Notes on First Order Differential Equations for Math 22B,
based on Chapter 2 of Boyce and DiPrima’s Elementary Differential Equations and Boundary Value Problems.

Method of Integrating Factors. (Section 2.1)
Assume we have a first order linear equation of the form
\[ y' + p(t)y = g(t) \quad (*) \]
Define the integrating factor
\[ \mu(t) = \exp \left( \int p(t) \, dt \right). \]
Multiplying both sides of equation (*) by \( \mu(t) \) then integrating with respect to \( t \), we obtain the following general solution
\[ y = \frac{1}{\mu(t)} \left[ \int \mu(t)g(t) \, dt + c \right] \]
of the original equation, where \( c \) is an arbitrary constant. We may use an initial condition \( y(t_0) = y_0 \) to determine the constant \( c \).

Separable Equations. (Section 2.2)
Assume we have a first order differential equation which we can write in the form
\[ M(x) + N(y)(dy/dx) = 0. \]
This equation is said to be separable, and we may write
\[ \int M(x) \, dx = - \int N(y) \, dy. \]
Given an initial condition \( y(x_0) = y_0 \), the solution to this initial value problem is then given by
\[ \int_{x_0}^{x} M(s) \, ds + \int_{y_0}^{y} N(s) \, ds = 0. \]

Exact Equations and Integrating Factors. (Section 2.6.)
[NOTE: Section 2.6 is not listed in the class syllabus. I’m including it here for those who are curious about exact equations.]

**Theorem 2.6.1.** Say we are given a first order differential equation of the form
\[ M(x, y) + N(x, y)y' = 0 \quad (**) \]
Assume the functions \( M, N, M_y, \) and \( N_x, \) where subscripts denote partial derivatives, are continuous in the rectangular region \( R : \alpha < x < \beta, \gamma < y < \delta. \) (Actually, it suffices to assume that the region is simply connected.) Then, there exists a function \( \psi \) satisfying
\[ \psi_x(x, y) = M(x, y) \quad \text{and} \quad \psi_y(x, y) = N(x, y) \]
if and only if \( M \) and \( N \) satisfy
\[ M_y(x, y) = N_x(x, y) \]
at each point of \( R. \) In this case, (*) is said to be an exact differential equation in \( R, \) and \( \psi(x, y) \) is an implicitly defined solution to (*)

If we do not have \( M_y(x, y) = N_x(x, y), \) we may be able to multiply the equation (*) by an integrating factor \( \mu(x, y) \) to convert it into an exact differential equation. Two particular cases of this are when \( \mu \) depends only on \( x \) and when \( \mu \) depends only on \( y. \)

(a) If \( (M_y - N_x)/N \) is a function of \( x \) only, then solving the (linear and separable) differential equation
\[ \frac{d\mu}{dx} = \frac{M_y - N_x}{N} \mu \]
gives an integrating factor $\mu(x)$, which we may use to convert $(\ast)$ into an exact differential equation.

(b) If $(N_x - M_y)/M$ is a function of $y$ only, then solving the (linear and separable) differential equation

$$\frac{d\mu}{dx} = \frac{N_x - M_y}{M} \mu$$

gives an integrating factor $\mu(y)$, which we may use to convert $(\ast)$ into an exact differential equation.

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** Differences Between Linear and Nonlinear Equations. (Section 2.4)**

**Theorem 2.4.1.** Assume we are given the initial value problem

$$\begin{aligned}
& y' + p(t)y = g(t), \\
& y(t_0) = y_0,
\end{aligned} \tag{1}$$

where $y_0$ is an arbitrary prescribed initial value. If the functions $p$ and $g$ are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation (1) for each $t$ in $I$, and that also satisfies the initial condition (2).

**Theorem 2.4.2.** Assume we are given the initial value problem

$$\begin{aligned}
& y' = f(t, y) \\
& y(t_0) = y_0
\end{aligned} \tag{*}$$

where $y_0$ is an arbitrary prescribed initial value. If the functions $f$ and $\partial f/\partial y$ are continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point $(t_0, y_0)$, then in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$ there is a unique solution $y = \phi(t)$ of the initial value problem $(\ast)$.

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** Euler’s Method. (Section 2.7)**

Given an first order initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0,$$

we may approximate the solution $y_n$ of $y(t)$ at $t = t_n$ by iteratively computing

$$y_{k+1} = y_k + f(t_k, y_k) \cdot h,$$

for $k = 1, \ldots, n$, where $h = (t_n - t_0)/n$ and $t_k = t_{k-1} + h$.

**Picard’s Iteration Method (Section 2.8.)**

Given an first order initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0,$$

we may approximate a solution $y = \phi(t)$ by iteratively computing

$$\phi_k(t) = \int_0^t f[s, \phi_{k-1}(s)] \, ds,$$

then taking the limit as $k \to \infty$. We often choose the initial approximation $\phi_0(t)$ to be zero.