Discussion problems. The problems below should be completed in class.

(D1) Building block designs from block designs. Consider the following 12 subsets of \( \{1, \ldots, 9\} \).

\[
\begin{align*}
\{1, 2, 3\} & \quad \{1, 4, 7\} & \quad \{1, 5, 9\} & \quad \{1, 6, 8\} \\
\{4, 5, 6\} & \quad \{2, 5, 8\} & \quad \{2, 6, 7\} & \quad \{2, 4, 9\} \\
\{7, 8, 9\} & \quad \{3, 6, 9\} & \quad \{3, 4, 8\} & \quad \{3, 5, 7\}
\end{align*}
\]

(a) For which \( t \) is the above a \( t \)-design? What are the values of \( v, k, r \) and \( b \) in each case?
(b) Replace each set in the above example with its complement in \( \{1, \ldots, 9\} \). For which \( t \) does this form a \( t \)-design? These are called complementary designs. What are the values of \( v, k, r \) and \( b \) in each case?
(c) Based on your observation in part (c), state a conjecture for arbitrary 2-designs. Don’t worry about proving your conjecture; you will do this on your homework!
(d) Find an explicit formula for \( v, k, r \) and \( b \) in part (d) case (your formulas will depend on \( v, k, r \) and \( b \) in the original example).

(D2) Designs from finite fields. The goal of this problem is to prove the following theorem.

**Theorem.** If \( q = p^r \) for \( p \) prime and \( r \geq 1 \), there is a 2-design with \( v = q^2 \), \( k = q \), \( r = 1 \).

(a) Let’s consider the case \( q = 3 \). Draw the 2-dimensional vector space \( V = \mathbb{Z}_3^2 \) over \( \mathbb{Z}_3 \).
(b) Find all lines (not necessarily containing the origin) in \( \mathbb{Z}_3^2 \), written as sets of points. You may find it easier to shorten points from \((2, 1)\) to simply “21” in your list.
(c) Does this collection of sets constitute a 2-design? What are \( v, k, r \), and \( b \)?
(d) We now generalize the above construction from \( \mathbb{Z}_3 \) to any finite field. Suppose \( \mathbb{F}_q \) is a field with \( q \) elements, let \( V = \mathbb{F}_q^2 \) denote a 2-dimensional vector space over \( \mathbb{F}_q \), and let \( B_1, \ldots, B_b \subset V \) denote the set of lines in \( V \). For these to form a 2-design \((q^2, q, 1)\),

(i) each block must have the same size \( q \),
(ii) every element must lie in the same number of blocks, and
(iii) any pair of elements must occur together in exactly 1 block.

Restate each of the above requirements for \( B_1, \ldots, B_b \) in terms of geometry.
(e) Any line \( L \subset V \) can be written as follows for some \( a, b, c \in \mathbb{F}_q \):

\[
L = \{(x, y) \in V : ax + by = c\}
\]

Prove that any two lines with more than one point in common must be equal as sets. Hint: given \((x_1, y_1), (x_2, y_2) \in L\) distinct, express \( a, b \) and \( c \) terms of \( x_1, y_1, x_2 \) and \( y_2 \).
(f) Prove that any line \( L \subset V \) contains exactly \( q \) points.
(g) Which requirement listed in part (d) follows from the other two? With this in mind, find the number of lines containing each point of \( V \) (in terms of \( q \)).
**Required problems.** As the name suggests, you must submit all required problem with this homework set in order to receive full credit.

(R1) For each triple \((v, k, r)\) below, construct a block design (1-design) with parameters \((v, k, r)\), or prove that no such design exists.

(a) \((v, k, r) = (7, 6, 6)\)
(b) \((v, k, r) = (6, 3, 1)\)
(c) \((v, k, r) = (5, 2, 1)\)
(d) \((v, k, r) = (9, 6, 4)\)

(R2) Suppose that there exists a 5-design with parameters \((v, k, r) = (12, 6, 1)\). Such a design is also an \(s\)-design for any \(s \leq 5\); find the corresponding parameters for each \(s\). How many blocks must this design have?

(R3) Find a 5-design with parameters \((v, k, r) = (6, 5, 1)\), or argue that no such design exists.

(R4) Is it possible there exists a 3-design with parameters \((v, k, r) = (15, 6, 2)\)? What about a 4-design with parameters \((v, k, r) = (11, 5, 1)\)?

(R5) Find a 2-design with parameters \((v, k, r) = (16, 4, 1)\) using the theorem in problem (D2). Hint: what field should you use, and what do the elements of that field look like?

**Selection problems.** You are required to submit all parts of one selection problem with this problem set. You may submit additional selection problems if you wish, but please indicate what you want graded. Although I am happy to provide written feedback on all submitted work, no extra credit will be awarded for completing additional selection problems.

(S1) Fix a 2-design \(B_1, \ldots, B_b \subset \{1, \ldots, v\}\) with parameters \((v, k, r)\), and let \(B_i^c = \{1, \ldots, v\} \setminus B_i\) for each \(i \leq b\). Prove that the blocks \(B_1^c, \ldots, B_b^c\) also form a 2-design. Is the complement of a 3-design always a 3-design?

(S2) Fix a 2-design of the form \((v, 3, 1)\). Prove that \(v\) must have the form \(6n + 1\) or \(6n + 3\) for some \(n \geq 0\).

**Challenge problems.** Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Fix a finite field \(\mathbb{F}_q\), and let \(V = \mathbb{F}_q^3\) denote a 3-dimensional vector space over \(\mathbb{F}_q\). A hyperplane is a subset \(H \subset V\) given by

\[ H = \{(x, y, z) \in V : ax + by + cz = d\} \]

for some \(a, b, c, d \in \mathbb{F}_q\) (in particular, note that a hyperplane need not contain the origin).

(a) Find the number of hyperplanes in \(V\) (as a function of \(q\)).
(b) Determine for which \(t\) the collection of all hyperplanes in \(V\) forms a \(t\)-design.