Section 2.1: Determinants by Cofactor Expansion

Definition. If $A$ is a square matrix, then the minor of entry $a_{ij}$ is denoted by $M_{ij}$, and is defined to be the determinant of the submatrix that remains after the $i$th row and the $j$th column are deleted from $A$. The number $(-1)^{i+j}M_{ij}$ is denoted by $C_{ij}$ and is called the cofactor of entry $a_{ij}$.

Theorem. If $A$ is any $m \times n$ matrix, then regardless of which row or column of $A$ is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same.

Definition. If $A$ is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of $A$ by the corresponding cofactors and adding the resulting products is called the determinant of $A$, and the sums themselves are called the cofactor expansions of $A$. That is, $\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{nn}C_{nn}$ (cofactor expansion along the $j$th column) and $\det(A) = a_{11}C_{11} + a_{21}C_{21} + \cdots + a_{nn}C_{nn}$ (cofactor expansion along the $i$th row).

Theorem. If $A$ is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then $\det(A)$ is the product of the entries on the main diagonal of the matrix; that is, $\det(A) = a_{11}a_{22}\cdots a_{nn}$.

Section 2.2: Evaluating Determinants by Row Reduction

Theorem. Let $A$ be a square matrix. If $A$ has a row of zeros or a column of zeros, then $\det(A) = 0$.

Theorem. Let $A$ be a square matrix. Then $\det(A) = \det(A^T)$.

Theorem. Let $A$ be an $n \times n$ matrix.
(a) If $B$ is the matrix that results when a single row or single column of $A$ is multiplied by a scalar $k$, then $\det(B) = k \det(A)$.
(b) If $B$ is the matrix that results when two rows or two columns of $A$ are interchanged, then $\det(B) = -\det(A)$.
(c) If $B$ is the matrix that results when a multiple of one row of $A$ is added to another row or when a multiple of one column of $A$ is added to another column, then $\det(B) = \det(A)$.

Theorem. Let $E$ be an $n \times n$ elementary matrix.
(a) If $E$ results from multiplying a row of $I_n$ by $k$, then $\det(E) = k$.
(b) If $E$ results from interchanging two rows of $I_n$, then $\det(E) = -1$.
(c) If $E$ results from adding a multiple of one row of $I_n$ to another, then $\det(E) = 1$.

Theorem. If $A$ is a square matrix with two proportional rows or two proportional columns, then $\det(A) = 0$.

Section 2.3: Properties of Determinants

• For $A$ an $n \times n$ matrix, $\det(kA) = k^n\det(A)$.

Theorem. Let $A$, $B$, and $C$ be $n \times n$ matrices that differ only in a single row, say the $r$th, and assume that the $r$th row of $C$ can be obtained by adding corresponding entries in the $r$th rows of $A$ and $B$. Then $\det(C) = \det(A) + \det(B)$. The same result holds for columns.

Lemma. If $B$ is an $n \times n$ matrix and $E$ is an $n \times n$ elementary matrix, then $\det(EB) = \det(A)\det(B)$. The same result holds for columns.

Theorem. A square matrix $A$ is invertible if and only if $\det(A) \neq 0$.

Theorem. If $A$ and $B$ are square matrices of the same size, then $\det(AB) = \det(A)\det(B)$.

Theorem. If $A$ is invertible, $\det(A^{-1}) = 1/\det(A)$.

Theorem. Equivalent Statements: If $A$ is an $n \times n$ matrix, then the following statements are equivalent:
(a) $A$ is invertible
(b) $Ax = 0$ has only the trivial solution
(c) the reduced row-echelon form of $A$ is $I_n$
(d) $A$ is expressible as a product of elementary matrices
(e) $Ax = b$ is consistent for every $n \times 1$ matrix $b$
(f) $Ax = b$ has exactly one solution for every $n \times 1$ matrix $b$
(g) $\det(A) \neq 0$