NOTES FROM ELEMENTARY LINEAR ALGEBRA, 10TH EDITION, BY ANTON AND RORRES

CHAPTER 2: DETERMINANTS

Section 2.1: Determinants by Cofactor Expansion

Definition. If A is a square matrix, then the **minor of entry a_{ij}** is denoted by M_{ij}, and is defined to be the determinant of the submatrix that remains after the ith row and the jth column are deleted from A. The number $(-1)^{i+j}M_{ij}$ is denoted by C_{ij} and is called the **cofactor of entry a_{ij}**.

Theorem. If A is any $m \times n$ matrix, then regardless of which row or column of A is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same.

Definition. If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the **determinant of A**, and the sums themselves are called the **cofactor expansions of A**. That is, $\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$ (cofactor expansion along the jth column) and $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$ (cofactor expansion along the ith row).

Theorem. If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then $\det(A)$ is the product of the entries on the main diagonal of the matrix; that is, $\det(A) = a_{11}a_{22}\cdots a_{nn}$.

Section 2.2: Evaluating Determinants by Row Reduction

Theorem. Let A be a square matrix. If A has a row of zeros or a column of zeros, then $\det(A) = 0$.

Theorem. Let A be a square matrix. Then $\det(A) = \det(A^T)$.

Theorem. Let A be an $n \times n$ matrix.
(*a*) If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, $\det(B) = k\det(A)$.
(*b*) If B is the matrix that results when two rows or two columns of A are interchanged, then $\det(B) = -\det(A)$.
(*c*) If B is the matrix that results when a multiple of one row of A is added to another row or when a multiple of one column of A is added to another column, then $\det(B) = \det(A)$.

Theorem. Let E be an $n \times n$ elementary matrix.
(*a*) If E results from multiplying a row of I_n by k, then $\det(E) = k$.
(*b*) If E results from interchanging two rows of I_n, then $\det(E) = -1$.
(*c*) If E results from adding a multiple of one row of I_n to another, then $\det(E) = 1$.

Theorem. If A is a square matrix with two proportional rows or two proportional columns, then $\det(A) = 0$.

Section 2.3: Properties of Determinants

- For A an $n \times n$ matrix, $\det(kA) = k^n\det(A)$.

Theorem. Let A, B, and C be $n \times n$ matrices that differ only in a single row, say the rth, and assume that the rth row of C can be obtained by adding corresponding entries in the rth rows of A and B. Then $\det(C) = \det(A) + \det(B)$. The same result holds for columns.

Lemma. If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then $\det(EB) = \det(E)\det(B)$.

Theorem. A square matrix A is invertible if and only if $\det(A) \neq 0$.

Theorem. If A and B are square matrices of the same size, then $\det(AB) = \det(A)\det(B)$.

Theorem. If A is invertible, $\det(A^{-1}) = 1/\det(A)$

Theorem. **Equivalent Statements**: If A is an $n \times n$ matrix, then the following statements are equivalent:
(*a*) A is invertible
(*b*) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
(*c*) the reduced row-echelon form of A is I_n
(*d*) A is expressible as a product of elementary matrices
(*e*) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$
(*f*) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$
(*g*) $\det(A) \neq 0$