NOTES FROM ELEMENTARY LINEAR ALGEBRA, 10TH EDITION, BY ANTON AND RORRES

CHAPTER 4: GENERAL VECTOR SPACES

Section 4.1: Real Vector Spaces

Definition. Let V be an arbitrary nonempty set of objects on which two operations are defined: addition, and multiplication by scalars (numbers). By **addition**, we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in V an object $\mathbf{u} + \mathbf{v}$, called the sum of $\mathbf{u}$ and $\mathbf{v}$; by **scalar multiplication**, we mean a rule for associating with each scalar k and each object $\mathbf{u}$ in V an object $k\mathbf{u}$, called the **scalar multiple** of $\mathbf{u}$ by k. If the following axioms are satisfied by all objects $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all scalars k and m, then we call V a **vector space** and we call the objects in V **vectors**.
1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u} + \mathbf{v}$ is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. There is an object $\mathbf{0}$ in V, called a **zero vector** for V, such that $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u}$ in V.
5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a **negative** of $\mathbf{u}$, such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$.
6. If k is any scalar and $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V.
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)(\mathbf{u})$
10. $1\mathbf{u} = \mathbf{u}$

- **To Show that a Set with Two Operations is a Vector Space**:
  Step 1: Identify the set V of objects that will become vectors.
  Step 2: Identify the addition and scalar multiplication operations on V.
  Step 3: Verify Axioms 1 and 6; that is, adding two vectors in V produces a vector in V, and multiplying a vector in V by a scalar also produces a vector in V. Axion 1 is **called closure under addition** and Axiom 6 is called **closure under multiplication**.
  Step 4: Confirm that Axioms 2, 3, 4, 5, 7, 8, 9, and 10 hold.

Theorem. Let V be a vector space, $\mathbf{u}$ a vector in V, and k a scalar; then:
(*a*) $0\mathbf{u} = \mathbf{0}$
(*b*) $k\mathbf{0} = \mathbf{0}$
(*c*) $(-1)\mathbf{u} = -\mathbf{u}$
(*d*) If $k\mathbf{u} = \mathbf{0}$ then $k = 0$ or $\mathbf{u} = \mathbf{0}$.

Section 4.2: Subspaces

Definition. A subset W of a vector space V is called a **subspace** of V if W itself is a vector space under the addition and scalar multiplication defined on V.

Theorem. If W is a set of one or more vectors from a vector space V, then W is a subspace of V if and only if the following conditions hold:
(*a*) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u} + \mathbf{v}$ is in W.
(*b*) If k is any scalar and $\mathbf{u}$ is any vector in W, then $k\mathbf{u}$ is in W.

Theorem. If $W_1, W_2, \dots W_r$ are subspaces of a vector space V, then the intersection of these subspaces is also a subspace of V.

Definition. If $\mathbf{w}$ is a vector in a vector space V, then $\mathbf{w}$ is said to be a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ if it can be expressed in the form $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \dots + k_r\mathbf{v}_r$ where $k_1, k_2, \dots, k_r$ are scalars. These scalars are called the **coefficients** of the linear combination.

Theorem. If $S = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space V, then the set W of all possible linear combinations of the vectors in S is a subspace of V. This set W is the "smallest" subspace of V that contains all of the vectors in S in the sense that any other subspace of V that contains those vectors contains W.

Definition. The subspace of a vector space V that is formed from all possible linear combinations of the vectors in a nonempty set S is called the **span of *S***, and we say that the vectors in S **span** that subspace. If $S = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_r\}$, then we denote the span of S by $\text{span}\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_r\}$ or span(S).

Theorem. The solution set of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ in n unknowns is a subspace of $\boldsymbol{R}^n$.

Theorem. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ and $S' = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k\}$ are nonempty sets of vectors in a vector space V, then $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\} = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k\}$ if and only if each vector in S is a linear combination of those in S' and each vector in S' is a linear combination of those in S.

Section 4.3: Linear Independence

Definition. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is a nonempty set of vectors, then the vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \dots + k_r\mathbf{v}_r = \mathbf{0}$ has at least one solution, namely $k_1 = 0, k_2 = 0, \dots, k_r = 0$. If this is the only solution, then S is called a **linearly independent** set. If there are other solutions, then S is called a **linearly dependent** set.

Theorem. A set S with two or more vectors is:
(*a*) Linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other

vectors in S.
(b) Linearly independent if and only if no vector in S is expressible as a linear combination of the other vectors in S.

Theorem. A finite set that contains **0** is linearly dependent. A set with exactly one vector is linearly independent if and only if that vector is **0**. A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

Theorem. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ be a set of vectors in $\boldsymbol{R}^n$. If $r > n$, then S is linearly dependent.

Definition. If $\mathbf{f}_1 = f_1(x)$, $\mathbf{f}_2 = f_2(x)$, … , $\mathbf{f}_n = f_n(x)$ are functions that are $n - 1$ times differentiable on the interval $(-\infty, \infty)$, then the determinant $W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{vmatrix}$ is called the **Wronskian** of $f_1, f_2, \dots, f_n$.

Theorem. If the functions $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n$ have $n - 1$ continuous derivatives on the interval $(-\infty, \infty)$, and if the Wronskian of these functions is not identically zero on $(-\infty, \infty)$, then they form a linearly independent set of vectors in $C^{(n-1)}(-\infty, \infty)$.

Section 4.4: Coordinates and Basis

Definition. If V is any vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a set of vectors in V, then S is called a **basis** for V if S is linearly independent and S spans V.

- A nonzero vector space V is called **finite-dimensional** if it contains a finite set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ that forms a basis. If no such set exists, V is called **infinite-dimensional**. In addition, we shall regard the zero vector space to be finite dimensional.

Theorem. **Uniqueness of Basis Representation**: If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n$ in exactly one way.

Definition. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V, and $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n$ is the expression for a vector $\mathbf{v}$ in terms of the basis S, then the scalars $c_1, c_2, \dots, c_n$ are called the **coordinates** of $\mathbf{v}$ relative to the basis S. The vector $(c_1, c_2, \dots, c_n)$ in R^n constructed from these coordinates is called the **coordinate vector of v relative to *S***; it is denoted by $(\mathbf{v})_S = (c_1, c_2, \dots, c_n)$.

Section 4.5: Dimension

Theorem. All bases for a finite-dimensional vector space have the same number of vectors.

Theorem. Let V be a finite-dimensional vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be any basis. If a set has more than n vectors, then it is linearly dependent. If a set has fewer than n vectors, then it does not span V.

Definition. The **dimension** of a finite-dimensional vector space V, denoted by $\dim(V)$, is defined to be the number of vectors in a basis for V. In addition, the zero vector space is defined to have dimension zero.

Theorem. **Plus/Minus Theorem**: Let S be a nonempty set of vectors in a vector space V.
(a) If S is a linearly independent set, and if $\mathbf{v}$ is a vector in V that is outside of span(S), then the set $S \cup \{\mathbf{v}\}$ that results by inserting $\mathbf{v}$ into S is still linearly independent.
(b) If $\mathbf{v}$ is a vector in S that is expressible as a linear combination of other vectors in S, and if $S - \{\mathbf{v}\}$ denotes the set obtained by removing $\mathbf{v}$ from S, then S and $S - \{\mathbf{v}\}$ span the same space; that is, $\text{span}(S) = \text{span}(S - \{\mathbf{v}\})$.

Theorem. Let V be an n-dimensional vector space, and let S be a set in V with exactly n vectors. Then S is a basis for V if either S spans V or S is linearly independent.

Theorem. Let S be a finite set of vectors in a finite-dimensional vector space V.
(a) If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.
(b) If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

Theorem. If W is a subspace of a finite-dimensional vector space V, then:
(a) W is finite-dimensional
(b) $\dim(W) \leq \dim(V)$
(c) $W = V$ if and only if $\dim(W) = \dim(V)$

Section 4.6: Change of Basis

- **The Change-of-Basis Problem**: If $\mathbf{v}$ is a vector in a finite-dimensional vector space V, and if we change the basis for V from a basis B to a basis B', how are the coordinate vectors $[\mathbf{v}]_B$ and $[\mathbf{v}]_{B'}$ related?
- **Solution of the Change-of-Basis Problem**: If we change the basis for a vector space V from an old basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ to a new basis $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \dots, \mathbf{u}'_n\}$, then for each vector $\mathbf{v}$ in V, the old coordinate vector $[\mathbf{v}]_B$ is related to the new coordinate vector $[\mathbf{v}]_{B'}$ by the equation $[\mathbf{v}]_B = P[\mathbf{v}]_{B'}$, where the columns of P are the coordinate vectors of the new basis vectors relative to the old basis; that is, the column vectors of P are $[\mathbf{u}'_1]_B, [\mathbf{u}'_2]_B, \dots, [\mathbf{u}'_n]_B$.
- $P_{B' \to B} = [\, [\mathbf{u}'_1]_B \mid [\mathbf{u}'_2]_B \mid \dots \mid [\mathbf{u}'_n]_B \,]$
- $P_{B \to B'} = [\, [\mathbf{u}_1]_{B'} \mid [\mathbf{u}_2]_{B'} \mid \dots \mid [\mathbf{u}_n]_{B'} \,]$

- The columns of the transition matrix from an old basis to a new basis are the coordinate vectors of the old basis relative to the new basis.
- $[\mathbf{v}]_{B'} = P_{B' \to B}\,[\mathbf{v}]_{B'}$
- $[\mathbf{v}]_{B'} = P_{B \to B'}\,[\mathbf{v}]_{B'}$

Theorem. If P is the transition matrix from a basis B' to a basis B for a finite-dimensional vector space V, then P is invertible and P^{-1} is the transition matrix from B to B'.

- **A Procedure for Computing $P_{B \to B'}$:**
  Step 1: Form the matrix $[\,B' \mid B\,]$.
  Step 2: Use elementary row operations to reduce the matrix in Step 1 to reduced row echelon form.
  Step 3: The resulting matrix will be $[\,I \mid P_{B \to B'}\,]$.
  Step 4: Extract the matrix $P_{B \to B'}$ from the right side of the matrix in Step 3.
- [new basis | old basis] → (row operations) [I | transition from old to new]

Theorem. Let $B' = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ be any basis for the vector space R^n and let $S = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be the standard basis for R^n. If the vectors in these bases are written in column form, then $P_{B' \to S} = [\mathbf{u}_1 \mid \mathbf{u}_2 \mid \dots \mid \mathbf{u}_n]$.

Section 4.7: Row Space, Column Space, and Null Space

Definition. For an $m \times n$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ the vectors

$$\mathbf{r}_1 = [a_{11}\ \ a_{12}\ \ \cdots\ a_{1n}]$$
$$\mathbf{r}_2 = [a_{21}\ \ a_{22}\ \ \cdots\ a_{2n}]$$
$$\vdots \qquad \vdots$$
$$\mathbf{r}_m = [a_{m1}\ \ a_{m2}\ \cdots\ a_{mn}]$$

in $\boldsymbol{R}^n$ formed from the rows of A are called the **row vectors** of A, and the vectors

$$\mathbf{c}_1 = [a_{11}\ \ a_{21}\ \ \cdots\ \ a_{m1}]^t$$
$$\mathbf{c}_2 = [a_{12}\ \ a_{22}\ \ \cdots\ \ a_{m2}]^t$$
$$\vdots \qquad \vdots$$
$$\mathbf{c}_n = [a_{1n}\ \ a_{2n}\ \ \cdots\ \ a_{mn}]^t$$

in $\boldsymbol{R}^m$ formed from the columns of A are called the **column vectors** of A.

Definition. If A is a $m \times n$ matrix, then the subspace of $\boldsymbol{R}^n$ spanned by the row vectors of A is called the **row space** of A, and the subspace of $\boldsymbol{R}^m$ spanned by the column vectors of A is called the **column space** of A. The solution space of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$, which is a subspace of $\boldsymbol{R}^n$, is called the **null space** of A.

- **Question 1**: What relationships exist among the solutions of a linear system $A\mathbf{x} = \mathbf{b}$ and the row space, column space, and null space of the coefficient matrix A?
- **Question 2**: What relationships exist among the row space, column space, and null space of a matrix?

Theorem. A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

Theorem. If $\mathbf{x}_0$ denotes any single solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$, and if $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ form a basis for the nullspace of A—that is, the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$—then every solution of $A\mathbf{x} = \mathbf{b}$ can be expressed in the form $\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k$. Conversely, for all choices of scalars $c_1, c_2, \dots, c_k$, the vector $\mathbf{x}$ in this formula is a solution of $A\mathbf{x} = \mathbf{b}$.

- The general solution of a consistent linear system can be expressed as the sum of a particular solution of that system and the general solution of the corresponding homogeneous system.

Theorem. Elementary row operations do not change the null space of a matrix.

Theorem. Elementary row operations do not change the row space of a matrix.

Theorem. If a matrix R is in row-echelon form, then the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of R, and the column vectors with the leading 1's of the row vectors form a basis for the column space of R.

Theorem. If A and B are row equivalent matrices, then:
(a) A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.
(b) A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.

- **Problem**: Given a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ in R^n, find a subset of these vectors that forms a basis for span(S), and express those vectors that are not in that basis as a linear combination of the basis vectors.
- **Basis for Span(S)**:
  Step 1: Form the matrix A having vectors in $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ as column vectors.

Step 2: Reduce the matrix A to reduced row echelon form R.
Step 3: Denote the column vectors of R by $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k\}$.
Step 4: Identify the columns of R that contain the leading 1's. The corresponding column vectors of A form a basis for span(S).
This completes the first part of the problem.
Step 5: Obtain a set of dependency equations by expressing each column vector of R that does not contain a leading 1 as a linear combination of preceding column vectors that do contain leading 1's.
Step 6: Replace the column vectors of R that appear in the dependency equations by the corresponding column vectors of A.
This completes the second part of the problem.

Section 4.8: Rank, Nullity, and the Fundamental Matrix Spaces

Theorem. The row space and column space of a matrix A have the same dimension.

Definition. The common dimension of the row space and column space of a matrix A is called the **rank** of A, and is denoted by rank(A); the dimension of the null space of A is called the **nullity** of A and is denoted by nullity(A).

Theorem. **Dimension Theorem for Matrices**: If A is a matrix with n columns, then rank(A) + nullity(A) = n.

Theorem. If A is an $m \times n$ matrix, then: rank(A) = the number of leading variables in the solution of $A\mathbf{x} = \mathbf{0}$ and nullity(A) = the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$.

Theorem. If $A\mathbf{x} = \mathbf{b}$ is a consistent linear system of m equations in n unknowns, and if A has rank r, then the general solution of the system contains $n - r$ parameters.

Theorem. Let A be an $m \times n$ matrix.
Overdetermined Case: If $m > n$, then the linear system $A\mathbf{x} = \mathbf{b}$ is inconsistent for at least one vector $\mathbf{b}$ in $\boldsymbol{R}^n$.
Underdetermined Case: If $m < n$, then for each vector $\mathbf{b}$ in $\boldsymbol{R}^m$ the linear system $A\mathbf{x} = \mathbf{b}$ is either inconsistent or has infinitely many solutions.

Theorem. If A is any matrix, then rank(A) = rank(A^T).

- rank(A) + nullity(A^T) = m
- If rank(A) = r, then:
  dim[row(A)] = r
  dim[col(A)] = r
  dim[null(A)] = $n - r$
  dim[null(A^T)] = $m - r$

Definition. If W is a subspace of $\boldsymbol{R}^n$, then the set of all vectors in $\boldsymbol{R}^n$ that are orthogonal to every vector in W is called the **orthogonal complement** of W and is denoted by the symbol $W^\perp$.

Theorem. If W is a subspace of $\boldsymbol{R}^n$, then:
(a) $W^\perp$ is a subspace of $\boldsymbol{R}^n$.
(b) The only vector common to W and $W^\perp$ is $\mathbf{0}$.
(c) The orthogonal complement of $W^\perp$ is W.

Theorem. If A is a $m \times n$ matrix, then:
(a) The null space of A and the row space of A are orthogonal complements in $\boldsymbol{R}^n$.
(b) The null space of A^T and the column space of A are orthogonal complements in $\boldsymbol{R}^m$.

Theorem. **Equivalent Statements**: If A is an $n \times n$ matrix, then the following statements are equivalent:
(a) A is invertible
(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
(c) the reduced row-echelon form of A is I_n
(d) A is expressible as a product of elementary matrices
(e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$
(f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$
(g) det(A) $\neq 0$
(h) The column vectors of A are linearly independent.
(i) The row vectors of A are linearly independent.
(j) The column vectors of A span $\boldsymbol{R}^n$.
(k) The row vectors of A span $\boldsymbol{R}^n$.
(l) The column vectors of A form a basis for $\boldsymbol{R}^n$.
(m) The row vectors of A form a basis for $\boldsymbol{R}^n$.
(n) A has rank n
(o) A has nullity 0
(p) The orthogonal complement of the null space of A is $\boldsymbol{R}^n$.
(q) The orthogonal complement of the row space of A is $\mathbf{0}$.

Section 4.9: Matrix Transformations from $\boldsymbol{R}^n$ to $\boldsymbol{R}^m$

Definition. If V and W are vector spaces, and if f is a function with domain V and codomain W, then we say that f is a **transformation** from V to W or that f **maps** V to W, which we denote by writing $f: V \to W$. In the special case where $V = W$, the transformation is called an **operator** on V.

Theorem. For every matrix A the matrix transformation $T_A: \boldsymbol{R}^n \to \boldsymbol{R}^m$ has the following properties for all vectors **u** and **v** in $\boldsymbol{R}^n$ and for every scalar k:
(a) $T_A(\mathbf{0}) = \mathbf{0}$
(b) $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$ (Homogeneity property)
(c) $T_A(\mathbf{u} + \mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$ (Additivity property)
(d) $T_A(\mathbf{u} - \mathbf{v}) = T_A(\mathbf{u}) - T_A(\mathbf{v})$

- $T_A(k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \ldots + k_r\mathbf{u}_r) = k_1T_A(\mathbf{u}_1) + k_2T_A(\mathbf{u}_2) + \ldots + k_rT_A(\mathbf{u}_r)$

Theorem. If $T_A: \boldsymbol{R}^n \to \boldsymbol{R}^m$ and $T_B: \boldsymbol{R}^n \to \boldsymbol{R}^m$ are matrix transformations, and if $T_A(\mathbf{x}) = T_B(\mathbf{x})$ for every vector **x** in $\boldsymbol{R}^n$, then $A = B$.

- $A = [T_A(\mathbf{e}_1) \mid T_A(\mathbf{e}_2) \mid \ldots \mid T_A(\mathbf{e}_n)]$.
- **Finding the Standard Matrix for a Matrix Transformation**:
  Step 1: Find the images of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ for $\boldsymbol{R}^n$ in column form.
  Step 2: Construct the matrix that has the images obtained in Step 1 as its successive columns. This matrix is the standard matrix for the transformation.

Section 4.10: Properties of Matrix Transformations

Definition. A matrix transformation $T: \boldsymbol{R}^n \to \boldsymbol{R}^m$ is said to be **one-to-one** if T maps distinct vectors (points) in $\boldsymbol{R}^n$ to distinct vectors (points) in R^m.

Theorem. If A is an $n \times n$ matrix and $T_A: \boldsymbol{R}^n \to \boldsymbol{R}^n$ is the corresponding matrix operator, then the following statements are equivalent:
(a) A is invertible
(b) The range of T_A is R^n
(c) T_A is one-to-one.

- $T_{A^\wedge\{-1\}} = T^{-1}{}_{\mathrm{A}}$
- $[T^{-1}] = [T]^{-1}$
- **Question**: Are there algebraic properties for a transformation $T: \boldsymbol{R}^n \to \boldsymbol{R}^m$ that can be used to determine whether T is a matrix transformation?

Theorem. $T: \boldsymbol{R}^n \to \boldsymbol{R}^m$ is a matrix transformation if and only if the following relationships hold for all vectors **u** and **v** in $\boldsymbol{R}^n$ and for every scalar k:
(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (Additivity property)
(ii) $T(k\mathbf{u}) = kT(\mathbf{u})$ (Homogeneity property)

Theorem. Every linear transformation from $\boldsymbol{R}^n$ to $\boldsymbol{R}^m$ is a matrix transformation, and, conversely, every matrix transformation from $\boldsymbol{R}^n$ to $\boldsymbol{R}^m$ is a linear transformation.

Theorem. **Equivalent Statements**: If A is an $n \times n$ matrix, then the following statements are equivalent:
(a) A is invertible
(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
(c) the reduced row-echelon form of A is I_n
(d) A is expressible as a product of elementary matrices
(e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix **b**
(f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix **b**
(g) $\det(A) \neq 0$
(h) The column vectors of A are linearly independent.
(i) The row vectors of A are linearly independent.
(j) The column vectors of A span $\boldsymbol{R}^n$.
(k) The row vectors of A span $\boldsymbol{R}^n$.
(l) The column vectors of A form a basis for $\boldsymbol{R}^n$.
(m) The row vectors of A form a basis for $\boldsymbol{R}^n$.
(n) A has rank n
(o) A has nullity 0
(p) The orthogonal complement of the null space of A is $\boldsymbol{R}^n$.
(q) The orthogonal complement of the row space of A is $\mathbf{0}$.
(r) The range of T_A is $\boldsymbol{R}^n$
(s) T_A is one-to-one.