Discussion problems. The problems below should be completed in class.

(D1) Building block designs from block designs. Consider the following 12 subsets of \{1, \ldots, 9\}.

\begin{align*}
\{1, 2, 3\} & \quad \{1, 4, 7\} & \quad \{1, 5, 9\} & \quad \{1, 6, 8\} \\
\{4, 5, 6\} & \quad \{2, 5, 8\} & \quad \{2, 6, 7\} & \quad \{2, 4, 9\} \\
\{7, 8, 9\} & \quad \{3, 4, 8\} & \quad \{3, 5, 7\} & \\
\end{align*}

(a) For which \( t \) is the above a \( t \)-design? What are the values of \( v, k, r \), and \( b \) in each case? Note that some of these values will be different for each \( t \).

(b) Fix \( s \leq t \). We saw in lecture that any \( s \)-design \((v, k, r)\) is also an \( s \)-design \((v, k, r_s)\) for some parameter \( r_s \). Find a formula for \( r_1 \) in terms of \( v, k, \) and \( r_2 \).

(c) For \( t \geq 2 \), find a formula for \( r_{t-1} \) in terms of \( v, k, r_t, \) and \( t \). Start by finding a 3-design \((v, k, r_3) = (5, 3, 1)\), and compute \( r_2 \) and \( r_1 \).

(d) Is there an 8-design \((v, k, r_8)\) in which \( r_8 = r_7 = \cdots = r_1 = 1 \)?

(e) Replace each set in the above example with its complement in \{1, \ldots, 9\} (you should have 12 sets with 6 elements each). For which \( t \) does this form a \( t \)-design? These are called complementary designs. What are the values of \( v, k, r, \) and \( b \) in each case?

(f) How do these values of \( v, k, r, \) and \( b \) relate to the corresponding values in part (a)? State a conjecture about complementary 2-designs. Don’t worry about proving your conjecture; you will do this on your homework!

(D2) Designs from finite fields. The goal of this problem is to prove the following theorem.

**Theorem.** If \( q = p^r \) for \( p \) prime and \( r \geq 1 \), there is a 2-design with \((v, k, r) = (q^2, q, 1)\).

(a) Let’s consider the case \( q = 3 \). Draw the 2-dimensional vector space \( \mathbb{Z}_3^2 \) over \( \mathbb{Z}_3 \).

(b) Find all lines (not necessarily containing the origin) in \( \mathbb{Z}_3^2 \), written as sets of points. You may find it easier to shorten points from \((2, 1)\) to simply “21” in your list.

(c) Does this collection of sets constitute a 2-design? What are \( v, k, r, \) and \( b \)?

(d) We now generalize the above construction from \( \mathbb{Z}_3^2 \) to any finite field. Suppose \( \mathbb{F}_q \) is a field with \( q \) elements, let \( V = \mathbb{F}_q^2 \) denote a 2-dimensional vector space over \( \mathbb{F}_q \), and let \( B_1, \ldots, B_6 \subset V \) denote the set of lines in \( V \). For these to form a 2-design \((q^2, q, 1)\),

(i) each block must have the same size \( q \),

(ii) every element must lie in the same number of blocks, and

(iii) any pair of elements must occur together in exactly 1 block.

Restate each of the above requirements for \( B_1, \ldots, B_6 \) in terms of geometry.

(e) The goal of the rest of this problem is to prove the geometric statements from part (d). Any line \( L \subset V \) can be written uniquely as either

\[ L = \{ (x, y) \in V : x = b \} \quad \text{or} \quad L = \{ (x, y) \in V : y = ax + b \} \]

for some \( a, b \in \mathbb{F}_q \). Over the finite field \( \mathbb{F}_3 = \mathbb{Z}_3 \), find \( a \) and \( b \) corresponding to the lines given by \( \text{span}\{(1, 2)\}, \langle 4, 2 \rangle + \text{span}\{(1, 2)\}, \) and \( \langle 1, 4 \rangle + \text{span}\{(0, 2)\} \) in \( \mathbb{F}_3^2 \).

(f) Prove that any line \( L \subset V \) contains exactly \( q \) points.

(g) Prove that any two lines with more than one point in common must be equal as sets.

Hint: given distinct \((x_1, y_1), (x_2, y_2) \in L\), express \( a, b \) and \( c \) terms of \( x_1, y_1, x_2 \) and \( y_2 \).

(h) Find the number of lines containing each point of \( V \) (in terms of \( q \)). Hint: based on your knowledge of 2-designs, why does this follow from the other two items in part (d)?
**Required problems.** As the name suggests, you must submit all required problem with this homework set in order to receive full credit.

(R1) For each triple \((v, k, r)\) below, construct a block design (1-design) with parameters \((v, k, r)\), or state why no such design exists.

(a) \((v, k, r) = (7, 6, 6)\)  
(b) \((v, k, r) = (6, 3, 1)\)  
(c) \((v, k, r) = (5, 2, 1)\)  
(d) \((v, k, r) = (9, 6, 4)\)

(R2) Suppose that there exists a 5-design with parameters \((v, k, r) = (12, 6, 1)\). Such a design is also an \(s\)-design for each \(s \leq 5\); find the corresponding parameters for each \(s\). How many blocks must this design have?

(R3) Find a 5-design with parameters \((v, k, r) = (6, 5, 1)\), or argue that no such design exists.

(R4) Is it possible there exists a 3-design with parameters \((v, k, r) = (15, 6, 2)\)? What about a 4-design with parameters \((v, k, r) = (11, 5, 1)\)?

(R5) Find a 2-design with parameters \((v, k, r) = (16, 4, 1)\) using the theorem in problem (D2). Hint: what field should you use, and what do the elements of that field look like?

**Selection problems.** You are required to submit all parts of one selection problem with this problem set. You may submit additional selection problems if you wish, but please indicate what you want graded. Although I am happy to provide written feedback on all submitted work, no extra credit will be awarded for completing additional selection problems.

(S1) Fix a 2-design \(B_1, \ldots, B_b \subset \{1, \ldots, v\}\) with parameters \((v, k, r)\), and let \(B_c^i = \{1, \ldots, v\} \setminus B_i\) for each \(i \leq b\). Prove that the blocks \(B_1^c, \ldots, B_b^c\) also form a 2-design. Is the complement of a 3-design always a 3-design?

(S2) Prove that for any 2-design of the form \((v, 3, 1)\), \(v\) must have the form \(6n + 1\) or \(6n + 3\) for some \(n \geq 0\).

**Challenge problems.** Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Suppose \(F_q\) is a finite field of size \(q\), and let \(F_q^3\) be a 3-dimensional vector space over \(F_q\). A hyperplane \(H \subset F_q^3\) is a (possibly translated) 2-dimensional subspace of \(F_q^3\). Any hyperplane can be expressed as

\[ H = t + \text{span}\{v, w\} \]

for some \(t \in F_q^3\) and linearly independent \(v, w \in F_q^3\) (in particular, note that a hyperplane need not contain the origin).

(a) Does the collection of all hyperplanes in \(F_q^3\) form a 3-design? Explain using geometry.
(b) Find the number of hyperplanes in \(F_q^3\) (as a function of \(q\)). Prove your formula holds.