Notes on Systems of First Order Linear Equations for Math 22B

based on Chapter 7 of Boyce and DiPrima's *Elementary Differential Equations and Boundary Value Problems*.

Section 7.1: Introduction.

Notation. In the following, we will denote the independent variable by t, and we will let $x_1, x_2, x_3, \ldots$ represent dependent variables that are functions of t. Differentiation with respect to t will be denoted by a prime.

To transform an arbitrary nth order equation $y^{(n)} = F(t, y, y', \ldots, y^{(n-1)})$ into a system of n first order equations: Introduce the variables $x_1, x_2, \ldots, x_n$ defined by

$$\begin{cases} x_1 = y \\ x_2 = y' \\ x_3 = y'' \\ \quad\vdots \\ x_n = y^{(n-1)} \end{cases}$$

Then,

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \quad\vdots \\ x_{n-1}' = x_n \\ x_n' = F(t, x_1, x_2, \ldots, x_n) \end{cases}$$

Definition. A **solution** of the system

$$\begin{cases} x_1' = F_1(t, x_1, x_2, \ldots, x_n) \\ x_2' = F_2(t, x_1, x_2, \ldots, x_n) \\ \quad\vdots \\ x_n' = F_n(t, x_1, x_2, \ldots, x_n) \end{cases} \qquad (*_1)$$

on the interval $I : \alpha < t < \beta$ is a set of n functions

$$\begin{cases} x_1 = \phi_1(t) \\ x_2 = \phi_2(t) \\ \quad\vdots \\ x_n = \phi_n(t) \end{cases} \qquad (*_2)$$

that are differentiable at all points in the interval I and that satisfy the above system of of ordinary differential equations at all points in this interval. Further, **initial conditions** for this system are of the form

$$\begin{cases} x_1(t_0) = x_1^0 \\ x_2(t_0) = x_2^0 \\ \quad\vdots \\ x_n(t_0) = x_n^0 \end{cases} \qquad (*_3)$$

where $t_0 \in I$, and the x_i^0 are (real) numbers.

If each of the functions F_i is a linear function of the dependent variables x_j, then the system of equations is **linear**; otherwise, it is **nonlinear**. Thus, the most general system of n first order linear equations has the form:

$$\begin{cases} x_1' = p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t) \\ \quad\vdots \\ x_n' = p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t) \end{cases} \qquad (*_4)$$

If each of the functions $g_i(t)$ is zero for all $t \in I$, then the system is **homogeneous**; otherwise, it is **nonhomogeneous**.

Theorem 7.1.1. If each of the functions $F_1, \dots, F_n$ and the partial derivatives

$$\begin{gathered}\partial F_1/\partial x_1, \partial F_1/\partial x_2, \dots, \partial F_1/\partial x_n,\\ \partial F_2/\partial x_1, \partial F_2/\partial x_2, \dots, \partial F_2/\partial x_n,\\ \vdots \\ \partial F_n/\partial x_1, \partial F_n/\partial x_2, \dots, \partial F_n/\partial x_n\end{gathered}$$

are all continuous in a region R of $tx_1x_2\cdots x_n$-space defined by

$$\alpha < t < \beta, \qquad \alpha_1 < x_1 < \beta_1, \qquad \dots, \qquad \alpha_n < x_n < \beta_n,$$

and the point $(t_0, x_1^0, x_2^0, \dots, x_n^0)$ is in R, then there is an interval $|t - t_0| < h$ in which there exists a unique solution $(*_2)$ of the above system $(*_1)$ of differential equations that also satisfies the initial conditions $(*_3)$.

Theorem 7.1.2. If the functions $p_{11}, p_{12}, \dots, p_{nn}, g_1, \dots, g_n$ are continuous on an open interval $I : \alpha < t < \beta$, then there exists a unique solution $(*_2)$ of the system $(*_4)$ that also satisfies the above initial conditions $(*_3)$, where t_0 is any point in I, and $x_1^0, \dots, x_n^0$ are any prescribed numbers. Further, the solution exists throughout the interval I.

Section 7.2: Review of Matrices.

Definition. Let

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad \text{and} \qquad A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{m1}(t) & \cdots & a_{mn}(t) \end{bmatrix}.$$

The matrix $A(t)$ is said to be **continuous** at $t = t_0$ or on an interval $\alpha < t < \beta$ if each element of A is a continuous function at the given point or on the given interval. Similarly, $A(t)$ **differentiable** if each of its elements is differentiable, and its derivative dA/dt is defined by $dA/dt = (da_{ij}/dt)$. In the same way, the **integral** of a matrix function is defined as $\int_a^b A(t)\ dt = (\int_a^b a_{ij}(t)\ dt)$.

Fact. We have

(i) $d/dt(CA) = C(dA/dt)$, where C is a constant matrix

(ii) $d/dt(A + B) = dA/dt + dB/dt$

(iii) $d/dt(AB) = A(dB/dt) + (dA/dt)B$

Section 7.3: Systems of Linear Algebraic Equations; Linear Independence, Eigenvalues, Eigenvectors.

Systems of Linear Algebraic Equations. A set of n simultaneous linear algebraic equations in n variables

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \quad\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

can be written as $A\mathbf{x} = \mathbf{b}$, where $A = (a_{ij})$ and $\mathbf{b} = (b_k)$ are given, and $\mathbf{x} = (x_\ell)$ is to be determined. If $\mathbf{b} = \mathbf{0}$, then the system is **homogeneous**; otherwise, it is **nonhomogeneous**.

If A is invertible, then the unique solution is $\mathbf{x} = A^{-1}\mathbf{b}$. However, if A is not invertible, then solutions of this system either do not exist or do exist but are not unique. If the system is homogeneous, then there are infinitely many solutions. However, if the system is nonhomogeneous, then it has no solution unless the vector $(\mathbf{b}, \mathbf{y}) = 0$ for all vectors $\mathbf{y}$ satisfying $\overline{A}^T\mathbf{y} = \mathbf{0}$. In this case, it has infinitely many solutions, which are of the form $\mathbf{x} = \mathbf{x}^{(0)} + \vec{\xi}$, where $\mathbf{x}^{(0)}$ is a particular solution of the above system, and $\vec{\xi}$ is the most general solution of the homogeneous system.

Definition. Let

$$\mathbf{x}^{(1)}(t), \dots, \mathbf{x}^{(k)}(t)$$

be a set of vector functions defined on an interval $\alpha < t < \beta$. These vectors are said to be **linearly dependent** on $\alpha < t < \beta$ if there exists a set of constants $c_1, \dots, c_k$, not all of which are zero, such that

$$c_1\mathbf{x}^{(1)}(t) + \cdots + c_k\mathbf{x}^{(k)}(t) = 0$$

for all t in the interval. Otherwise, $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(k)}(t)$ are linearly independent.

Fact. Hermitian matrices are matrices for which $\overline{A}^T = A$. For these matrices,

(i) All eigenvalues are real.

(ii) There always exists a full set of n linearly independent eigenvectors, regardless of the algebraic multiplicities of the eigenvalues.

(iii) If $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are eigenvectors that correspond to different eigenvalues, then $(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = 0$. Thus, if all eigenvalues are simple, then the associated eigenvectors form an orthogonal set of vectors.

(iv) Corresponding to an eigenvalue of algebraic multiplicity m, it is possible to choose m eigenvectors that are mutually orthogonal. Thus the full set of n eigenvectors can always be chosen to be orthogonal as well as linearly independent.

Section 7.4: Basic Theory of Systems of First Order Linear Equations.

Notation. We may write the system $(*_4)$ of n first order linear equations in matrix notation, with

(i) $p_{11}(t), \ldots, p_{nn}(t)$ the elements of an $n \times n$ matrix $P(t)$;

(ii) $g_1(t), \ldots, g_n(t)$ the components of a vector $\mathbf{g}(t)$; and

(iii) $x_1 = \phi_1(t), \ldots, x_n = \phi_n(t)$ the components of a vector $\mathbf{x} = \phi(t)$.

This equation then takes the form $\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t)$.

We will also use the notation

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} x_{11}(t) \\ \vdots \\ x_{n1}(t) \end{bmatrix}, \ldots, \mathbf{x}^{(k)}(t) = \begin{bmatrix} x_{1k}(t) \\ \vdots \\ x_{nk}(t) \end{bmatrix}, \ldots,$$

to designate specific solutions of the system. Note that $x_{ij}(t) = x_i^{(j)}(t)$ refers to the ith component of the jth solution $\mathbf{x}^{(j)}(t)$.

Note. Throughout this section, we assume that P and $\mathbf{g}$ are continuous on some interval $\alpha < t < \beta$.

Theorem 7.4.1. If the vector functions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ are solutions of the homogeneous equation

$$\mathbf{x}' = P(t)\mathbf{x},$$

then the linear combination

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + \cdots + c_k\mathbf{x}^{(k)}(t)$$

is also a solution for any constants $c_1, \ldots, c_k$.

Definition. Let $X(t)$ be the matrix whose columns are the vectors $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$. Then, the **Wronskian** of these n solutions is

$$W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}](t) = \det X(t).$$

Theorem 7.4.2. If the vector functions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent solutions of the homogeneous equation $\mathbf{x}' = P(t)\mathbf{x}$ for each point in the interval $\alpha < t < \beta$, then each solution $x = \phi(t)$ of the system can be expressed as a linear combination of $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$,

$$\phi(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + cn\mathbf{x}^{(n)}(t),$$

in exactly one way.

Definition. If the vector functions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent solutions of the homogeneous equation $\mathbf{x}' = P(t)\mathbf{x}$ for each point in the interval $\alpha < t < \beta$, then we call

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t),$$

where the constants $c_1, \ldots, c_n$ are arbitrary, the **general solution** of this homogeneous equation. Any set of solutions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ that is linearly independent at each point in the interval $\alpha < t < \beta$ is said to be a **fundamental set of solutions** for that interval.

Theorem 7.4.3. If $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are solutions of the homogeneous equation $\mathbf{x}' = P(t)\mathbf{x}$ on the interval $\alpha < t < \beta$, then in this interval $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ either is identically zero or else never vanishes.

Abel's formula. We have

$$W(t) = c\exp\int [p_{11}(t) + \cdots + p_{nn}(t)]\, dt.$$

Theorem 7.4.4. Let $\mathbf{e}^i$ be the column vector with 1 in the ith place and zeros elsewhere. Further, let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ be the solutions of the homogeneous equation $\mathbf{x}' = P(t)\mathbf{x}$ that satisfy the initial condition

$$\mathbf{x}^{(1)}(t_0) = \mathbf{e}^{(1)}, \ldots, \mathbf{x}^{(n)}(t_0) = \mathbf{e}^{(n)},$$

respectively, where t_0 is any point in $\alpha < t < \beta$. Then $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ form a fundamental set of solutions of the system.

Theorem. Consider the system $\mathbf{x}' = P(t)\mathbf{x}$, where each element of P is a real-valued continuous function. If

$$\mathbf{x} = \mathbf{u}(t) + i\mathbf{v}(t)$$

is a complex-valued solution, then its real part $\mathbf{u}(t)$ and its imaginary part $\mathbf{v}(t)$ are also solutions of this equation.

Section 7.5: Homogeneous Linear Systems with Constant Coefficients.

Let

$$\mathbf{x}' = A\mathbf{x} \qquad (*)$$

where A is a constant $n \times n$ matrix with real entries and with $\det A \neq 0$, be a system of homogeneous linear equations with constant coefficients. Then, the vector

$$\mathbf{x} = \vec{\xi} e^{rt},$$

where r is a constant and $\vec{\xi}$ is a vector, is a solution, provided that r is an eigenvalue and $\vec{\xi}$ an associated eigenvector of the coefficient matrix A, i.e. r and $\vec{\xi}$ satisfy

$$(A - rI)\vec{\xi} = \mathbf{0}.$$

To find solutions of the general differential equation $\mathbf{x}' = A\mathbf{x}$: Find the eigenvalues $r_1, \ldots, r_n$ (which need not all be different) and eigenvectors of A from the associated algebraic system $(A - rI)\vec{\xi} = \mathbf{0}$. Since we assumed that A is a real-valued matrix, we have the following possibilities for the eigenvalues of A:

(i) All eigenvalues are real and different from each other.

(ii) Some eigenvalues occur in complex conjugate pairs.

(iii) Some eigenvalues, either real or complex, are repeated.

In case (i), associated with each eigenvalue r_i is a real eigenvector $\vec{\xi}^{(i)}$, and the n eigenvectors $\vec{\xi}^{(1)}, \ldots, \vec{\xi}^{(n)}$ are linearly independent. The general solution is thus

$$\mathbf{x} = c_1 \vec{\xi}^{(1)} e^{r_1 t} + \cdots + c_n \vec{\xi}^{(n)} e^{r_n t}.$$

If A is real and symmetric, then all eigenvalues $r_1, \ldots, r_n$ are real, and even if some are repeated there is always a full set of n linearly independent eigenvectors $\vec{\xi}^{(1)}, \ldots, \vec{\xi}^{(n)}$. Hence the general solution is again

$$\mathbf{x} = c_1 \vec{\xi}^{(1)} e^{r_1 t} + \cdots + c_n \vec{\xi}^{(n)} e^{r_n t},$$

where r_i is the eigenvalue corresponding to $\vec{\xi}^{(i)}$.

The remaining cases will be discussed in later sections.

Section 7.6: Complex Eigenvalues.

Let

$$\mathbf{x}' = A\mathbf{x},$$

where A is a constant $n \times n$ matrix with real entries and with $\det A \neq 0$, be a system of homogeneous linear equations with constant coefficients. Since the elements of A are real, any complex eigenvalues must occur in complex conjugate pairs. Suppose A has two complex conjugate eigenvalues

$$r_1 = \lambda + i\mu \qquad \text{and} \qquad r_2 = \lambda - i\mu.$$

The corresponding eigenvectors are complex conjugates, and may be written as $\vec{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$ and $\vec{\xi}^{(2)} = \mathbf{a} - i\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are real. Then, we have $\mathbf{x}^{(1)}(t) = (\mathbf{a} + i\mathbf{b})e^{(\lambda + i\mu)t}$ and $\mathbf{x}^{(2)}(t) = (\mathbf{a} - i\mathbf{b})e^{(\lambda - i\mu)t}$. We can show that the vectors

$$\mathbf{u}(t) = e^{\lambda t}(\mathbf{a} \cos \mu t - \mathbf{b} \sin \mu t) \qquad \text{and} \qquad v(t) = e^{\lambda t}(\mathbf{a} \sin \mu t + \mathbf{b} \cos \mu t)$$

are real-valued, linearly independent, solutions of the above equation. Further, $r_3, \ldots, r_n$ are all real and distinct eigenvalues, with corresponding eigenvectors $\vec{\xi}^{(3)}, \ldots, \vec{\xi}^{(n)}$, then the general solution of the above equation is

$$\mathbf{x} = c_1\mathbf{u}(t) + c_2\mathbf{v}(t) + c_3\vec{\xi}^{(3)}e^{r_3t} + \cdots + c_n\vec{\xi}^{(n)}e^{r_nt}.$$

Section 7.7: Fundamental Matrices.

Suppose that $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ form a fundamental set of solutions for the equation

$$\mathbf{x}' = P(t)\mathbf{x} \qquad (*)$$

on some interval $\alpha < t < \beta$. The matrix

$$\Psi(t) = \begin{bmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{bmatrix},$$

whose columns are the vectors $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$, is said to be a **fundamental matrix** for the above system. Then, the general solution of the above equation $(*)$ may be written as

$$\mathbf{x} = \Psi(t)\mathbf{c},$$

where $\mathbf{c}$ is a constant vector with arbitrary components $c_1, \ldots, c_n$. For an initial value problem consisting of $(*)$ and the initial condition

$$\mathbf{x}(t_0) = \mathbf{x}^0,$$

where t_0 is a given point in $\alpha < t < \beta$ and $\mathbf{x}^0$ is a given initial vector, we see that

$$\mathbf{c} = \Psi^{-1}(t_0)\mathbf{x}^0.$$

The **special fundamental matrix**, denoted by $\Phi(t)$, has columns the vectors $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ designated in Theorem 7.4.4. Hence, $\Phi(t_0) = I$. In this case, the solution of the above initial value problem is

$$\mathbf{x} = \Phi(t)\mathbf{x}^0.$$

Further,

$$\Phi(t) = \Psi(t)\Psi^{-1}(t_0).$$

Definition. Let A be an $n \times n$ constant matrix. We define

$$\exp(At) = I + \sum_{n=1}^{\infty} \frac{t^n}{n!}A^n.$$

Each element of this matrix sum converges for all t as $n \to \infty$. Further, we have

$$\frac{d}{dt}\exp(At) = A\exp(At) \qquad \text{and} \qquad \exp(At)\big|_{t=0} = I.$$

Fact. It can be shown that the fundamental matrix $\Phi(t)$ for the initial value problem

$$\mathbf{x}' = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}^0$$

and $\exp(At)$ are the same. Hence, we can write the solution of the above initial value problem as

$$\mathbf{x} = \exp(At)\mathbf{x}^0.$$

Section 7.8: Repeated Eigenvalues.

Let

$$\mathbf{x}' = A\mathbf{x},$$

where A is a constant $n \times n$ matrix with real entries and with $\det A \neq 0$, be a system of homogeneous linear equations with constant coefficients. Suppose that $r = \rho$ is an m-fold root of the characteristic equation $\det(A - rI) = 0$, and that there are fewer than m linearly independent eigenvectors corresponding to ρ. Then, there will be fewer than m solutions of the above equation of the form $\vec{\xi}e^{\rho t}$ associated with this eigenvalue. For example, suppose that $r = \rho$ is a double eigenvalue of A, but that there is only one corresponding eigenvector $\vec{\xi}$. Then one solution is

$$\mathbf{x}^{(1)}(t) = \vec{\xi}e^{\rho t},$$

where $\vec{\xi}$ satisfies

$$(A - \rho I)\vec{\xi} = \mathbf{0}.$$

We find that a second solution is

$$\mathbf{x}^{(2)}(t) = \vec{\xi}te^{\rho t} + \vec{\eta}e^{\rho t},$$

where $\vec{\xi}$ satisfies the above equation and $\vec{\eta}$ is determined from

$$(A - \rho I)^2\vec{\eta} = \mathbf{0}.$$

It can be shown that it is always possible to solve this equation for $\vec{\eta}$. The vector $\vec{\eta}$ is called a **generalized eigenvector** of the matrix A corresponding to the eigenvalue ρ.

Section 7.9: Nonhomogeneous Linear Systems.

Consider the nonhomogeneous system

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t),$$

where the $n \times n$ matrix $P(t)$ and $n \times 1$ vector $\mathbf{g}(t)$ are continuous for $\alpha < t < \beta$. The general solution of this equation can be expressed as

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) + \mathbf{v}(t),$$

where $c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$ is the general solution of the homogeneous system $\mathbf{x}' = P(t)\mathbf{x}$, and $\mathbf{v}(t)$ is a particular solution of the above nonhomogeneous system.

Methods for determining a particular solution $\mathbf{v}(t)$.

Diagonalization. Let $P(t) = A$ be an $n \times n$ diagonalizable constant matrix. Let T be the matrix with columns the eigenvectors $\vec{\xi}^{(1)}, \ldots, \vec{\xi}^{(n)}$ of A, and define $\mathbf{y}$ by $\mathbf{x} = T\mathbf{y}$. Then, we obtain the equation

$$\mathbf{y}' = (T^{-1}AT)\mathbf{y} + T^{-1}\mathbf{g}(t) = D\mathbf{y} + \mathbf{h}(t),$$

where $\mathbf{h}(t) = T^{-1}\mathbf{g}(t)$ and D is the diagonal matrix whose diagonal entries are the eigenvalues $r_1, \ldots, r_n$ of A, arranged in the same order as the corresponding eigenvectors $\vec{\xi}^{(1)}, \ldots, \vec{\xi}^{(n)}$ that appear as columns of T. This is a system of n uncoupled first-order linear equations for $y_1(t), \ldots, y_n(t)$, and hence the equations can be solved separately. Then, the solution of the original system can be found by the relation $\mathbf{x} = T\mathbf{y}$.

Undetermined Coefficients. The method of undetermined coefficients is applied similarly to as in Section 3.5 for linear second order equations. This method is applicable only if the coefficient matrix $P(t) = A$ is a constant matrix, and if the components of $\mathbf{g}(t)$ are polynomial, exponential, or sinusoidal functions, or sums or products of these. The procedure for choosing the form of the solution is the same as that given in Section 3.5, except, e.g., in the case of a nonhomogeneous term of the form $\mathbf{u}e^{\lambda t}$, where λ is a simple root of the characteristic equation. Then, we need to assume a solution of the form $\mathbf{a}te^{\lambda t} + \mathbf{b}e^{\lambda t}$.

Variation of Parameters. Let

$$\mathbf{x}' = P(t)x + \mathbf{g}(t),$$

where $P(t)$ and $\mathbf{g}(t)$ are continuous on $\alpha < t < \beta$. Assume that a fundamental matrix $\Psi(t)$ for the corresponding homogeneous system $\mathbf{x}' = P(t)\mathbf{x}$ has been found. Then, the general solution of the above system is given by

$$\mathbf{x} = \Psi(t)\mathbf{c} + \Psi(t)\int_{t_1}^{t} \Psi^{-1}(s)\mathbf{g}(s)\, ds,$$

where t_1 is any point in the interval (α, β). The first term on the right side of this equation is the general solution of the corresponding homogeneous system, and the second term is a particular solution.

If we have the initial value problem consisting of the above differential equation and the initial condition $\mathbf{x}(t_0) = \mathbf{x}^0$, then we can find the solution of this problem most conveniently if we choose the lower limit of integration in the above integral to be the initial point t_0, and we obtain

$$\mathbf{c} = \Psi^{-1}(t_0)\mathbf{x}^0.$$

This solution takes a slightly simpler form if we use the fundamental matrix $\Phi(t)$ satisfying $\Phi(t_0) = I$. In this case, we have

$$\mathbf{x} = \Phi(t)\mathbf{x}^0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\mathbf{g}(s)\, ds.$$

Laplace Transforms. Since the Laplace transform is an integral, the transform of a vector is computed component by component. Thus $\mathcal{L}\{\mathbf{x}(t)\}$ is the vector whose components are the transforms of the respective components of $\mathbf{x}(t)$.