Discussion problems. The problems below should be completed in class.

(D1) Rings of functions. Determine which of the following sets \((R, +, \cdot)\) forms a ring under the given addition and multiplication. For each ring \(R\), determine whether \(R\) is a field.

(a) The set \(R = \{\ldots, -2, 0, 2, \ldots\} \subset \mathbb{Z}\) of even integers, under the usual addition and multiplication of integers.

(b) The set \(R = \{r_n x^n + \cdots + r_1 x + r_0 : r_i \in \mathbb{R}\}\) of polynomials in a variable \(x\) with real coefficients, under the usual addition and multiplication.

(c) The subset of \(R\) from part (b) consisting of those polynomials with degree at most \(d\).

(d) The set \(R = \{f(x)/g(x) : f(x), g(x) \text{ polynomials, } g \neq 0\}\) of rational functions under the usual addition and multiplication of fractions. (Here, cancelling common factors in the numerator and denominator yields the same element of \(R\).)

(e) The subset of \(R\) in part (d) whose denominator takes only positive values over \(\mathbb{R}\).

(f) The set \(R = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous}\}\) of continuous real-valued functions on \(\mathbb{R}\), under the usual addition and multiplication of functions.

(D2) Zero divisors. A nonzero ring element \(r \in R\) is called a zero divisor if there exists a nonzero element \(r' \in R\) with \(r \cdot r' = 0\).

(a) For which \(n\) does the ring \(\mathbb{Z}_n\) have zero divisors?

(b) Which rings from problem (D1) have zero divisors?

(c) Give an example of a ring with no zero divisors that is not a field.

(D3) Gaussian integers. Consider the ring \(\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}\) under usual addition and multiplication of complex numbers (i.e. using \(i^2 = -1\)).

(a) Verify that \(\mathbb{Z}[i]\) is a ring. Is it a field? Which elements are invertible?

(b) Verify that the set \(U\) of units of \(\mathbb{Z}[i]\) forms a group under multiplication. Which (more familiar) group is this?

(c) Does \(\mathbb{Z}[i]\) have any zero divisors?

(D4) Cartesian products. Recall that the Cartesian product of two rings \(R_1\) and \(R_2\) is the set \(R_1 \times R_2 = \{(a,b) : a \in R_1, b \in R_2\}\) with addition \((a,b) + (a',b') = (a+a',b+b')\) and multiplication \((a,b) \cdot (a',b') = (a \cdot a', b \cdot b')\).

(a) How many elements does \(\mathbb{Z}_n \times \mathbb{Z}_m\) have?

(b) What is the characteristic of \(\mathbb{Z}_n \times \mathbb{Z}_m\)? What about \(R_1 \times R_2\) for rings \(R_1, R_2\)?

(c) The order of an element \(a \in R\) is the smallest integer \(k\) such that adding \(a\) to itself \(k\) times yields \(0\) (or, written more compactly, \(ka = 0\)).

(i) Which elements of \(\mathbb{Z}_5 \times \mathbb{Z}_3\) have highest order? What about \(\mathbb{Z}_6 \times \mathbb{Z}_4\)? \(\mathbb{Z}_n \times \mathbb{Z}_m\)?

(ii) Express the order of \((a,b) \in R_1 \times R_2\) in terms of the orders of \(a \in R_1\) and \(b \in R_2\).

(iii) Is there a relationship between the order of \(a \in R\) and the characteristic of \(R\)?
**Required problems.** As the name suggests, you must submit all required problem with this homework set in order to receive full credit.

(R1) Find all Abelian groups with exactly 4 elements. Argue that there are no more.

(R2) Fix a field $F$. Prove that the set $F \setminus \{0\}$ forms a group under multiplication. Is the same true if $F$ is any ring?

(R3) Show that the group $(\mathbb{Z}_{13} \setminus \{0\}, \cdot)$ is cyclic.

(R4) Fix a ring $R$. Prove that if $r \in R$ is a zero-divisor, then it is not invertible.

**Selection problems.** You are required to submit all parts of one selection problem with this problem set. You may submit additional selection problems if you wish, but please indicate what you want graded. Although I am happy to provide written feedback on all submitted work, no extra credit will be awarded for completing additional selection problems.

(S1) Fix two rings $R_1$ and $R_2$.

   (a) Characterize the units of $R_1 \times R_2$ in terms of the units of $R_1$ and $R_2$.
   
   (b) Characterize the zero divisors of $R_1 \times R_2$ in terms of the zero divisors of $R_1$ and $R_2$.

(S2) Under what conditions on $m$ and $n$ is the group $(\mathbb{Z}_n \times \mathbb{Z}_m, +)$ cyclic? Prove your assertion.

**Challenge problems.** Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Give an example of a ring in which the set of non-unit elements is closed under both addition and multiplication.