n$ vertices and no Hamiltonian cycles. What is the maximum number of edges $G$ can have?

**Challenge problems.** Challenge problems are not required for submission, but bonus points will be awarded on top of your score for submitting a partial attempt or a complete solution.

(C1) Up to isomorphism, how many 4-regular graphs are there with exactly 10 edges?