Notes on Second Order Differential Equations for Math 22B,

based on Chapter 3 of Boyce and DiPrima's *Elementary Differential Equations and Boundary Value Problems***.**

Section 3.1: Homogeneous Equations with Constant Coefficients.

A **second order linear ordinary differential equation** has the form

$$y'' + p(t)y' + q(t)y = g(t).$$

For a second order differential equation, we need two initial conditions,

$$y(t_0) = y_0 \qquad \text{and} \qquad y'(t_0) = y_0'.$$

A second order linear equation is **homogeneous** if $g(t)$ is zero for all t, and $g(t)$ is sometimes called the **nonhomogeneous** term. In the special case of **constant coefficients**, the above equation becomes

$$ay'' + by' + cy = 0 \qquad (*_1)$$

where a, b, and c are given constants. The **characteristic equation** for the differential equation $(*_1)$ is then

$$ar^2 + br + c = 0.$$

If the characteristic equation are real and different, say $r_1 \neq r_2$, then

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t},$$

where c_1 and c_2 are constants, is a solution of $(*_1)$.

Section 3.2: Solutions of Linear Homogeneous Equations; the Wronskian.

Let p and q be continuous functions on an open interval I, say $\alpha < t < \beta$ (note that we may have $\alpha = -\infty$ and/or $\beta = \infty$). Then, for any function ϕ that is twice differentiable on I, we define the differential operator L by the equation

$$L[\phi] = \phi'' + p\phi' + q\phi.$$

So, the value of $L[\phi]$ at a point t in I is $L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t)$. We will also use the notation

$$L[y] = y'' + p(t)y' + q(t)y = 0 \qquad (*_2)$$

for a second order linear homogeneous differential equation, where $y = \phi(t)$.

Theorem 3.2.1. (Existence and Uniqueness Theorem.) Consider the initial value problem

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t), \\ y(t_0) = y_0, \\ y'(t_0) = y_0', \end{cases}$$

where p, q, and g are continuous on an open interval I that contains the point t_0. Then, there is exactly one solution $y = \phi(t)$ of this problem, and the solution exists throughout the interval I.

Theorem 3.2.2. (Principle of Superposition) If y_1 and y_2 are two solutions of $(*_2)$, then the linear combination

$$c_1 y_1 + c_2 y_2$$

is also a solution of $(*_2)$ for any values of the constants c_1 and c_2.

Definition. The solutions y_1 and y_2 are said to form a **fundamental set of solutions** of this equation if and only if their Wronskian

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2$$

is nonzero.

Theorem 3.2.3. Suppose that y_1 and y_2 are two solutions of $(*_2)$ and that the initial conditions

$$y(t_0) = y_0 \qquad \text{and} \qquad y'(t_0) = y_0'$$

are assigned. Then, it is always possible to choose the constants c_1, c_2 so that

$$y = c_1 y_1(t) + c_2 y_2(t)$$

satisfies $(*_2)$ and the above initial conditions if and only if the Wronskian $W(y_1, y_2)$ is not zero at t_0.

Theorem 3.2.4. Suppose that y_1 and y_2 are two solutions of $(*_2)$. Then, the family of solutions

$$y = c_1 y_1(t) + c_2 y_2(t)$$

with arbitrary coefficients c_1 and c_2 includes every solution of this equation if and only if there is a point t_0 where the Wronskian $W(y_1, y_2)$ is not zero.

Theorem 3.2.5. Assume that in $(*_2)$ the coefficient functions p and q are continuous on some open interval I. Choose some point t_0 in I. Let y_1 be the solution of the above equation that also satisfies the initial conditions

$$y(t_0) = 1 \qquad \text{and} \qquad y'(t_0) = 0,$$

and let y_2 be the solution which satisfies

$$y(t_0) = 0 \qquad \text{and} \qquad y'(t_0) = 1.$$

Then, y_1 and y_2 form a fundamental set of solutions of this equation.

Theorem 3.2.6. Assume that in $(*_2)$ the coefficient functions p and q are continuous real-valued functions. If

$$y = u(t) + iv(t)$$

is a complex-valued solution, then its real part u and its imaginary part v are also solutions of this equation, and hence the complex conjugate $\overline{y}$ is also a solution.

Theorem 3.2.7. (Abel's Theorem) If y_1 and y_2 are solutions of $(*)$, where p and q are continuous on an open interval I, then their Wronskian is given by

$$W(y_1, y_2)(t) = c \exp\left[-\int p(t)\, dt \right],$$

where c is a certain constant that depends on y_1 and y_2, but not on t. Further, $W(y_1, y_2)(t)$ either is zero for all t in I (if $c = 0$) or else is never zero in I (if $c \neq 0$).

Summary. To find the general solution of the differential equation

$$y'' + p(t)y' + q(t)y = 0, \qquad \text{where} \qquad \alpha < t < \beta,$$

we first find two functions y_1 and y_2 that satisfy this differential equation on $\alpha < t < \beta$. Then we check that there is a point t_0 in this interval where the Wronskian $W(y_1, y_2)(t_0) \neq 0$. In this case, y_1 and y_2 form a fundamental set of solutions, and the general solution is

$$y = c_1 y_1(t) + c_2 y_2(t),$$

where c_1 and c_2 are arbitrary constants. If initial conditions are prescribed at a point in $\alpha < t < \beta$, then c_1 and c_2 can be chosen so as to satisfy these conditions.

Section 3.3: Complex Roots of the Characteristic Equation.

Say the roots of the characteristic equation for the differential equation

$$ay'' + by' + cy = 0 \qquad (*_3)$$

are conjugate complex numbers

$$r_1 = \lambda + i\mu \qquad \text{and} \qquad r_2 = \lambda - i\mu,$$

where λ and μ are real and $\mu \neq 0$. Then,

$$y = c_1 \exp[(\lambda + i\mu)t] + c_2 \exp[(\lambda - i\mu)t]$$

is the general solution for $(*_3)$. Here, we use **Euler's equation**

$$e^{i\theta} = \cos\theta + i \sin\theta$$

to interpret these complex-valued functions. We may use Theorem 3.2.6 to show that

$$y = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t$$

also gives the general solution to $(*_3)$.

Section 3.4: Repeated Roots; Reduction of Order.

Say the two roots of the characteristic equation for the differential equation

$$ay'' + by' + cy = 0 \qquad (*_4)$$

are are equal, $r = -b/2a$. Then, we may use Theorem 3.2.6 to show that

$$y = c_1 e^{-bt/2a} + c_2 t e^{-bt/2a}$$

gives the general solution to $(*_4)$.

Section 3.5: Nonhomogeneous Equations; Method of Undetermined Coefficients.

Let p, q, and g be given continuous functions on the open interval I, and write

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \qquad (*_5)$$

for a second order linear nonhomogeneous equation. The equation

$$L[y] = 0,$$

i.e. when $g(t) = 0$, is called the **corresponding homogeneous equation**.

Theorem 3.5.1. If Y_1 and Y_2 are two solutions of the nonhomogeneous equation $(*_5)$, then their difference $Y_1 - Y_2$ is a solution of the corresponding homogeneous equation. If, in addition, y_1 and y_2 are a fundamental set of solutions of the homogeneous equation, then

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t),$$

where c_1 and c_2 are certain constants.

Theorem 3.5.2. The general solution of the nonhomogeneous equation $(*_5)$ can be written in the form

$$y = \phi(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t),$$

where y_1 and y_2 are a fundamental set of solutions of the corresponding homogeneous equation, c_1 and c_2 are arbitrary constants, and Y is some specific solution of the nonhomogeneous equation.

Summary: To solve the nonhomogeneous equation, we must

1. Find the general solution

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t)$$

of the corresponding homogeneous equation, which is called the **complementary solution**.

2. Find some single solution $Y(t)$, called a **particular solution**, of the nonhomogeneous equation.

3. Form the sum

$$y(t) = y_c(t) + Y(t).$$

This gives the general solution to $(*_5)$.

The Method of Undetermined Coefficients. Given the nonhomogeneous equation $(*_5)$, we may attempt to find the general solution using the method of undetermined coefficients, as follows.

1. Check that the function $g(t)$ involves nothing more than exponential functions, sines, cosines, polynomials, or sums or products of such functions. If this is not the case, use the method of variation of parameters (see the next section).

2. If $g(t) = g_1(t) + \cdots + g_n(t)$, then form n subproblems $L[y] = g_i(t)$, where i runs from 1 to n.

3. For the ith subproblem, assume a particular solution $Y_i(t)$ as follows.

(i) If $g_i(t) = P_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n$, then choose

$$Y_i(t) = t^s(A_0 t^n + A_1 t^{n-1} + \cdots + A_n).$$

(ii) If $g_i(t) = P_n(t)e^{\alpha t}$, then choose

$$Y_i(t) = t^s(A_0 t^n + A_1 t^{n-1} + \cdots + A_n)e^{\alpha t}.$$

(iii) If $g_i(t) = P_n(t)e^{\alpha t} \begin{cases} \sin \beta t \\ \cos \beta t \end{cases}$, then choose

$$Y_i(t) = t^s[(A_0 t^n + A_1 t^{n-1} + \cdots + A_n)e^{\alpha t}\cos\beta t + (B_0 t^n + B_1 t^{n-1} + \cdots + B_n)e^{\alpha t}\sin\beta t].$$

Here, s is the smallest nonnegative integer ($s = 0, 1$, or 2 for a second order differential equation) which will ensure that no term in $Y_i(t)$ is a solution of the corresponding homogeneous equation. Equivalently, for the three cases, s is the number of times 0 is a root of the characteristic equation, α is a root of the characteristic equation, and $\alpha + i\beta$ is a root of the characteristic equation, respectively.

4. Find a particular solution $Y_i(t)$ for each of the subproblems. Then, the sum $Y_1(t) + \cdots + Y_n(t)$ is a particular solution of the full nonhomogeneous equation, which we may add to the complementary solution $y_c(t)$ to obtain the general solution

$$y(t) = y_c(t) + Y_1(t) + \cdots + Y_n(t)$$

of $(*)$.

Section 3.6: Variation of Parameters.

Theorem 3.6.1. If the functions p, q, and g are continuous on an open interval I, and if the functions y_1 and y_2 are a fundamental set of solutions of the homogeneous equation corresponding to the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t),$$

then a particular solution of the nonhomogeneous equation is

$$Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)}\,ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)(s)}\,ds,$$

where t_0 is any conveniently chosen point in I. The general solution is then

$$y = c_1 y_1(t) + c_2 y_2(t) + Y(t).$$