CHAPTER 9: NUMERICAL METHODS

Section 9.1: *LU*-Decompositions

- **The Method of *LU*-Decomposition.**
  Step 1. Rewrite the system $A\mathbf{x} = \mathbf{b}$ as $LU\mathbf{x} = \mathbf{b}$ (*).
  Step 2. Define a new $n \times 1$ matrix $\mathbf{y}$ by $U\mathbf{x} = \mathbf{y}$ (**).
  Step 3. Use (**) to rewrite (*) as $L\mathbf{y} = \mathbf{b}$ and solve this system for $\mathbf{y}$ (**forward substitution**).
  Step 4. Substitute $\mathbf{y}$ in (**) and solve for $\mathbf{x}$ (**back substitution**).

Definition. A factorization of a square matrix A as $A = LU$, where L is lower triangular and U is upper triangular, is called an *LU*-Decomposition (or *LU*-Factorization) of A.

Theorem. If A is a square matrix that can be reduced to a row echelon form U by Gaussian elimination without row interchanges, then A can be factored as $A = LU$, where L is a lower triangular matrix.

- **Procedure for Constructing a *LU*-Decomposition.**
  Step 1. Reduce A to a row echelon form U by Gaussian elimination without row interchanges, keeping track of the multipliers used to introduce the leading 1's and the multipliers used to introduce zeros below the leading 1's.
  Step 2. In each position along the main diagonal of L, place the reciprocal of the multiplier that introduced the leading 1 in that position in U.
  Step 3. In each position below the main diagonal of L, place the negative of the multiplier used to introduce the zero in that position in U.
  Step 4. Form the decomposition $A = LU$.
- In an ***LDU*-decomposition** of A, we have $A = LDU$, where L is a lower triangular matrix with 1's on the diagonal, D is a diagonal matrix, and U is an upper triangular matrix with 1's on the diagonal.
- In a ***PLU*-decomposition** of A, we have $A = PLU$, where P is a permutation matrix (needed for row interchanges), L is a lower triangular matrix, and U is an upper triangular matrix.