NOTES FROM ELEMENTARY LINEAR ALGEBRA, 10TH EDITION, BY ANTON AND RORRES

CHAPTER 3: EUCLIDEAN VECTOR SPACES

Section 3.1: Vectors in 2-Space, 3-Space, and n-Space

Definition. If n is a positive integer, then an **ordered n-tuple** is a sequence of n real numbers $(a_1, a_2, \ldots, a_n)$. The set of all ordered n-tuples is called **n-space** and is denoted by $\boldsymbol{R}^n$.

Definition. Two vectors $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ in $\boldsymbol{R}^n$ are said to be **equivalent** (also **equal**) if $v_1 = w_1, v_2 = w_2, \ldots, v_n = w_n$. We indicate this by writing $\mathbf{v} = \mathbf{w}$.

Definition. If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ are vectors in $\boldsymbol{R}^n$, and if k is any scalar, then we define $\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$; $k\mathbf{v} = (kv_1, kv_2, \ldots, kv_n)$; $-\mathbf{v} = (-v_1, -v_2, \ldots, -v_n)$; $\mathbf{w} - \mathbf{v} = \mathbf{w} + (-\mathbf{v}) = (w_1 - v_1, w_2 - v_2, \ldots, w_n - v_n)$

Theorem. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\boldsymbol{R}^n$, and if k and m are scalars, then:
(a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
(b) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
(c) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
(d) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
(e) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
(f)$(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
(g) $k(m\mathbf{u}) = (km)\mathbf{u}$
(h) $1\mathbf{u} = \mathbf{u}$

Theorem. If $\mathbf{v}$ is a vector in R^n and k is a scalar, then:
(a) $0\mathbf{v} = \mathbf{0}$
(b) $\mathrm{k}\mathbf{0} = \mathbf{0}$
(c) $(-1)\mathbf{v} = -\mathbf{v}$

Definition. If $\mathbf{w}$ is a vector in R^n, then $\mathbf{w}$ is said to be a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in $\boldsymbol{R}^n$ if it can be expressed in the form $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r$ where $k_1, k_2, \ldots, k_r$ are scalars. These scalars are called the **coefficients** of the linear combination. In the case where $r = 1$, $\mathbf{w} = k_1\mathbf{v}_1$, so that a linear combination of a single vector is just a scalar multiple of that vector.

Section 3.2: Norm, Dot Product, and Distance in R^n

Definition. If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in $\boldsymbol{R}^n$, then the **norm** of $\mathbf{v}$ (also called the **length** of $\mathbf{v}$ or the **magnitude** of $\mathbf{v}$) is denoted by $||\mathbf{v}||$, and is defined by the formula $||\mathbf{v}|| = \sqrt{}(v_1^2 + v_2^2 + v_3^2 + \ldots + v_n^2)$

Theorem. If $\mathbf{v}$ is a vector in $\boldsymbol{R}^n$, and if k is any scalar, then: $||\mathbf{v}|| \geq 0$, $||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = \mathbf{0}$, $||k\mathbf{v}|| = |k|\ ||\mathbf{v}||$

- unit vector $\mathbf{u} = (1/||\mathbf{v}||)\mathbf{v}$
- $\mathbf{v} = (v_1, v_2, \ldots, v_n) = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \ldots + v_n\mathbf{e}_n$

Definition. If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are points in $\boldsymbol{R}^n$, then we denote the distance between $\mathbf{u}$ and $\mathbf{v}$ by $d(\mathbf{u},\mathbf{v})$ and define it to be $d(\mathbf{u},\mathbf{v}) = ||\mathbf{u} - \mathbf{v}|| = \sqrt{}[(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2]$

Definition. If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\boldsymbol{R}^2$ or $\boldsymbol{R}^3$, and if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the **dot product** (also called the **Euclidean inner product**) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u}\bullet\mathbf{v}$ and is defined as $\mathbf{u}\bullet\mathbf{v} = ||\mathbf{u}||\ ||\mathbf{v}|| \cos\theta$. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u}\bullet\mathbf{v}$ to be 0.

Definition. If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are any vectors in R^n, then the **dot product** (also called the **Euclidean inner product**) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u}\bullet\mathbf{v}$ is defined by $\mathbf{u}\bullet\mathbf{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n$

- $||\mathbf{v}|| = \sqrt{}(\mathbf{v}\bullet\mathbf{v})$

Theorem. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\boldsymbol{R}^n$, and if k is a scalar, then:
(a) $\mathbf{u}\bullet\mathbf{v} = \mathbf{v}\bullet\mathbf{u}$ (symmetry property)
(b) $\mathbf{u}\bullet(\mathbf{v}+\mathbf{w}) = \mathbf{u}\bullet\mathbf{v} + \mathbf{u}\bullet\mathbf{w}$ (distributive property)
(c) $k(\mathbf{u}\bullet\mathbf{v}) = (k\mathbf{u})\bullet\mathbf{v}$ (homogeneity property)
(d) $\mathbf{v}\bullet\mathbf{v} \geq 0$ and $\mathbf{v}\bullet\mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$ (positivity property)

Theorem. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\boldsymbol{R}^n$, and if k is a scalar, then:
(a) $\mathbf{0}\bullet\mathbf{v} = \mathbf{v}\bullet\mathbf{0} = 0$
(b) $(\mathbf{u}+\mathbf{v})\bullet\mathbf{w} = \mathbf{u}\bullet\mathbf{w} + \mathbf{v}\bullet\mathbf{w}$
(c) $\mathbf{u}\bullet(\mathbf{v}-\mathbf{w}) = \mathbf{u}\bullet\mathbf{v} - \mathbf{u}\bullet\mathbf{w}$
(d) $(\mathbf{u}-\mathbf{v})\bullet\mathbf{w} = \mathbf{u}\bullet\mathbf{w} - \mathbf{v}\bullet\mathbf{w}$
(e) $k(\mathbf{u}\bullet\mathbf{v}) = \mathbf{u}\bullet(k\mathbf{v})$

- $\theta = \cos^{-1}[(\mathbf{u}\bullet\mathbf{v})/(||\mathbf{u}||\ ||\mathbf{v}||)]$

Theorem. **Cauchy-Schwarz Inequality**: If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in $\boldsymbol{R}^n$, then: $|\mathbf{u}\bullet\mathbf{v}| \leq ||\mathbf{u}||\ ||\mathbf{v}||$

Theorem. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\boldsymbol{R}^n$, then: $||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$ (**Triangle Inequality for Vectors**) and $d(\mathbf{u},\mathbf{v}) \leq d(\mathbf{u},\mathbf{w}) + d(\mathbf{w},\mathbf{v})$ (**Triangle Inequality for Distances**)

Theorem. **Parallelogram Equation for Vectors**: If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\boldsymbol{R}^n$, then: $||\mathbf{u} + \mathbf{v}||^2 + ||\mathbf{u} - \mathbf{v}||^2 = 2(||\mathbf{u}||^2 + ||\mathbf{v}||^2)$

Theorem. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\boldsymbol{R}^n$ with the Euclidean inner product, then $\mathbf{u}\bullet\mathbf{v} = ¼\|\mathbf{u} + \mathbf{v}\|^2 – ¼\|\mathbf{u} – \mathbf{v}\|^2$

- $\mathbf{u}\bullet\mathbf{v} = \mathbf{v}^T\mathbf{u}$
- $A\mathbf{u}\bullet\mathbf{v} = \mathbf{u}\bullet A^T\mathbf{v}$
- $\mathbf{u}\bullet A\mathbf{v} = A^T\mathbf{u}\bullet\mathbf{v}$

Section 3.3: Orthogonality

Definition. Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\boldsymbol{R}^n$ are said to be **orthogonal** (or **perpendicular**) if $\mathbf{u}\bullet\mathbf{v} = 0$. We will also agree that the zero vector in $\boldsymbol{R}^n$ is orthogonal to every vector in $\boldsymbol{R}^n$. A nonempty set of vectors in $\boldsymbol{R}^n$ is called an **orthogonal set** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set of unit vectors is called an **orthonormal set**.

- **Point-Normal Form of the Equation of a Line** for $P_0(x_0,y_0)$ and $\mathbf{n} = (a,b)$: $a(x – x_0) + b(y – y_0) = 0$
- **Point-Normal Form of the Equation of a Plane** for $P_0(x_0,y_0,z_0)$ and $\mathbf{n} = (a,b,c)$: $a(x – x_0) + b(y – y_0) + c(z – z_0) = 0$

Theorem. (*a*) If a and b are constants that are not both zero, then an equation of the form $ax + by + c = 0$ represents a line in R^2 with normal $\mathbf{n} = (a,b)$.
(*b*) If a, b, and c are constants that are not all zero, then an equation of the form $ax + by + cz + d = 0$ represents a plane in $\boldsymbol{R}^3$ with normal $\mathbf{n} = (a,b,c)$.

- $\mathbf{n}\bullet\mathbf{x} = 0$

Theorem. **Projection Theorem**: If $\mathbf{u}$ and $\mathbf{a}$ are vectors in $\boldsymbol{R}^n$, and if $\mathbf{a} \neq 0$, then $\mathbf{u}$ can be expressed in exactly one way in the form $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is a scalar multiple of $\mathbf{a}$ and $\mathbf{w}_2$ is orthogonal to $\mathbf{a}$.

- $\text{proj}_{\mathbf{a}}\mathbf{u} = (\mathbf{u}\bullet\mathbf{a}/\|\mathbf{a}\|^2)\mathbf{a}$ (vector component of $\mathbf{u}$ along $\mathbf{a}$)
- $\mathbf{u} – \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} –(\mathbf{u}\bullet\mathbf{a}/\|\mathbf{a}\|^2)\mathbf{a}$ (vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$)
- $\|\text{proj}_{\mathbf{a}}\mathbf{u}\| = |\mathbf{u}\bullet\mathbf{a}|/\|\mathbf{a}\| = \|\mathbf{u}\|\ |\cos\theta|$ where θ is the angle between $\mathbf{u}$ and $\mathbf{a}$.

Theorem. **Theorem of Pythagoras in $\boldsymbol{R}^n$**: If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors in $\boldsymbol{R}^n$ with the Euclidean inner product, then $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$

Theorem. (*a*) In R^2, the distance between the point $P_0(x_0, y_0)$ and the line $ax + by + c = 0$ is $D = |ax_0 + by_0 + c|/\sqrt{(a^2 + b^2)}$
(*b*) In R^3, the distance between the point $P_0(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is
$D = |ax_0 + by_0 + cz_0 + d|/\sqrt{(a^2 + b^2 + c^2)}$

Section 3.4: The Geometry of Linear Systems

Theorem. Let L be the line in $\boldsymbol{R}^2$ or $\boldsymbol{R}^3$ that contains the point $\mathbf{x}_0$ and is parallel to the nonzero vector $\mathbf{v}$. Then the equation of the line through $\mathbf{x}_0$ that is parallel to $\mathbf{v}$ is $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$. If $\mathbf{x}_0 = \mathbf{0}$, then the line passes through the origin and the equation has the form $\mathbf{x} = t\mathbf{v}$.

Theorem. Let W be the plane in $\boldsymbol{R}^3$ that contains the point $\mathbf{x}_0$ and is parallel to the noncollinear vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Then an equation of the plane through $\mathbf{x}_0$ that is parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$ is given by $\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2$. If $\mathbf{x}_0 = \mathbf{0}$, then the plane passes through the origin and the equation has the form $\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$.

Definition. If $\mathbf{x}_0$ and $\mathbf{v}$ are vectors in $\boldsymbol{R}^n$, and if $\mathbf{v}$ is nonzero, then the equation $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ defines the **line through $\mathbf{x}_0$ that is parallel to $\mathbf{v}$**. In the special case that $\mathbf{x}_0 = \mathbf{0}$, the line is said to **pass through the origin**.

Definition. If $\mathbf{x}_0$, $\mathbf{v}_1$, and $\mathbf{v}_2$ are vectors in $\boldsymbol{R}^n$, and if $\mathbf{v}_1$ and $\mathbf{v}_2$ are not collinear, then the equation $\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2$ defines the **plane through $\mathbf{x}_0$ that is parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$**. In the special case that $\mathbf{x}_0 = \mathbf{0}$, the line is said to **pass through the origin**.

Definition. If $\mathbf{x}_0$ and $\mathbf{x}_1$ are vectors in $\boldsymbol{R}^n$, then the equation $\mathbf{x} = \mathbf{x}_0 + t(\mathbf{x}_1 – \mathbf{x}_0)$, $(0 \leq t \leq 1)$, defines the **line segment from $\mathbf{x}_0$ to $\mathbf{x}_1$**. When convenient, this equation can be written as $\mathbf{x} = (1 – t)\mathbf{x}_0 + t\mathbf{x}_1$, $(0 \leq t \leq 1)$.

Theorem. If A is an $m \times n$ matrix, then the solution set of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ consists of all vectors in $\boldsymbol{R}^n$ that are orthogonal to every row vector of A.

Theorem. The general solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$ can be obtained by adding any specific solution of $A\mathbf{x} = \mathbf{b}$ to the general solution of $A\mathbf{x} = \mathbf{0}$.

Section 3.5: Cross Product

Definition. If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in 3-space, then the **cross product $\mathbf{u} \times \mathbf{v}$** is the vector defined by $\mathbf{u} \times \mathbf{v} = (u_2v_3 – u_3v_2, u_3v_1 – u_1v_3, u_1v_2 – u_2v_1)$.

Theorem. **Relationships Involving Cross Product and Dot Product**: If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in 3-space, then:
(*a*) $\mathbf{u}\bullet(\mathbf{u}\times\mathbf{v}) = \mathbf{0}$ ($\mathbf{u}\times\mathbf{v}$ is orthogonal to $\mathbf{u}$)
(*b*) $\mathbf{v}\bullet(\mathbf{u}\times\mathbf{v}) = \mathbf{0}$ ($\mathbf{u}\times\mathbf{v}$ is orthogonal to $\mathbf{v}$)
(*c*) $\|\mathbf{u}\times\mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 – (\mathbf{u}\bullet\mathbf{v})^2$ (Lagrange's Identity)
(*d*) $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\bullet\mathbf{w})\mathbf{v} – (\mathbf{u}\bullet\mathbf{v})\mathbf{w}$ (relationship between cross and dot products)
(*e*) $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = (\mathbf{u}\bullet\mathbf{w})\mathbf{v} – (\mathbf{v}\bullet\mathbf{w})\mathbf{u}$ (relationship between cross and dot products)

Theorem. **Properties of Cross Product**: If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors in 3-space and k is any scalar, then:
(*a*) $\mathbf{u}\times\mathbf{v} = –(\mathbf{v}\times\mathbf{u})$
(*b*) $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = (\mathbf{u}\times\mathbf{v}) + (\mathbf{u}\times\mathbf{w})$

(c) $(\mathbf{u}+\mathbf{v})\times\mathbf{w} = (\mathbf{u}\times\mathbf{w}) + (\mathbf{v}\times\mathbf{w})$
(d) $k(\mathbf{u}\times\mathbf{v}) = (k\mathbf{u})\times\mathbf{v} = \mathbf{u}\times(k\mathbf{v})$
(e) $\mathbf{u}\times\mathbf{0} = \mathbf{0}\times\mathbf{u} = \mathbf{0}$
(f) $\mathbf{u}\times\mathbf{u} = \mathbf{0}$

Theorem. **Area of a Parallelogram**: If **u** and **v** are vectors in 3-space, then $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta$ is equal to the area of the parallelogram determined by **u** and **v**.

Definition. If **u, v,** and **w** are vectors in 3-space, then $\mathbf{u}\bullet(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ is called the **scalar triple product** of **u, v,** and **w.**

Theorem. (a) The absolute value of the determinant $\det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$ is equal to the area of the parallelogram in 2-space determined by the vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$.

(b) The absolute value of the determinant $\det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$ is equal to the volume of the parallelepiped in 3-space determined by the vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$.

Theorem. If the vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$ have the same initial point, then they lie in the same plane if and only if $\mathbf{u}\bullet(\mathbf{v}\times\mathbf{w}) = 0$.