NOTES FROM ELEMENTARY LINEAR ALGEBRA, 10TH EDITION, BY ANTON AND RORRES

CHAPTER 1: SYSTEMS OF LINEAR EQUATIONS AND MATRICES

Section 1.1: Introduction to Systems of Linear Equations

- Every system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

Section 1.2: Gaussian Elimination

Definition. If a linear system has infinitely many solutions, then a set of parametric equations from which all solutions can be obtained by assigning numerical values to the parameters is called a **general solution** of the system.

Theorem. **Free Variable Theorem for Homogeneous Systems**: If a homogeneous linear system has n unknowns, and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has $n - r$ free variables.

Theorem. A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions.

Section 1.3: Matrices and Matrix Operations

Definition. A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** in the matrix.

Definition Two matrices are defined to be **equal** if they have the same size and their corresponding entries are equal.

Definition. If A and B are matrices of the same size, then the **sum** $A + B$ is the matrix obtained by adding the entries of B to the corresponding entries of A, and the **difference** $A - B$ is the matrix obtained by subtracting the entries of B from the corresponding entries of A. Matrices of different sizes cannot be added or subtracted.

- For $A = [a_{ij}]$ and $B = [b_{ij}]$ having the same size, $(A + B)_{ij} = (A)_{ij} + (B)_{ij} = a_{ij} + b_{ij}$ and $(A - B)_{ij} = (A)_{ij} - (B)_{ij} = a_{ij} - b_{ij}$.

Definition. If A is any matrix and c is any scalar, the the **product** cA is the matrix obtained by multiplying each entry of the matrix A by c. The matrix cA is said to be a **scalar multiple** of A.

- For $A = [a_{ij}]$, $(cA)_{ij} = c(A)_{ij} = ca_{ij}$

Definition. If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the **product** AB is the $m \times n$ matrix whose entries are determined as follows: To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and column together, and then add up the resulting products.

- $(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \ldots + a_{ir}b_{rj}$

Definition. If $A_1, A_2, \ldots A_r$ are matrices of the same size, and $c_1, c_2, \ldots c_r$ are scalars, then an expression of the form $c_1A_1 + c_2A_2 + \ldots c_rA_r$ is called a **linear combination** of $A_1, A_2, \ldots A_r$ with **coefficients** $c_1, c_2, \ldots c_r$.

Theorem. If A is an $m \times n$ matrix, and if $\mathbf{x}$ is an $n \times 1$ column vector, then the product $A\mathbf{x}$ can be expressed as a linear combination of the column vectors of A in which the coefficients are the entries of $\mathbf{x}$.

Definition. If A is any $m \times n$ matrix, then the **transpose of** $\boldsymbol{A}$, denoted by A^T, is defined to be the $n \times m$ matrix that results from interchanging the rows and columns of A; that is, the first column of A^T is the first row of A, the second column of A^T is the second row of A, and so forth.

- $(A^T)_{ij} = (A)_{ji}$

Definition. If A is a square matrix, then the **trace of** $\boldsymbol{A}$, denoted by $\mathrm{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.

Section 1.4: Inverses; Algebraic Properties of Matrices

Theorem. **Properties of Matrix Arithmetic**: Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid:

(*a*) $A + B = B + A$ **(commutative law for addition)**
(*b*) $A + (B + C) = (A + B) + C$ **(associative law for addition)**
(*c*) $A(BC) = (AB)C$ **(associative law for multiplication)**
(*d*) $A(B + C) = AB + AC$ **(left distributive law)**
(*e*) $(B + C)A = BA + CA$ **(right distributive law)**
(*f*) $A(B - C) = AB - AC$
(*g*) $(B - C)A = BA - CA$
(*h*) $a(B + C) = aB + aC$
(*i*) $a(B - C) = aB - aC$
(*j*) $(a + b)C = aC + bC$
(*k*) $(a - b)C = aC - bC$
(*l*) $a(bC) = (ab)C$
(*m*) $a(BC) = (aB)C = B(aC)$

Theorem. **Properties of Zero Matrices**: If c is a scalar, and the sizes of the matrices are such that the operations can be performed, then:
(a) $A + 0 = 0 + A = A$
(b) $A - 0 = A$
(c) $A - A = A + (-A) = 0$
(d) $0A = 0$
(e) If $cA = 0$, then $c = 0$ or $A = 0$

• For A any $m \times n$ matrix, $AI_n = A$ and $I_mA = A$

Theorem. If R is the reduced row-echelon form of an $n \times n$ matrix A, then either R has a row of zeroes or R is the identity matrix I_n.

Definition. If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be **invertible** (or **nonsingular**) and B is called an **inverse** of A. If no such matrix B can be found, then A is said to be **singular**.

Theorem. If B and C are both inverses of the matrix A, then $B = C$.

• $AA^{-1} = I$ and $A^{-1}A = I$

Theorem. The matrix $A = [a\ \ b\ ;\ c\ \ d]$ is invertible if $ad - bc \neq 0$, in which case the inverse is given by the formula
$A^{-1} = \{1/(ad - bc)\}[d\ \ -b\ ;\ -c\ \ \ a]$

Theorem. If A and B are invertible matrices of the same size, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

• A product of any number of invertible matrices is invertible, and the inverse of the product is the product of the inverses in the reverse order.

• If A is a square matrix, then we define the nonnegative integer powers of A to be $A^0 = I$ and $A^n = AA\cdots A$ with n factors, for $n > 0$. Moreover, if A is invertible, then we define the negative integer powers to be $A^{-n} = (A^{-1})^n = A^{-1}A^{-1}\cdots A^{-1}$ with n factors. For r and s integers, $A^rA^s = A^{r+s}$ and $(A^r)^s = A^{rs}$.

Theorem. If A is invertible and n is a nonnegative integer, then:
(a) A^{-1} is invertible and $(A^{-1})^{-1} = A$
(b) A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$ for $n = 0, 1, 2, \ldots$
(c) kA is invertible for any nonzero scalar k, and $(kA)^{-1} = k^{-1}A^{-1} = 1/k\ A^{-1}$

Theorem. **Properties of the Transpose:** If the sizes of the matrices are such that the stated operations can be performed, then:
(a) $((A)^T)^T = A$
(b) $(A + B)^T = A^T + B^T$
(c) $(A - B)^T = A^T - B^T$
(d) $(kA)^T = kA^T$, where k is any scalar
(e) $(AB)^T = B^TA^T$

• The transpose of a product of any number of matrices is equal to the product of their transposes in the reverse order.

Theorem. If A is an invertible matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$

Section 1.5: Elementary Matrices and a Method for Finding A^{-1}

Definition. Matrices A and B are said to be **row equivalent** if either (hence each) can be obtained from the other by a sequence of elementary row operations.

Definition. An $n \times n$ matrix is called an **elementary matrix** if it can be obtained from the $n \times n$ identity matrix I_n by performing a single elementary row operation.

Theorem. **Row Operations by Matrix Multiplication**: If the elementary matrix E results from performing a certain row operation on I_n and if A is a $m \times n$ matrix, then the product EA is the matrix that results when this same row operation is performed on A.

Theorem. Every elementary matrix is invertible, and the inverse is also an elementary matrix.

Theorem. **Equivalent Statements**: If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all true or all false:
(a) A is invertible
(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
(c) the reduced row-echelon form of A is I_n
(d) A is expressible as a product of elementary matrices

• **Inversion Algorithm**: To find the inverse of an invertible matrix A, find a sequence of elementary row operations that reduces A to the identity and then perform this same sequence of operations on I_n to obtain A^{-1}.

Section 1.6: More on Linear Systems and Invertible Matrices

Theorem. A system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

Theorem. If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\mathbf{b}$, the system of equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$.

Theorem. Let A be a square matrix. If B is a square matrix satisfying $BA = I$, then $B = A^{-1}$. If B is a square matrix satisfying $AB = I$, then $B = A^{-1}$.

Theorem. **Equivalent Statements**: If A is an $n \times n$ matrix, then the following statements are equivalent:
(a) A is invertible
(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
(c) the reduced row-echelon form of A is I_n
(d) A is expressible as a product of elementary matrices
(e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$
(f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$

Theorem. Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.

Section 1.7: Diagonal, Triangular, and Symmetric Matrices

- A square matrix in which all the entries above the main diagonal are zero is called **lower triangular**, and a square matrix in which all the entries below the main diagonal are zero is called **upper triangular**. A matrix that is either upper triangular or lower triangular is called **triangular**.
- To multiply a matrix A on the left by a diagonal matrix D, one can multiply successive rows of A by the successive diagonal entries of D, and to multiply A on the right by D, one can multiply successive columns of A by the successive diagonal entries of D.

Theorem. The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular. The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular. A triangular matrix is invertible if and only if its diagonal entries are all nonzero. The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.

Definition. A square matrix A is **symmetric** if $A = A^T$.

- A square matrix $A = [a_{ij}]$ us symmetric if and only if $(A)_{ij} = (A)_{ji}$ for all values of i and j.

Theorem. If A and B are symmetric matrices with the same size, and if k is any scalar, then:
(a) A^T is symmetric
(b) $A + B$ and $A - B$ are symmetric
(c) kA is symmetric.

Theorem. The product of two symmetric matrices is symmetric if and only if the matrices commute.

Theorem. If A is an invertible symmetric matrix, then A^{-1} is symmetric.

Theorem. If A is an invertible symmetric matrix, then AA^T and A^TA are also invertible.