Notes from Elementary Linear Algebra, 10th Edition, by Anton and Rorres

Chapter 5: Eigenvalues and Eigenvectors

Section 5.1: Eigenvalues and Eigenvectors

Definition. If A is an $n \times n$ matrix, then a nonzero vector $\mathbf{x}$ in $\boldsymbol{R}^n$ is called an **eigenvector** of A (or of the matrix operator T_A) if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$; that is, if $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. The scalar λ is called an **eigenvalue** of A, and $\mathbf{x}$ is said to be an **eigenvector of A corresponding to λ**.

Theorem. If A is an $n \times n$ matrix, then λ is an eigenvalue of A if and only if it satisfies the equation $\det(\lambda I - A) = 0$. This is called the **characteristic equation** of A.

Theorem. If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are the entries on the main diagonal of A.

Theorem. If A is an $n \times n$ matrix and λ is a real number, then the following are equivalent:
(*a*) λ is an eigenvalue of A
(*a*) the system of equations $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nontrivial solutions
(*a*) there is a nonzero vector $\mathbf{x}$ in $\boldsymbol{R}^n$ such that $A\mathbf{x} = \lambda\mathbf{x}$
(*a*) λ is a solution of the characteristic equation $\det(\lambda I - A) = 0$

Theorem. If k is a positive integer, λ is an eigenvalue of a matrix A, and $\mathbf{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\mathbf{x}$ is a corresponding eigenvector.

Theorem. A square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

Theorem. **Equivalent Statements**: If A is an $n \times n$ matrix, then the following statements are equivalent:
(*a*) A is invertible
(*b*) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
(*c*) the reduced row-echelon form of A is I_n
(*d*) A is expressible as a product of elementary matrices
(*e*) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$
(*f*) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$
(*g*) $\det(A) \neq 0$
(*h*) The column vectors of A are linearly independent.
(*i*) The row vectors of A are linearly independent.
(*j*) The column vectors of A span $\boldsymbol{R}^n$.
(*k*) The row vectors of A span $\boldsymbol{R}^n$.
(*l*) The column vectors of A form a basis for $\boldsymbol{R}^n$.
(*m*) The row vectors of A form a basis for $\boldsymbol{R}^n$.
(*n*) A has rank n
(*o*) A has nullity 0
(*p*) The orthogonal complement of the null space of A is $\boldsymbol{R}^n$.
(*q*) The orthogonal complement of the row space of A is $\mathbf{0}$.
(*r*) The range of T_A is $\boldsymbol{R}^n$
(*s*) T_A is one-to-one.
(*t*) $\lambda = 0$ is not an eigenvalue of A.

Section 5.2: Diagonalization

• **Problem 1**: Given an $n \times n$ matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is diagonal?
Problem 2: Given an $n \times n$ matrix A, does A have n linearly independent eigenvectors?

Definition. If A and B are square matrices, then we say that ***B* is similar to *A*** if there is an invertible matrix P such that $B = P^{-1}AP$.

Definition. A square matrix A is said to be **diagonalizable** if it is similar to some diagonal matrix; that is, if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal. In this case the matrix P is said to **diagonalize** A.

Theorem. If A is a $n \times n$ matrix, then the following are equivalent:
(*a*) A is diagonalizable;
(*b*) A has n linearly independent eigenvectors.

• **Procedure for Diagonalizing a Matrix**:
Step 1: Confirm that the matrix is actually diagonalizable by finding n linearly independent eigenvectors. One way to do this is by finding a basis for each eigenspace and merging these basis vectors inso a single set S. If this set has fewer than n vectors, then the matrix is not diagonalizable.
Step 2: Form the matrix $P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \ldots \quad \mathbf{p}_n]$ that has the vectors in S as its column vectors.
Step 3: The matrix $P^{-1}AP$ will be diagonal and have the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ corresponding to the eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ as its successive diagonal entries.

Theorem. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are eigenvectors of a matrix A corresponding to distinct eigenvalues, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a linearly independent set.

Theorem. If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Theorem. If λ is an eigenvalue of a square matrix A and $\mathbf{x}$ is a corresponding eigenvector, and if k is any positive integer, then λ^k is an eigenvalue of A^k and $\mathbf{x}$ is a corresponding eigenvector.

Theorem. **Geometric and Algebraic Multiplicity**:
If A is a square matrix, then:
(*a*) For every eigenvalue of A, the geometric multiplicity is less than or equal to the algebraic multiplicity.
(*b*) A is diagonalizable if and only if, for every eigenvalue, the geometric multiplicity is equal to the algebraic multiplicity.