Topological Recursion for Generalized *bc*-Motzkin Numbers and Some Related Results

By

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Abstract

In this dissertation, I show that a higher genus generalization of the bc-Motzkin numbers satisfies a topological recursion. These bc-Motzkin numbers are themselves a generalization of Catalan numbers, and, remarkably, the topological recursion satisfied by these generalized bc-Motzkin numbers is identical, up to a change of variable, to the topological recursion which had previously been proved for generalized Catalan numbers by Olivia Dumitrescu and Motohico Mulase. This topological recursion is an example of the Eynard-Orantin topological recursion, and it is one of multiple topological recursion formulas which have come from counting problems in combinatorics.

In the process of obtaining this topological recursion, I show that my higher genus generalization of bc-Motzkin numbers, which can be defined via an analogy with coloring graphs with v vertices on the genus g surface, satisfies a recursive formula, and, further, that the discrete Laplace transform of these numbers satisfies a differential recursion. It has been observed that the discrete Laplace transform of edge contraction operations in many graph counting problems corresponds to a topological recursion, and my example of the bc-Motzkin numbers is no exception.

Additionally, this definition of a higher genus generalization of bc-Motzkin numbers also leads to some identities and closed-form expressions for generalized Catalan numbers, in the case of small genus g and small number of vertices v. These results can be proved using the generating functions for these generalized bc-Motzkin numbers, and in this dissertation I show that these generating functions satisfy a recursive formula. Using the computer algebra system Mathematica to aid in computations, I obtain explicit generating functions for the generalized bc-Motzkin numbers for some cases of small (g, v). Then, based on these examples, I form a conjecture regarding the general form of the generating functions of these generalized bc-Motzkin numbers.

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CHAPTER 1

Introduction

Topological recursion was first introduced by Chekhov, Eynard, and Orantin in their 2006 paper [**4**] (see also [**12**] for a more precise initial definition of topological recursion), but instances of such formulas started appearing earlier. In their paper, the recursion structure was used to calculate multi-resolvent correlation functions of random matrices. However, a topological recursion-like formula had also appeared in a geometry problem before this. Mirzakhani's recursion formula for the Weil-Petersson volume of the moduli space of genus g bordered Riemann surfaces with n geodesic boundaries, which she proved in her thesis in 2004 (see [**17**] and [**18**]) was shown by Eynard and Orantin to also satisfy a topological recursion after applying the Laplace transform (see [**13**]).

Since then, many examples of topological recursion formulas have been discovered. As discussed in the preface to [**16**], such formulas have appeared in topological quantum field theory and cohomological field theory, intersection numbers of cohomology classes on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves, Gromov-Witten theory, A-polynomials and polynomial invariants of hyperbolic knots, WKB analysis of classical ordinary differential equations, enumeration of Hurwitz numbers, Witten-Kontsevich intersection numbers, and moduli spaces of Higgs bundles (see, for example, [**2**], [**5**], [**8**], [**11**], [**22**], [**23**]).

Further, topological recursion formulas have also come from various different counting problems in combinatorics, such as counting Grothendieck's dessins d'enfants, and more general counting problems related to graphs drawn on surfaces (see [**3**], [**9**], [**6**], [**19**], [**21**]). One example of this is a generalization of the Catalan numbers, which is described in detail in [**7**] and is also discussed briefly in Chapter 2 of this dissertation. This topological recursion for the generalized Catalan numbers is what prompted this author to look for a topological recursion based on bc-Motzkin numbers, which can be viewed as a generalization of the Catalan numbers. It has been observed in [**9**] that the discrete Laplace transform of edge contraction operations in many graph counting problems corresponds to a topological recursion, and this can be seen in the examples mentioned above, as well as for the bc-Motzkin numbers, which will be described later in this dissertation. More generally, the Laplace transform can be identified as a mirror symmetry between the A-model side of enumerative geometry and the B-model side of holomorphic geometry.

Now, what, precisely, is topological recursion? We will not discuss the original definition of topological recursion in detail in this dissertation. The reader is referred to the excellent survey papers [**10**] by Eynard and [**1**] by Borot to learn more about topological recursion. We will, however, give a short definition of topological recursion, tailored to the specific case of our combinatorial examples. This combinatorial definition of topological recursion, which is a special case of the more general definition, is as given in [**9**].

Definition 1.0.1. *Let t be a choice of coordinate on the one-dimensional complex projective space $\mathbb{P}^1$. The spectral curve of genus 0 is the data (Σ, π) consisting of an open Riemann surface Σ of genus 0 realized as an open subset of $\mathbb{P}^1$, which has a simply ramified holomorphic map*

$$\pi : \Sigma \ni t \mapsto \pi(t) = x \in \mathbb{P}^1$$

such that its differential dx has only simple zeros. Let $R = \{p_1, p_2, \ldots, p_r\} \subset \Sigma$ denote the ramification points, and let $U = \sqcup_{j=1}^{r} U_j$ denote the disjoint union of small neighborhoods U_j around each p_j, such that $\pi : U_j \to \pi(U_j) \subset \mathbb{P}^1$ is a double-sheeted covering ramified only at p_j. We denote by $\bar{t}$ the local Galois conjugate of $t \in U_j$ (i.e., interchanging the two sheets). The canonical sheaf of Σ (i.e., the sheaf of holomorphic 1-forms on Σ) is denoted by $\mathcal{K}$. Because of our choice of coordinate t, we have a preferred basis dt for $\mathcal{K}$ and $\partial/\partial t$ for $\mathcal{K}^{-1}$. The meromorphic differential forms $W_{g,v}(t_1, t_2, \ldots, t_v)$ on Σ^v are said to satisfy the Eynard-Orantin topological recursion if the following conditions are satisfied:

(1) $W_{0,1}(t) \in H^0(\Sigma, \mathcal{K})$.

(2) We have

$$W_{0,2}(t_1, t_2) = \frac{dt_1 \cdot dt_2}{(t_1 - t_2)^2} - \pi^* \frac{dx_1 \cdot dx_2}{(x_1 - x_2)^2} \in H^0(\Sigma \times \Sigma, \mathcal{K}^{\otimes 2}(2\Delta)),$$

where Δ is the diagonal of $\Sigma \times \Sigma$.

(3) The recursion kernel $K_j(t,t_1) \in H^0(U_j \times \Sigma, (\mathcal{K}_{U_j}^{-1} \otimes \mathcal{K})(\Delta))$ *for* $t \in U_j$ *and* $t_1 \in \Sigma$ *is defined by*

$$K_j(t,t_1) = \frac{1}{2}\frac{\int_t^{\bar{t}} W_{0,2}(\cdot,t_1)}{W_{0,1}(\bar{t}) - W_{0,1}(t)}.$$

The kernel is an algebraic operator that multiplies dt_1 *while contracting* dt.

(4) The general differential forms $W_{g,n}(t_1,t_2,\ldots,t_v) \in H^0(\Sigma^v, \mathcal{K}(*R)^{\otimes v})$ *are meromorphic symmetric differential forms with poles only at the ramification points* R *for* $2g-2+v>0$, *and are given by the recursion formula*

$$\begin{aligned} W_{g,v}(t_1,t_2,\ldots,t_v) = \frac{1}{2\pi i}\sum_{j=1}^r \oint_{U_j} K_j(t,t_1)\Big[& W_{g-1,v+1}(t,\bar{t},t_2,\ldots,t_v) \\ & + \sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{no } (0,1) \text{ terms}} W_{g_1,|I|+1}(t,t_I)W_{g_2,|J|+1}(\bar{t},t_J)\Big]. \end{aligned}$$

Here, the integration is taken with respect to $t_j \in U_j$ *along a positively oriented simple closed loop around* p_j, *and* $t_I = (t_i)_{i\in I}$ *for a subset* $I \subset \{1,2,\ldots,v\}$.

(5) The differential form $W_{1,1}(t_1)$ *requires a separate treatment since* $W_{0,2}(t_1,t_2)$ *is regular at the ramification points but has poles elsewhere.*

$$\begin{aligned} W_{1,1}(t_1) &= \frac{1}{2\pi i}\sum_{j=1}^r \oint_{U_j} K_j(t,t_1)\left[W_{0,2}(u,v) + \pi^*\frac{dx(u)\cdot dx(v)}{(x(u)-x(v))^2}\right]\Bigg|_{u=t,v=\bar{t}} \\ &= \frac{1}{2\pi i}\sum_{j=1}^r \oint_{U_j} K_j(t,t_1)\left[\frac{dt\cdot d\bar{t}}{(t-\bar{t})^2}\right]. \end{aligned}$$

Let $y : \Sigma \to \mathbb{C}$ *be a holomorphic function defined by the equation*

$$W_{0,1}(t) = y(t)\ dx(t).$$

Equivalently, we can define the function by contraction $y = i_X W_{0,1}$, *where* X *is the vector field on* Σ *dual to* $dx(t)$ *with respect to the coordinate* t. *Then, we have an embedding*

$$\Sigma \ni t \mapsto (x(t),y(t)) \in \mathbb{C}^2.$$

(6) If the spectral curve has at most two branches, then we choose a preferred coordinate t with the branch points located at $t = \infty$ and $t = 0$. This results in differentials $W_{g,v}$ that are Laurent polynomials in t and serves to simplify many of the residue calculations.

1.1. Main Results

For the main result of this dissertation, we show that a higher genus generalization of the *bc*-Motzkin numbers, which were originally defined by Zhi-Wei Sun in 2014 as a generalization of the Motzkin numbers (see [**24**]), satisfies a topological recursion. It is known (see [**7**]) that the Catalan numbers can be given a higher genus analogue, and this generalization satisfies a topological recursion. Thus, it is a natural question to ask whether the *bc*-Motzkin numbers, which are defined in terms of Catalan numbers, also satisfy a topological recursion.

We have proved that this is indeed true, as given in the following theorem.

Theorem 1.1.1. *Define symmetric v-linear differential forms on $(\mathbb{P}^1)^v$ for $2g-2+v>0$, called Eynard-Orantin differential forms, by*

$$W_{g,v}^{\widetilde{M}(b,c)}(t_1, t_2, \ldots, t_v) = d_{t_1} \cdots d_{t_v} F_{g,v}^{\widetilde{M}(b,c)}(t_1, t_2, \ldots, t_v),$$

and for $(g, v) = (0, 2)$ by

$$W_{0,2}^{\widetilde{M}(b,c)}(t_1, t_2) = \frac{dt_1\, dt_2}{(t_1 - t_2)^2}.$$

Here, $F_{g,v}^{\widetilde{M}(b,c)}(t_1, t_2, \ldots, t_v)$ is the discrete Laplace transform of the generalized bc-Motzkin numbers.

Then, these differential forms satisfy the following integral recursion formula:

$$\begin{aligned}
W_{g,v}^{\widetilde{M}(b,c)}(t_1, t_2, \ldots, t_v) = -\frac{1}{64}\frac{1}{2\pi i}\int_\gamma \left(\frac{1}{t+t_1} + \frac{1}{t-t_1}\right)\frac{(t^2-1)^3}{t^2}\,\frac{1}{dt}\,dt_1 \\
\cdot \Bigg[\sum_{j=2}^{v} \Bigg(W_{0,2}^{\widetilde{M}(b,c)}(t, t_j) W_{g,v-1}^{\widetilde{M}(b,c)}(-t, t_2, \ldots, \widehat{t_j}, \ldots, t_v) \\
+ W_{0,2}^{\widetilde{M}(b,c)}(-t, t_j) W_{g,v-1}^{\widetilde{M}(b,c)}(t, t_2, \ldots, \widehat{t_j}, \ldots, t_v) \Bigg) \\
+ W_{g-1,v+1}^{\widetilde{M}(b,c)}(t, -t, t_2, \ldots, t_v) \\
+ \sum_{g_1+g_2=g,\ I \sqcup J = \{2,\ldots,v\},\ stable} W_{g_1,|I|+1}^{\widetilde{M}(b,c)}(t, t_I) W_{g_2,|J|+1}^{\widetilde{M}(b,c)}(-t, t_J) \Bigg]
\end{aligned}$$

where the curve γ *is as given in Figure 1.1. Here,* $t_I = (t_{i_1}, t_{i_2}, \ldots, t_{i_{|I|}})$ *for an index set* I*, and the notation* $\widehat{t_j}$ *means that we delete* t_j *from this sequence. The last sum in the above formula is for all partitions of* g *and all set partitions of* $\{2, \ldots, v\}$*, and the "stable" summation means* $2g_1 + |I| - 1 > 0$ *and* $2g_2 + |J| - 1 > 0$.

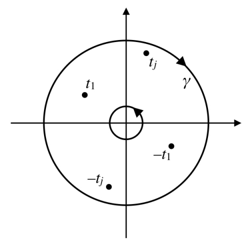

FIGURE 1.1. The curve γ.

Our topological recursion for generalized bc-Motzkin numbers gives yet another example of a topological recursion formula that comes from a combinatorial counting problem. What is remarkable is that this topological recursion is identical to the topological recursion for generalized Catalan numbers given in [**7**] (also see Chapter 2 of this dissertation), with the only difference being that we have made a change of variable depending on b and c. All this will be described in more detail in subsequent chapters of this dissertation.

In the Catalan case, an analogy with counting graphs on the genus g surface was used to generalize Catalan numbers to the case of higher genus and greater number of vertices. The kth Catalan number is given by

$$C_k = \frac{1}{k+1}\binom{2k}{k}.$$

The first few Catalan numbers for $n = 0, 1, 2, 3, \ldots$ are

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, \ldots$$

Catalan numbers appear in numerous different counting problems. For example, the kth Catalan number (roughly) counts the number of graphs drawn on a sphere with one vertex and k edges, where one of the half-edges incident on this vertex is chosen to be marked with an arrow. (See Figure 1.2.)

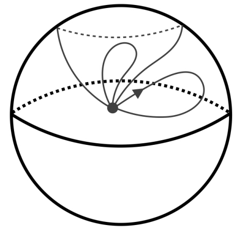

FIGURE 1.2. Catalan graph with three edges.

This graph analogy leads to the definition of generalized Catalan numbers, from which a differential recursion formula on the discrete Laplace transform of these numbers can be proved, and this subsequently gives a topological recursion (see [**7**]).

Thus, it is natural to ask if any other numbers frequently appearing in combinatorial problems admit a generalization of this kind, and if this generalization satisfies a topological recursion. Our focus in this dissertation is to study bc-Motzkin numbers, which are themselves a generalization of Motzkin numbers.

The nth Motzkin number is defined in terms of the Catalan numbers as

$$M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k.$$

The first few Motzkin numbers for $n = 0, 1, 2, 3, \ldots$ are

$$1, 1, 2, 4, 9, 21, 51, 127, 323, \ldots$$

Motzkin numbers were first introduced by Theodore Motzkin in 1948 in his paper [**25**]. There are many different well-known combinatorial interpretations of Motzkin numbers. For example, the nth Motzkin number counts the number of routes on a grid from the coordinate $(0, 0)$ to the coordinate $(n, 0)$ in n steps, subject to the requirement that the path does not cross below the x-axis. (See Figure 1.3.)

The bc-Motzkin numbers are defined by

$$M_n(b, c) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k b^{n-2k} c^k, \tag{1.1}$$

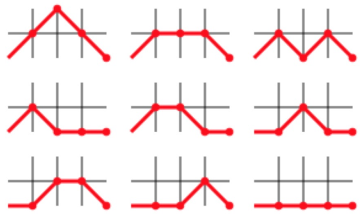

FIGURE 1.3. All possible Motzkin paths for $n = 4$. (Image credit: Wikipedia.)

where $b, c \in \mathbb{N}$. These numbers were introduced by Zhi-Wei Sun in 2014 (see [**24**]), and they appear, for example, in the work of lattice models of statistical physics.

In particular, when $(b, c) = (1, 1)$, we recover the definition of Motzkin numbers,

$$M_n(1,1) = M_n.$$

And, Catalan numbers are the special case of bc-Motzkin numbers when $(b, c) = (0, 1)$,

$$M_n(0,1) = C_n.$$

Thus, we can also consider these bc-Motzkin numbers as a generalization of Catalan numbers, as well as a generalization of Motzkin numbers.

This leads us to ask the following questions.

QUESTION 1.1.1. *Do the bc-Motzkin numbers admit a higher genus generalization?*

QUESTION 1.1.2. *If so, then does this generalization satisfy a Catalan-like recursion formula?*

QUESTION 1.1.3. *If we define the discrete Laplace transform for these higher genus bc-Motzkin numbers, does this satisfy a recursion formula? Do we also obtain a topological recursion?*

QUESTION 1.1.4. *How does the fact that these bc-Motzkin numbers are a generalization of Catalan numbers translate into the properties of these recursion formulas?*

In Chapters 3 through 6 of this dissertation, we answer all these questions affirmatively by giving concrete formulas. An unexpected result is that the recursion formula for the discrete Laplace transform, and hence the topological recursion, is (almost) identical to those for the Catalan numbers. The difference between these results for the Catalan numbers and the generalized bc-Motzkin numbers is the coordinate transformation between the defining variables for the discrete Laplace transform and the variables appearing in these recursion formulas. These coordinate transformations give a family of deformations of the "spectral curve" of the topological recursion. It is very interesting to see that a surprisingly simple deformation of the spectral curve produces a vast generalization of the Catalan numbers.

Note that Catalan numbers are a special case of bc-Motzkin numbers, so this is actually a two-parameter generalization of the results for generalized Catalan numbers.

Further, it is known (see, for example, [**27**]) that Catalan numbers satisfy the identity

$$C_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k 2^{n-2k}. \tag{1.2}$$

Since the right side of this equation is just the bc-Motzkin number $M_n(2,1)$, this identity can be proved by showing that the generating function for C_{n+1} is equal to the generating function for $M_n(2,1)$.

Thus, we see that, since the bc-Motzkin numbers are defined in terms of the Catalan numbers, they can be used to obtain identities on the Catalan numbers. This leads us to ask the following additional questions regarding our higher genus generalization of bc-Motzkin numbers.

QUESTION 1.1.5. *Do the generating functions for these higher genus bc-Motzkin numbers have a general form, where if we are given g and v then we can determine the formula for the generating function for the (g, v) generalized bc-Motzkin numbers?*

QUESTION 1.1.6. *Do these generating functions lead to identities for the generalized Catalan numbers, similar to the identity given above?*

QUESTION 1.1.7. *Do these generating functions lead to simple closed-form expressions for the generalized Catalan numbers?*

In Chapters 7 through 9 of this dissertation, we show that the generating functions for the higher genus bc-Motzkin numbers satisfy a recursion formula. We then use this recursion formula to determine the generating functions of these generalized bc-Motzkin numbers for some cases of small g and v, and to give a conjecture on the general form of these generating functions. For cases of small g and v, we also show that these generating functions can be used to give some identities for the generalized Catalan numbers and to determine some closed-form expressions for the generalized Catalan numbers.

1.2. Organization of this Dissertation

The remaining chapters of this dissertation are organized as follows.

In Chapter 2, we review the results in [**7**] regarding generalized Catalan numbers, first defining these generalized Catalan numbers via an analogy with graphs on the genus g surface. This gives a recursive definition of the generalized Catalan numbers. From this formula, one can then prove a differential recursion formula on the discrete Laplace transform of these numbers, and from this formula prove a subsequent topological recursion.

In Chapter 3, using an analogy with counting colored graphs on the genus g surface, we define the generalized bc-Motzkin numbers as follows.

Definition 1.2.1. *For $b, c \in \mathbb{R}$ with $b \geq 0$ and $c > 0$, we define the generalized bc-Motzkin numbers by*

$$\widetilde{M}_{g,v}(n_1, n_2, \ldots, n_v; b, c) = \sum_{\mu_1=0}^{n_1} \sum_{\mu_2=0}^{n_2} \cdots \sum_{\mu_v=0}^{n_v} \binom{n_1}{\mu_1}\binom{n_2}{\mu_2} \cdots \binom{n_v}{\mu_v} \cdot C_{g,v}(\mu_1, \mu_2, \ldots, \mu_v)\, b^{(n_1+n_2+\cdots+n_v)-(\mu_1+\mu_2+\cdots+\mu_v)} c^{\mu_1+\mu_2+\cdots+\mu_v}.$$

Using this definition, in Chapter 4 we give both a combinatorial proof and an algebraic proof of the following recursion formula for these generalized bc-Motzkin numbers:

Theorem 1.2.1. *The generalized bc-Motzkin numbers satisfy the following formula:*

$$\widetilde{M}_{g,v}(n_1, n_2, \ldots, n_v; b, c) - b\widetilde{M}_{g,v}(n_1 - 1, n_2, \ldots, n_v; b, c)$$
$$= c^2 \Bigg\{ \sum_{j=2}^{v} n_j\, \widetilde{M}_{g,v-1}(n_1 + n_j - 2, n_2, \ldots, \hat{n_j}, \ldots, n_v; b; c)$$
$$+ \sum_{\zeta+\xi=n_1-2} \Big[\widetilde{M}_{g-1,v+1}(\zeta, \xi, n_2, \ldots, n_v; b, c)$$

$$+\sum_{g_1+g_2=g}\sum_{I\sqcup J=\{2,\ldots,v\}}\widetilde{M}_{g_1,|I|+1}(\zeta,n_I;b,c)\widetilde{M}_{g_2,|J|+1}(\xi,n_J;b,c)\Bigg]\Bigg\}$$

Note that this is not truly a recursion formula, as the (g,v) term also appears on the right side in this formula.

Then, in Chapter 5, we show that this recursion formula does actually lead to a differential recursion formula on the discrete Laplace transform of these generalized bc-Motzkin numbers,

$$F_{g,v}^{\widetilde{M}(b,c)}(x_1,\ldots,x_v)=\begin{cases}\displaystyle\sum_{n=1}^{\infty}\frac{\widetilde{M}_{0,1}(n;b,c)}{n}x_1^{-n}-\widetilde{M}_{0,1}(0;b,c)\log x_1 \text{ if } (g,v)=(0,1),\\ \displaystyle\sum_{n_1=1}^{\infty}\cdots\sum_{n_v=1}^{\infty}\frac{\widetilde{M}_{g,v}(n_1,n_2,\ldots,n_v;b,c)}{n_1n_2\cdots n_v}x_1^{-n_1}\cdots x_v^{-n_v},\end{cases}$$

and with a particular choice of change of variable, this differential recursion formula is identical to the differential recursion formula for generalized Catalan numbers.

THEOREM 1.2.2. *The discrete Laplace transform* $F_{g,v}^{\widetilde{M}(b,c)}(t_1,t_2,\ldots,t_v)$ *satisfies the following differential recursion formula, for every* $(g,v)\neq(0,1),(0,2)$*:*

$$\begin{aligned}\frac{\partial}{\partial t_1}F_{g,v}^{\widetilde{M}(b,c)}(t_1,t_2,\ldots,t_v)=&-\frac{1}{16}\sum_{j=2}^{v}\left[\frac{t_j}{t_1^2-t_j^2}\left(\frac{(t_1^2-1)^3}{t_1^2}\frac{\partial}{\partial t_1}F_{g,v-1}^{\widetilde{M}(b,c)}(t_1,\ldots,\widehat{t_j},\ldots,t_v)\right.\right.\\&\left.\left.-\frac{(t_j^2-1)^3}{t_j^2}\frac{\partial}{\partial t_j}F_{g,v-1}^{\widetilde{M}(b,c)}(t_2,\ldots,t_v)\right)\right]\\&-\frac{1}{16}\sum_{j=2}^{v}\frac{(t_1^2-1)^2}{t_1^2}\left[\frac{\partial}{\partial t_1}F_{g,v-1}^{\widetilde{M}(b,c)}(t_1,\ldots,\widehat{t_j},\ldots,t_v)\right]\\&-\frac{1}{32}\frac{(t_1^2-1)^3}{t_1^2}\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}F_{g-1,v+1}^{\widetilde{M}(b,c)}(u_1,u_2,t_2,\ldots,t_v)\bigg|_{u_1=u_2=t_1}\\&-\frac{1}{32}\frac{(t_1^2-1)^3}{t_1^2}\sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ stable}\frac{\partial}{\partial t_1}F_{g_1,|I|+1}^{\widetilde{M}(b,c)}(t_1,t_I)\cdot\frac{\partial}{\partial t_1}F_{g_2,|J|+1}^{\widetilde{M}(b,c)}(t_1,t_J)\end{aligned}$$

where the "stable" summation means $2g_1+|I|-1>0$ *and* $2g_2+|J|-1>0$*.*

We have the initial conditions

$$\frac{\partial}{\partial t}F_{0,1}^{\widetilde{M}(b,c)}(t)=\frac{8t}{(t+1)(t-1)^3}$$

and

$$\frac{\partial}{\partial t_1} F_{0,2}^{\widetilde{M}(b,c)}(t_1,t_2) = \frac{(t_2+1)}{(t_1-1)(t_1+t_2)}.$$

Here, we have used the change of variable $t_i = t_i(x_i, b, c)$, for $i \in \{1, 2, \ldots, v\}$, defined by

$$\frac{x_i - b}{c} = 2 + \frac{4}{t_i^2 - 1}.$$

In Chapter 6, we show that this differential recursion formula for generalized bc-Motzkin numbers leads to the topological recursion given in Theorem 1.1.1 above.

Then, in Chapter 7 we obtain a recursive formula for the generating functions of these generalized bc-Motzkin numbers.

Theorem 1.2.3. *Let*

$$G_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_v=0}^{\infty} \widetilde{M}_{g,v}(n_1, n_2, \ldots, n_v; b, c)\, x_1^{n_1} x_2^{n_2} \cdots x_v^{n_v}$$

denote the generating function for the generalized bc-Motzkin numbers $\widetilde{M}_{g,v}(n_1, n_2, \ldots, n_v; b, c)$.

Then,

$$G_{0,1}^{\widetilde{M}(b,c)}(x) = \frac{1 - bx - \sqrt{1 - 2bx + (b^2 - 4c^2)x^2}}{2c^2x^2}$$

and

$$G_{0,2}^{\widetilde{M}(b,c)}(x_1, x_2) = [1 - 2bx_1 + (b^2 - 4c^2)x_1^2]^{-1/2} \left[\mathcal{I}\mathcal{I}_{0,2}^{\widetilde{M}(b,c)}(x_1, x_2) \right].$$

For $(g, v) \neq (0, 1), (0, 2)$*, the generating functions satisfy the recursive formula*

$$\begin{aligned} G_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) = [1 - 2bx_1 + (b^2 - 4c^2)x_1^2]^{-1/2} \Bigg[& \mathcal{I}\mathcal{I}_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) \\ & + 2c^2x_1^2 \sum_{j=2}^{v} G_{g,v-1}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, \hat{x}_j, \ldots, x_v) G_{0,2}^{\widetilde{M}(b,c)}(x_1, x_j) \\ & + c^2x_1^2\, G_{g-1,v+1}^{\widetilde{M}(b,c)}(x_1, x_1, x_2, \ldots, x_v) \\ & + c^2x_1^2 \sum_{g_1+g_2=g,\ I \sqcup J = \{2,\ldots,v\},\ stable} G_{g_1,|I|+1}^{\widetilde{M}(b,c)}(x_1, x_I) G_{g_2,|J|+1}^{\widetilde{M}(b,c)}(x_1, x_J) \Bigg] \end{aligned}$$

By "stable," we mean $2g_1 + |I| - 1 > 0$ *and* $2g_2 + |J| - 1 > 0$.

Here,

$$\begin{aligned}
&\mathcal{II}_{g,v}^{\widetilde{M}(b,c)}(x_1,x_2,\ldots,x_v)\\
&= c^2\sum_{j=2}^{v}\left\{\frac{x_1x_j}{(x_1-x_j)^2}\left[x_1^2\cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_1,x_2,\ldots,\hat{x_j},\ldots,x_v)-x_j^2\cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_j,x_2,\ldots,\hat{x_j},\ldots,x_v)\right]\right.\\
&\qquad\left.-\frac{x_1x_j}{x_1-x_j}\cdot\frac{\partial}{\partial x_j}\left[x_j^2\cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_j,x_2,\ldots,\hat{x_j},\ldots,x_v)\right]\right\}
\end{aligned}$$

and if $x_j = x_1$ for $j \neq 1$, we take the limit of the above expression for $\mathcal{II}_{g,v}^{\widetilde{M}(b,c)}(x_1,x_2,\ldots,x_v)$ as $x_j \to x_1$.

Using this theorem, and the computer algebra system Mathematica to aid in computations, we obtain explicit generating functions for the generalized bc-Motzkin numbers for some cases of small (g,v), in particular when $g+v=2$ and $g+v=3$. When $g+v=4$, we only compute the particular case when $(b,c)=(0,1)$, i.e. the case of generalized Catalan numbers, due to the lengthy computations involved when we work with general b and c.

Lastly, based on these examples computed for $G_{g,v}^{\widetilde{M}(b,c)}(x_1,x_2,\ldots,x_v)$, we form a conjecture regarding the general form of the generating functions of these generalized bc-Motzkin numbers.

Conjecture 1.2.1. *For notational convenience, let*

$$R^{(b,c)}(x_i) = [1-2bx_i+(b^2-4c^2)x_i^2].$$

If $(g,v) \neq (0,1),(0,2)$, then the generating functions $G_{g,v}^{\widetilde{M}(b,c)}(x_1,x_2,\ldots,x_v)$ for the generalized bc-Motzkin numbers are of the form

$$G_{g,v}^{\widetilde{M}(b,c)}(x_1,x_2,\ldots,x_v)=\frac{x_1x_2\cdots x_v}{[R^{(b,c)}(x_1)R^{(b,c)}(x_2)\cdots R^{(b,c)}(x_v)]^{\alpha(g,v)/2}}P_{g,v}^{\widetilde{M}(b,c)}(x_1,x_2,\ldots,x_v),$$

where:

(i) the exponent $\alpha(g,v)=6g-3+2v$,

(ii) $P_{g,v}^{\widetilde{M}(b,c)}(x_1,x_2,\ldots,x_v)$ is a polynomial, and

(iii) for $v \geq 2$, the polynomial $P_{g,v}^{\widetilde{M}(b,c)}(x_1,x_1,x_2,\ldots,x_{v-1})$ is divisible by $[R^{(b,c)}(x_1)]^{\beta(g,v)}$, where $\beta(g,v)=3g-3+v$.

In Chapter 8, we use the results of the previous chapter to obtain closed-form expressions for generalized Catalan numbers in the cases when $(g, v) = (1,1), (2,1)$, and $(3,1)$, as well as when $(g, v) = (0,3)$. This gives the following results.

PROPOSITION 1.2.1. *The* $(1,1)$ *Catalan numbers are given as follows.*

(i) $C_{1,1}(2k+1) = 0$ *for all* k.

(ii) $C_{1,1}(2k) = 0$ *if* $k \leq 1$.

(iii) $C_{1,1}(2k) = \dfrac{k(k-1)}{12}\dbinom{2k}{k}$ *if* $k \geq 2$.

PROPOSITION 1.2.2. *The* $(2,1)$ *Catalan numbers are given as follows.*

(i) $C_{2,1}(2k+1) = 0$ *for all* k.

(ii) $C_{2,1}(2k) = 0$ *if* $k \leq 3$.

(iii) $C_{2,1}(2k) = \dfrac{5k-2}{4 \cdot 5 \cdot 8 \cdot 9}k(k-1)(k-2)(k-3)\dbinom{2k}{k}$ *if* $k \geq 4$.

PROPOSITION 1.2.3. *The* $(3,1)$ *Catalan numbers are given as follows.*

(i) $C_{3,1}(2k+1) = 0$ *for all* k.

(ii) $C_{3,1}(2k) = 0$ *if* $k \leq 5$.

(iii)
$$\begin{aligned} C_{3,1}(2k) = &\left[\frac{1}{2^6 \cdot 7} + \frac{1}{2^6 \cdot 3 \cdot 5}(k-6) + \frac{29 \cdot 73}{2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 13}(k-6)(k-7)\right] \\ &\cdot k(k-1)(k-2)(k-3)(k-4)(k-5)\binom{2k}{k} \end{aligned} \quad \textit{if } k \geq 6$$

PROPOSITION 1.2.4. *The* $(0,3)$ *Catalan numbers are given as follows.*

(i) $C_{0,3}(a,b,c) = 0$ *if* a,b, *or* c *equals zero.*

(ii) $C_{0,3}(a,b,c) = 0$ *if* $a+b+c$ *is odd.*

(iii) $C_{0,3}(a,b,c) = (n \cdot m \cdot k)\dbinom{2n}{n}\dbinom{2m}{m}\dbinom{2k}{k}$ *if* $a = 2n$, $b = 2m$, *and* $c = 2k$ *are all even.*

(iv) $C_{0,3}(a,b,c) = \dfrac{1}{4}(n \cdot m \cdot k)\dbinom{2n}{n}\dbinom{2m}{m}\dbinom{2k}{k}$ *if* $a = 2n$ *is even, and* $b = 2m-1$ *and* $c = 2k-1$ *are odd.*

(v) $C_{0,3}(a,b,c) = \dfrac{1}{4}(n \cdot m \cdot k)\dbinom{2n}{n}\dbinom{2m}{m}\dbinom{2k}{k}$ *if* $b = 2m$ *is even, and* $a = 2n-1$ *and* $c = 2k-1$ *are odd.*

(vi) $C_{0,3}(a,b,c) = \dfrac{1}{4}(n \cdot m \cdot k)\dbinom{2n}{n}\dbinom{2m}{m}\dbinom{2k}{k}$ *if* $c = 2k$ *is even, and* $a = 2n-1$ *and* $b = 2m-1$ *are odd.*

In Chapter 9, we use the results of Chapter 7 to prove identities for generalized Catalan numbers in the cases of $(g, v) = (1, 1)$ and $(g, v) = (0, 2)$, similar to the identity given in equation (1.2) above for the case of $(g, v) = (0, 1)$.

THEOREM 1.2.4. *The* $(1, 1)$ *Catalan numbers satisfy the following relation, for all* n.

$$C_{1,1}(2n-4) = \sum_{\mu=0}^{n} \binom{n}{\mu} C_{1,1}(\mu) 2^{n-\mu}.$$

THEOREM 1.2.5. *The* $(0, 2)$ *Catalan numbers satisfy the following relation, for all* n.

$$\sum_{a+b=2n-2} C_{0,2}(a,b) = \sum_{n_1+n_2=n} \left[\sum_{\mu_1=0}^{n_1} \sum_{\mu_2=0}^{n_2} \binom{n_1}{\mu_1} \binom{n_1}{\mu_1} C_{0,2}(\mu_1, \mu_2)\, 2^{(n_1+n_2)-(\mu_1+\mu_2)} \right].$$

Finally, in Appendices A and B, we use the computer algebra system Mathematica to compute exact values of some generalized Catalan numbers and some generalized *bc*-Motzkin numbers, for small (g, v).

CHAPTER 2

Background: Higher Genus Catalan Numbers

In this chapter we summarize some results obtained by Dumitrescu and Mulase in [**7**], regarding generalized Catalan numbers.

As discussed in the introduction, the kth Catalan number (roughly) counts the number of graphs drawn on a sphere with one vertex and k edges, where one of the half-edges incident on this vertex is chosen to be marked with an arrow. (See Figure 1.2.)

This analogy with counting graphs on a sphere and having one vertex can be extended, and is used to define the generalized Catalan numbers

$$C_{g,v}(\mu_1, \mu_2, \ldots, \mu_v),$$

which count the number of graphs drawn on the (oriented) genus g surface with v vertices that give a cell decomposition of the surface, where the ith vertex has degree μ_i, and at each vertex one of the incident half-edges is chosen to be marked with an arrow (see [**7**] and [**26**]).

We will call such a graph a Catalan graph of degree $(\mu_1, \mu_2, \ldots, \mu_v)$ on the genus g surface. (See Figure 2.1 for an example of such a graph.)

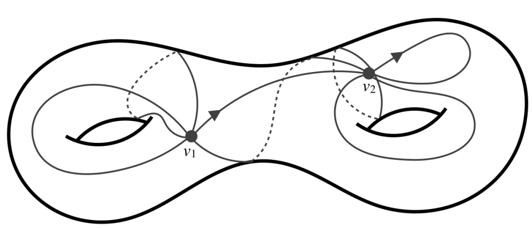

FIGURE 2.1. Catalan graph of degree $(6, 8)$ on the genus 2 surface.

Now, the Catalan numbers are known to satisfy the recursion formula,

$$C_n = \sum_{a+b=n-1} C_a C_b. \tag{2.1}$$

This formula has the higher genus analogue given in the following proposition.

Proposition 2.0.1. *The generalized Catalan numbers satisfy*

$$\begin{aligned} C_{g,v}(\mu_1,\mu_2,\ldots,\mu_v) = &\sum_{j=2}^{v} \mu_j C_{g,v-1}(\mu_1+\mu_j-2,\mu_2,\ldots,\widehat{\mu_j},\ldots,\mu_v) \\ &+ \sum_{\alpha+\beta=\mu_1-2} \Bigg[C_{g-1,v+1}(\alpha,\beta,\mu_2,\ldots,\mu_v) \\ &+ \sum_{g_1+g_2=g, I\sqcup J=\{2,\ldots,v\}} C_{g_1,|I|+1}(\alpha,\mu_I) C_{g_2,|J|+1}(\beta,\mu_J) \Bigg] \end{aligned} \tag{2.2}$$

where $\mu_I = (\mu_{i_1},\mu_{i_2},\ldots,\mu_{i_{|I|}})$ for an index set I, the notation $\widehat{\mu_j}$ means that we delete μ_j from this sequence, and the third sum in the above formula is for all partitions of g and set partitions of $\{2,\ldots,v\}$.

Remark 2.0.1. Observe that this is not truly a recursion formula, since the (g,v) term also appears on the right side of the equation.

This formula serves to define the generalized Catalan numbers.

Remark 2.0.2. With this definition, $C_{0,1}(\mu)$ is actually the aerated Catalan numbers,

$$C_{0,1}(\mu) = \begin{cases} C_{\mu/2} & \text{if } \mu \text{ is even,} \\ 0 & \text{if } \mu \text{ is odd.} \end{cases}$$

In Chapter 4, we will show that the generalized bc-Motzkin numbers satisfy a similar formula, which reduces to the Catalan formula when $b=0$ and $c=1$.

To prove this formula, we may proceed as follows. A proof of this result is also given in [**7**].

Proof. We start by contracting the edge E corresponding to the arrowed half-edge which is incident on v_1. There are two cases which we need to study.

Case 1. Assume E connects v_1 and v_j $(j \neq 1)$.

Contracting E replaces the two vertices v_1 and v_j with one vertex, of degree $\mu_1+\mu_j-2$. To make this counting bijective, we need to be able to go back to the original graph if we are given μ_1 and μ_j, which are the degrees of v_1 and v_j, respectively. We do this by putting an arrow on the half-edge next to E with respect to the counterclockwise cyclic ordering that comes from the orientation on the surface. (See Figure 2.2.) However, we must first delete the arrow that was assigned to a half-edge incident on v_j in the original graph. So, there are μ_j different graphs which produce the same result.

This gives the first term in the Catalan recursion.

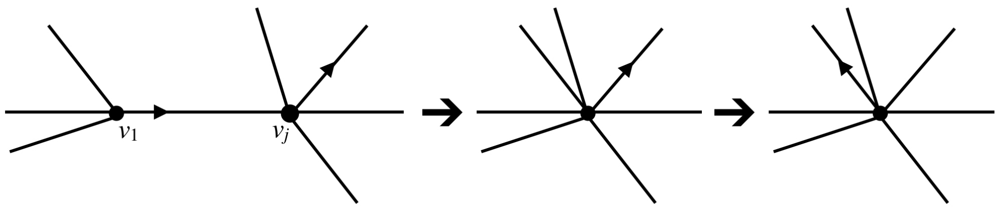

FIGURE 2.2.

Case 2. Assume E is a loop on v_1.

Contracting E separates the incident half-edges at v_1 into two collections, with α edges on one side and β edges on the other (note that α or β may be zero). Since E is a loop, contracting it causes pinching on the surface and produces a double point. We separate the double point into two new vertices. The result may be a surface of genus $g-1$, or two surfaces, of genus g_1 and g_2 with $g_1 + g_2 = g$. We assign an arrow to the half edge(s) next to E, again with respect to the counterclockwise cyclic ordering that comes from the orientation on the surface. (See Figure 2.3.)

This gives the remaining terms in the Catalan recursion.

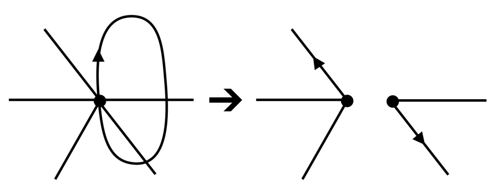

FIGURE 2.3.

□

Let us now look at some examples.

EXAMPLE 2.0.1. Contracting the arrowed half-edge in the Catalan graph of degree 4 on the genus 1 surface in Figure 2.4 gives a Catalan graph of degree $(1,1)$ on the genus 0 surface.

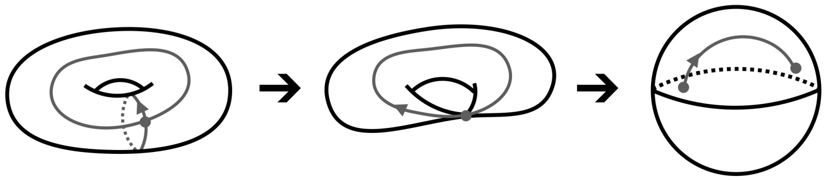

FIGURE 2.4.

EXAMPLE 2.0.2. Contracting the arrowed half-edge in the Catalan graph of degree $(4, 4)$ on the genus 1 surface in Figure 2.5 gives a Catalan graph of degree $(4, 1)$ on the genus 1 surface and a Catalan graph of degree 0 on the genus 0 surface.

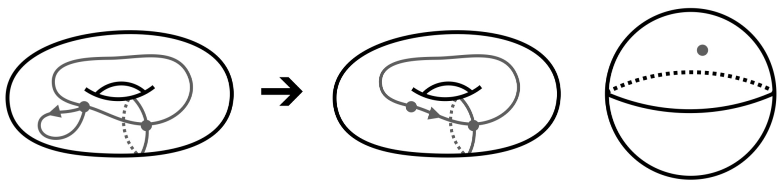

FIGURE 2.5.

EXAMPLE 2.0.3. Contracting the arrowed half-edge in the Catalan graph of degree 6 on the genus 0 surface in Figure 2.6 gives a Catalan graph of degree 2 on the genus 0 surface and a Catalan graph of degree 2 on the genus 0 surface.

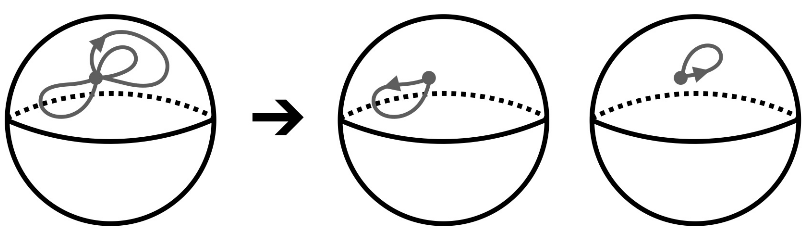

FIGURE 2.6.

To obtain a topological recursion from Proposition 2.0.1, we first need to look at the discrete Laplace transform for the generalized Catalan numbers.

This is defined by

$$F^C_{g,v}(x_1,x_2,\ldots,x_v)=\begin{cases}\displaystyle\sum_{n=1}^{\infty}\frac{C_{0,1}(n)}{n}x_1^{-n}-C_{0,1}(0)\log x_1 \text{ if } (g,v)=(0,1),\\ \displaystyle\sum_{\mu_1=1}^{\infty}\cdots\sum_{\mu_v=1}^{\infty}\frac{C_{g,v}(\mu_1,\mu_2,\ldots,\mu_v)}{\mu_1\mu_2\cdots\mu_v}x_1^{-\mu_1}x_2^{-\mu_2}\cdots x_v^{-\mu_v},\end{cases} \tag{2.3}$$

and it can be calculated by a recursion formula. The recursion formula is actually a "differential" recursion, on

$$\frac{\partial}{\partial t_1}F^C_{g,v}(t_1,t_2,\ldots,t_v),$$

where $(t_1,t_2,\ldots,t_v)$ is a particular choice of change of variables, given by

$$x_i=2+\frac{4}{t_i^2-1},\qquad i\in\{1,2,\ldots,v\}. \tag{2.4}$$

One may compute from the definition of the discrete Laplace transform, using Proposition 2.0.1 and the above choice of change of variable, that we have

$$\frac{\partial}{\partial t}F^C_{0,1}(t)=\frac{8t}{(t+1)(t-1)^3} \tag{2.5}$$

and

$$\frac{\partial}{\partial t_1}F^C_{0,2}(t_1,t_2)=\frac{(t_2+1)}{(t_1-1)(t_1+t_2)}. \tag{2.6}$$

More generally, the following result is given in [**7**].

Proposition 2.0.2 (Dumitrescu, Mulase, Safnuk, Sorkin '13). *The discrete Laplace transform $F^C_{g,v}(t_1,t_2,\ldots,t_v)$ satisfies the following differential recursion equation for every $(g,v)\neq(0,1),(0,2)$.*

$$\begin{aligned}\frac{\partial}{\partial t_1}F^C_{g,v}(t_1,t_2,\ldots,t_v)=&-\frac{1}{16}\sum_{j=2}^{v}\left[\frac{t_j}{t_1^2-t_j^2}\left(\frac{(t_1^2-1)^3}{t_1^2}\frac{\partial}{\partial t_1}F^C_{g,v-1}(t_1,\ldots,\widehat{t_j},\ldots,t_v)\right.\right.\\&\left.\left.-\frac{(t_j^2-1)^3}{t_j^2}\frac{\partial}{\partial t_j}F^C_{g,v-1}(t_2,\ldots,t_v)\right)\right]\\&-\frac{1}{16}\sum_{j=2}^{v}\frac{(t_1^2-1)^2}{t_1^2}\left[\frac{\partial}{\partial t_1}F^C_{g,v-1}(t_1,\ldots,\widehat{t_j},\ldots,t_v)\right]\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{32}\frac{(t_1^2-1)^3}{t_1^2}\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}F^C_{g-1,v+1}(u_1,u_2,t_2,\ldots,t_v)\Big|_{u_1=u_2=t_1}\\
&- \frac{1}{32}\frac{(t_1^2-1)^3}{t_1^2}\sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ stable}\frac{\partial}{\partial t_1}F^C_{g_1,|I|+1}(t_1,t_I)\cdot\frac{\partial}{\partial t_1}F^C_{g_2,|J|+1}(t_1,t_J),
\end{aligned}
\tag{2.7}
$$

where the "stable" summation means $2g_1+|I|-1>0$ and $2g_2+|J|-1>0$.

Remarkably, the discrete Laplace transform of the generalized bc-Motzkin numbers satisfies an identical differential recursion formula, with the only difference being that the change of variable from x_i to t_i depends also on b and c. When $b=0$ and $c=1$, it reduces to the same change of variables as in the Catalan case.

The proof of Proposition 2.0.2 is very similar to the proof in the bc-Motzkin numbers case (see Chapter 5), hence a proof of this result is not given here.

The differential recursion formula in Proposition 2.0.2 leads to a recursion in $2g-2+v$ on the Eynard-Orantin differential forms on $(\mathbb{P}^1)^v$, defined by

$$
W^C_{g,v}(t_1,t_2,\ldots,t_v)=d_{t_1}\cdots d_{t_n}F^C_{g,v}(t_1,t_2,\ldots,t_v). \tag{2.8}
$$

The resulting recursion formula is called the topological recursion for these generalized Catalan numbers.

Proposition 2.0.3 (Dumitrescu, Mulase, Safnuk, Sorkin '13). *For $2g-2+v>0$, these differential forms satisfy the following integral recursion equation:*

$$
\begin{aligned}
W^C_{g,v}(t_1,t_2,\ldots,t_v) &= -\frac{1}{64}\frac{1}{2\pi i}\int_\gamma\left(\frac{1}{t+t_1}+\frac{1}{t-t_1}\right)\frac{(t^2-1)^3}{t^2}\,\frac{1}{dt}\,dt_1\\
&\cdot\Bigg[\sum_{j=2}^{v}\Big(W^C_{0,2}(t,t_j)W^C_{g,v-1}(-t,t_2,\ldots,\widehat{t_j},\ldots,t_v)+W^C_{0,2}(-t,t_j)W^C_{g,v-1}(t,t_2,\ldots,\widehat{t_j},\ldots,t_v)\Big)\\
&+W^C_{g-1,v+1}(t,-t,t_2,\ldots,t_v)+\sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ stable}W^C_{g_1,|I|+1}(t,t_I)W^C_{g_2,|J|+1}(-t,t_J)\Bigg]
\end{aligned}
\tag{2.9}
$$

Here, γ is the contour in Figure 1.1.

To prove this result, one may apply the definition of the Eynard-Orantin differential forms to the differential recursion formula in Proposition 2.0.2 to obtain a formula for the $W^C_{g,v}(t_1,t_2,\ldots,t_v)$, then simplify the integral around the contour γ on the right side of the equation in Proposition 2.0.3 to show that the two formulas for $W^C_{g,v}(t_1, t_2, \ldots, t_v)$ are indeed equal.

Similarly to the case of the differential recursion formula, the generalized bc-Motzkin numbers satisfy an identical result, again with the only difference being that the change of variables from x_i to t_i now depends on b and c. Since the differential recursion formula is identical to the (more general) bc-Motzkin numbers case, a proof of the Catalan case is not given here. The proof of the topological recursion for generalized bc-Motzkin numbers is given in Chapter 6.

CHAPTER 3

Higher Genus bc-Motzkin Numbers

Recall that in the introduction we defined the bc-Motzkin numbers $M_n(b, c)$ by equation (1.1), following the definition in [**24**]. Now, we will define a slightly different version of bc-Motzkin numbers, which are better suited for our purposes in this dissertation, by

$$\widetilde{M}_{0,1}(n; b, c) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k b^{n-2k} c^{2k}, \tag{3.1}$$

where $b \in \mathbb{R}_{\geq 0}$ and $c \in \mathbb{R}_{>0}$.

The difference is that we now write c^{2k} rather than c^k, and we also allow b to be a nonnegative real number, and we allow c to be a positive real number, instead of restricting to the natural numbers.

Thus, we may write these $(0, 1)$-bc-Motzkin numbers in terms of the $(0, 1)$-Catalan numbers as

$$\widetilde{M}_{0,1}(n; b, c) = \sum_{\mu=0}^{n} \binom{n}{\mu} C_{0,1}(\mu) b^{n-\mu} c^{\mu}. \tag{3.2}$$

We would like to generalize the $(0, 1)$-bc-Motzkin numbers, just as was done for Catalan numbers. To do this, we first construct an analogy with colorings of Catalan graphs on the genus g surface. We are led to make the following new definition.

Definition 3.0.1. *Let*

$$\Gamma_{g,v}(n_1, n_2, \ldots, n_v; \mu_1, \mu_2, \ldots, \mu_v)$$

denote the number of ways to color Catalan graphs of degree $(\mu_1, \mu_2, \ldots, \mu_v)$ on the genus g surface, subject to the following requirements, for all $i \in \{1, 2, \ldots, v\}$.

(1) The degree μ_i of the ith vertex is less than or equal to the number of colors n_i with which the half-edges adjacent to that vertex can be colored.

(2) We choose μ_i colors from the set of n_i colors with which to color the half-edges adjacent to the ith vertex.

(3) The set of n_i colors associated with the ith vertex is ordered. Of the μ_i colors chosen from this set, the lowest-indexed color is assigned to the arrowed half-edge incident on that vertex, and the colors increase in ordering as we traverse the edges by going counterclockwise around the vertex.

Then, we define the generalized bc-Motzkin number to be equal to the following weighted sum of cardinalities of this set:

$$\widetilde{M}_{g,v}(n_1, n_2, \ldots, n_v; b, c) = \sum_{\mu_1=0}^{n_1} \sum_{\mu_2=0}^{n_2} \cdots \sum_{\mu_v=0}^{n_v} |\Gamma_{g,v}(n_1, n_2, \ldots, n_v; \mu_1, \mu_2, \ldots, \mu_v)| \cdot b^{(n_1+n_2+\cdots+n_v)-(\mu_1+\mu_2+\cdots+\mu_v)} c^{\mu_1+\mu_2+\cdots+\mu_v}. \quad (3.3)$$

Here, $b \in \mathbb{R}_{\geq 0}$ and $c \in \mathbb{R}_{>0}$.

EXAMPLE 3.0.1. The degree $(6, 8)$ Catalan graph in Figure 3.1 has been colored according to the requirements in Definition 3.0.1, with 6 colors $(k_2^{(1)}, k_3^{(1)}, k_5^{(1)}, k_6^{(1)}, k_7^{(1)}, k_{10}^{(1)})$ chosen from a set of 10 possible colors and assigned to the first vertex v_1, going counterclockwise around this vertex with respect to the orientation on this genus 2 surface and starting with the lowest indexed color on the arrowed half-edge in the underlying Catalan graph. And, 8 colors $(k_1^{(2)}, k_2^{(2)}, k_3^{(2)}, k_4^{(2)}, k_5^{(2)}, k_6^{(2)}, k_7^{(2)}, k_8^{(2)})$ have been chosen from a set of 8 possible colors and assigned to the second vertex v_2. (The superscript $^{(1)}$ on the first set of colors denotes that they are to be assigned to the vertex v_1, and similarly for v_2.)

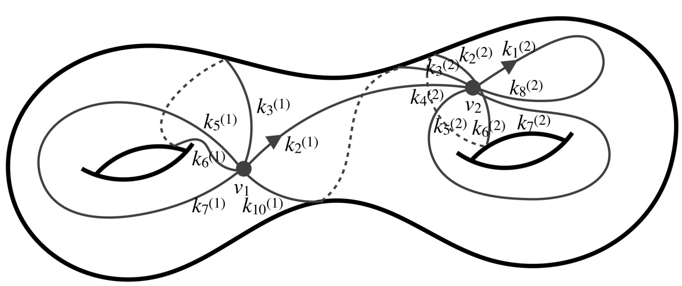

FIGURE 3.1.

Remark 3.0.1. In the case when $g = 0$ and $v = 1$, we see that Definition 3.0.1 gives

$$\widetilde{M}_{0,1}(n; b, c) = \sum_{\mu=0}^{n} |\Gamma_{0,1}(n; \mu)|\, b^{n-\mu} c^{\mu},$$

and $|\Gamma_{0,1}(n; \mu)| = \binom{n}{\mu} C_{0,1}(\mu)$, since once we have chosen μ different colors from the set of n possible colors, this uniquely determines the coloring on the Catalan graph of degree μ. Thus, this definition does indeed coincide with our earlier definition of $(0, 1)$-bc-Motzkin numbers in equation (3.2).

More generally, we have the following equivalent definition of the generalized bc-Motzkin numbers, in terms of the generalized Catalan numbers defined in Chapter 2.

Definition 3.0.2. *For $b, c \in \mathbb{R}$ with $b \geq 0$ and $c > 0$, we define the generalized bc-Motzkin numbers by*

$$\widetilde{M}_{g,v}(n_1, n_2, \ldots, n_v; b, c) = \sum_{\mu_1=0}^{n_1} \sum_{\mu_2=0}^{n_2} \cdots \sum_{\mu_v=0}^{n_v} \binom{n_1}{\mu_1}\binom{n_2}{\mu_2} \cdots \binom{n_v}{\mu_v}$$
$$\cdot\, C_{g,v}(\mu_1, \mu_2, \ldots, \mu_v)\, b^{(n_1+n_2+\cdots+n_v)-(\mu_1+\mu_2+\cdots+\mu_v)} c^{\mu_1+\mu_2+\cdots+\mu_v}. \quad (3.4)$$

Note that, as expected, this definition is symmetric in its arguments n_i, since the generalized Catalan numbers are symmetric in their arguments μ_i.

Remark 3.0.2. If $(b, c) = (0, 1)$, then we recover the generalized Catalan numbers,

$$\widetilde{M}_{g,v}(n_1, n_2, \ldots, n_v; 0, 1) = C_{g,v}(n_1, n_2, \ldots, n_v). \quad (3.5)$$

Hence these generalized bc-Motzkin numbers are also a generalization of Catalan numbers.

CHAPTER 4

The Recursion Formula for Generalized bc-Motzkin Numbers

As a first step towards obtaining a topological recursion for the generalized bc-Motzkin numbers, we wish to find a recursion formula for these numbers.

4.1. The $(0,1)$ Recursion Formula

It is known that the $(0,1)$-bc-Motzkin numbers satisfy a recursion formula, similar to the $(0,1)$-Catalan numbers. This recursion formula was initially proved by Yi Wang and Zhi-Hai Zhang in [**27**].

PROPOSITION 4.1.1. *For $n \geq 1$, the $(0,1)$-bc-Motzkin numbers $M_{0,1}(n;b,c)$ satisfy*

$$\widetilde{M}_{0,1}(n;b,c) - b\,\widetilde{M}_{0,1}(n-1;b,c) = c^2 \sum_{\alpha+\beta=n-2} \widetilde{M}_{0,1}(\alpha;b,c)\widetilde{M}_{0,1}(\beta;b,c). \tag{4.1}$$

Before proving this recursion formula, we first need to prove the following "Vandermonde-like" identity.

PROPOSITION 4.1.2. *For all $0 \leq i+j \leq n$, we have*

$$\sum_{a+b=n} \binom{a}{i}\binom{b}{j} = \binom{n+1}{i+j+1}. \tag{4.2}$$

PROOF. We will prove this statement using induction on n.

Base case. Assume $n=0$. This implies that we must have $i=j=0$, so

$$\sum_{a+b=0} \binom{a}{0}\binom{b}{0} = \binom{0}{0}\binom{0}{0} = 1 \cdot 1 = 1 = \binom{0+1}{0+0+1}.$$

Thus the claim holds for $n=0$.

Inductive case. Assume we know

$$\sum_{a+b=n} \binom{a}{i}\binom{b}{j} = \binom{n+1}{i+j+1}$$

is true for all $0 \le i+j \le n$.

Then, using Pascal's Identity and the inductive hypothesis,

$$\begin{aligned}
\binom{(n+1)+1}{i+j+1} &= \binom{n+1}{i+j} + \binom{n+1}{i+j+1} \\
&= \sum_{a+b=n} \binom{a}{i-1}\binom{b}{j} + \sum_{a+b=n} \binom{a}{i}\binom{b}{j} \\
&= \sum_{a+b=n} \left[\binom{a}{i-1} + \binom{a}{i}\right]\binom{b}{j} \\
&= \sum_{a+b=n} \binom{a+1}{i}\binom{b}{j} \\
&= \sum_{a+b=n+1} \binom{a}{i}\binom{b}{j}
\end{aligned}$$

which proves Proposition 4.1.2. □

Now, we may prove the recursion formula for $(0,1)$-bc-Motzkin numbers.

PROOF. Using equation (3.2) as the definition of the $(0,1)$-bc-Motzkin numbers, we may compute

$$\begin{aligned}
&\widetilde{M}_{0,1}(n;b,c) - b\,\widetilde{M}_{0,1}(n-1;b,c) \\
&= \sum_{\mu=0}^{n} \binom{n}{\mu} C_{0,1}(\mu)\, b^{n-\mu} c^{\mu} - b \sum_{\mu=0}^{n-1} \binom{n-1}{\mu} C_{0,1}(\mu)\, b^{n-1-\mu} c^{\mu} \\
&= C_{0,1}(n)\, c^{n} + \sum_{\mu=0}^{n-1} \left[\binom{n}{\mu} - \binom{n-1}{\mu}\right] C_{0,1}(\mu)\, b^{n-\mu} c^{\mu} \\
&= C_{0,1}(n)\, c^{n} + \sum_{\mu=0}^{n-1} \binom{n-1}{\mu-1} C_{0,1}(\mu)\, b^{n-\mu} c^{\mu}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\mu=0}^{n} \binom{n-1}{\mu-1} C_{0,1}(\mu)\, b^{n-\mu} c^{\mu} \\
&= \sum_{\mu=0}^{n} \binom{n-1}{\mu-1} \left[\sum_{i+j=\mu-2} C_{0,1}(i) C_{0,1}(j) \right] b^{n-\mu} c^{\mu} \\
&= c^2 \sum_{\mu-2=0}^{n-2} \sum_{i+j=\mu-2} \binom{(n-2)+1}{(\mu-2)+1} C_{0,1}(i) C_{0,1}(j) b^{(n-2)-(\mu-2)} c^{\mu-2} \\
&= c^2 \sum_{\mu=0}^{n-2} \sum_{i+j=\mu} \binom{(n-2)+1}{i+j+1} C_{0,1}(i) C_{0,1}(j) b^{(n-2)-(i+j)} c^{i+j} \\
&= c^2 \sum_{\mu=0}^{n-2} \sum_{i+j=\mu} \left[\sum_{\alpha+\beta=n-2} \binom{\alpha}{i} \binom{\beta}{j} \right] C_{0,1}(i) C_{0,1}(j) b^{(\alpha+\beta)-(i+j)} c^{i+j} \\
&= c^2 \sum_{\alpha+\beta=n-2} \left[\sum_{i=0}^{\alpha} \binom{\alpha}{i} C_{0,1}(i) b^{\alpha-i} c^{i} \right] \left[\sum_{j=0}^{\beta} \binom{\beta}{j} C_{0,1}(j) b^{\beta-j} c^{j} \right] \\
&= c^2 \sum_{\alpha+\beta=n-2} \widetilde{M}_{0,1}(\alpha; b, c) \widetilde{M}_{0,1}(\beta; b, c)
\end{aligned}
$$

Here, we recall from equation (2.1) that the Catalan numbers satisfy the recursion formula

$$
C_{0,1}(\mu) = \sum_{i+j=\mu-2} C_{0,1}(i) C_{0,1}(j),
$$

and we used the above "Vandermonde-like" identity.

This proves Proposition 4.1.1. ☐

4.2. The Recursion Formula for General (g, v)

We now wish to obtain a recursion formula for generalized bc-Motzkin numbers. To do this, we first observe that the generalized bc-Motzkin numbers are defined in terms of the generalized Catalan numbers in Definition 3.0.2, and Proposition 2.0.1 gives a "recursion" formula for the generalized Catalan numbers. Thus, we are led to prove the following result.

Theorem 4.2.1. *The generalized bc-Motzkin numbers satisfy the following formula:*

$$
\widetilde{M}_{g,v}(n_1, n_2, \ldots, n_v; b, c) - b\widetilde{M}_{g,v}(n_1 - 1, n_2, \ldots, n_v; b, c)
$$

$$
\begin{aligned}
&= c^2 \Bigg\{ \sum_{j=2}^{v} n_j \, \widetilde{M}_{g,v-1}(n_1+n_j-2, n_2, \ldots, \hat{n_j}, \ldots, n_v; b; c) \\
&\quad + \sum_{\zeta+\xi=n_1-2} \Bigg[\widetilde{M}_{g-1,v+1}(\zeta, \xi, n_2, \ldots, n_v; b, c) \\
&\quad + \sum_{g_1+g_2=g,\ I \sqcup J=\{2,\ldots,v\}} \widetilde{M}_{g_1,|I|+1}(\zeta, n_I; b, c) \widetilde{M}_{g_2,|J|+1}(\xi, n_J; b, c) \Bigg] \Bigg\}
\end{aligned} \tag{4.3}
$$

Observe that, as in the Catalan case, this is not truly a recursion formula, since the (g, v) term also appears on the right side of the equation.

Remark 4.2.1. The $(0, 1)$ case of Theorem 4.2.1, for the $(0, 1)$-bc-Motzkin numbers, is the same as in Proposition 4.1.1, as expected. Further, when $b = 0$ and $c = 1$, we recover the formula in Proposition 2.0.1 for generalized Catalan numbers.

The above theorem may be proved combinatorially, by using Definition 3.0.1 for generalized bc-Motzkin numbers in terms of the graph coloring analogy, and following the same approach as for the proof of the Catalan "recursion" formula. It can also be proved algebraically, by applying Definition 3.0.2 of the generalized bc-Motzkin numbers and Pascal's identity to simplify the left side of the above relation, then plugging in the Catalan formula from Proposition 2.0.1, simplifying, and re-writing the result entirely in terms of generalized bc-Motzkin numbers via some combinatorial identities.

We will first give the combinatorial proof, referencing the proof of Proposition 2.0.1 in Chapter 2 for the Catalan case.

Proof. Recall that $\widetilde{M}_{g,v}(n_1, n_2, \ldots, n_v; b, c)$ equals the number of ways to color all Catalan graphs of degree $(\mu_1, \mu_2, \ldots, \mu_v)$ on the genus g surface and having v vertices, subject to the coloring requirements in Definition 3.0.1, where $\mu_i \leq n_i$ for all i, and the sum of cardinalities of sets of colored Catalan graphs is weighted in a particular way by b and c.

Now, in each underlying Catalan graph, we fix the color on the arrowed half-edge incident on v_1 to be the lowest indexed color in the set of possible colors for v_1, call it $k_1^{(1)}$. Then, the resulting number of ways to color such Catalan graphs, subject to the necessary restrictions and weightings, is

$$\widetilde{M}_{g,v}(n_1, n_2, \ldots, n_v; b, c) - b\widetilde{M}_{g,v}(n_1 - 1, n_2, \ldots, n_v; b, c).$$

This can be seen by observing that $\widetilde{M}_{g,v}(n_1 - 1, n_2, \ldots, n_v; b, c)$ gives the number of ways to color the graphs in the definition of $\widetilde{M}_{g,v}(n_1, n_2, \ldots, n_v; b, c)$ without using the color $k_1^{(1)}$ on the vertex v_1. The exponent of b essentially counts the difference between the number of possible colors which can be assigned to a vertex and the degree of that vertex.

To show that this equals the right side of the above formula, we will to contract the edge E corresponding to the edge incident on v_1 which is colored by $k_1^{(1)}$. As in the proof of the analogous result for generalized Catalan numbers, there are two cases.

Case 1. Assume E connects v_1 and v_j $(j \neq 1)$.

Contracting E replaces the two vertices v_1 and v_j with one vertex, call it v. Since there were n_1 possible colors which could be assigned to v_1, and n_j possible colors which could be assigned to v_j, we see that there are $n_1 + n_j - 2$ possible colors which can be assigned to this resulting vertex v.

For notational convenience, assume that the colors on vertex v_1 are $(k_1^{(1)}, k_{a_2}^{(1)}, \ldots k_{a_{\mu_1}}^{(1)})$ and the colors on vertex v_j are $(k_{b_1}^{(j)}, k_{b_2}^{(j)}, \ldots k_{b_{\mu_j}}^{(j)})$, with the color on the other half-edge of E being $k_{b_\ell}^{(j)}$. We will make the convention that all colors on vertex v_j have higher index than those on vertex v_1, so that our list of colors on the vertex v is, in order,

$$(k_{a_2}^{(1)}, \ldots k_{a_{\mu_1}}^{(1)}, k_{b_1}^{(j)}, \ldots, \widehat{k_{b_\ell}^{(j)}}, \ldots k_{b_{\mu_j}}^{(j)}).$$

Then, we color the half-edges incident on the new vertex v starting with the lowest indexed color $k_{a_2}^{(1)}$ on the half-edge that was next to E with respect to the counterclockwise cyclic ordering coming from the orientation on the surface, and continue by going counterclockwise around the vertex v. Observe that this will not change the coloring on the half-edges that were originally assigned to v_1. (See Figure 4.1.)

Since there are n_j choices for the color $k_{b_\ell}^{(j)}$, we see that there are n_j different graphs which produce the same result, which gives the factor of n_j in the sum.

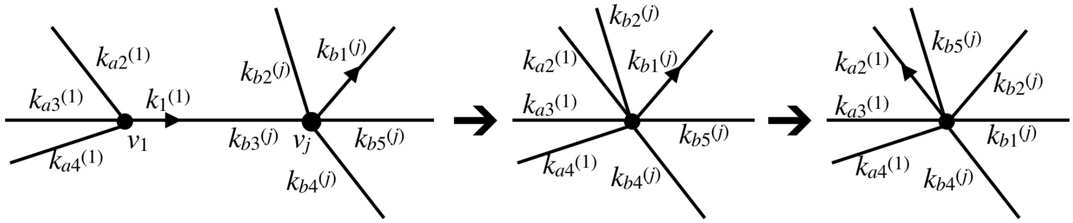

FIGURE 4.1.

Further, since the weight c counts the degree of the underlying Catalan graph, and the graph resulting from contracting edge E has degree two less than the original graph, we thus have a factor of c^2 in front of the resulting term.

This gives the first term in the above formula,

$$c^2 \sum_{j=2}^{v} n_j \, \widetilde{M}_{g,v-1}(n_1 + n_j - 2, n_2, \ldots, \hat{n_j}, \ldots, n_v; b; c).$$

Case 2. Assume E is a loop on v_1.

Just as in the proof of the analogous result for generalized Catalan numbers, contracting E separates the incident half-edges at v_1 into two collections, with α edges on one side and β edges on the other (note that α or β may be zero). Since E is a loop, contracting it causes pinching on the surface and produces a double point. We separate the double point into two new vertices. The result may be a surface of genus $g-1$, or two surfaces, of genus g_1 and g_2 with $g_1 + g_2 = g$. Note that we do not need to re-assign the colorings in this case.

Observe that, since we contracted an edge which was incident twice on v_1, there are two less colors with which we can color the resulting vertices. Further, the remaining colors must now be shared between two vertices. (See Figure 4.2.)

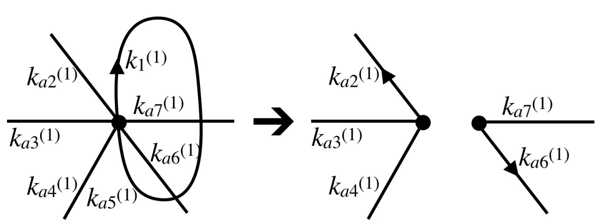

FIGURE 4.2.

As in the previous case, since the weight c counts the degree of the underlying Catalan graph, and the sum of the degrees of the graphs resulting from contracting edge E is two less than the degree of the original graph, we are left with a factor of c^2 in front of the resulting terms.

This gives the remaining terms in the above formula,

$$c^2 \sum_{\zeta+\xi=n_1-2} \Bigg[\widetilde{M}_{g-1,v+1}(\zeta,\xi,n_2,\ldots,n_v;b,c) \\ + \sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\}} \widetilde{M}_{g_1,|I|+1}(\zeta,n_I;b,c)\widetilde{M}_{g_2,|J|+1}(\xi,n_J;b,c) \Bigg]$$

□

We now present the algebraic proof of this recursion formula for generalized bc-Motzkin numbers.

Proof. We first apply Definition 3.0.2 of the generalized bc-Motzkin numbers, and Pascal's identity, to simplify the left side of the formula in Theorem 4.2.1.

$$\begin{aligned} &\widetilde{M}_{g,v}(n_1,n_2,\ldots,n_v;b,c) - b\widetilde{M}_{g,v}(n_1-1,n_2,\ldots,n_v;b,c) \\ &\qquad = \sum_{\mu_1=0}^{n_1}\sum_{\mu_2=0}^{n_2}\cdots\sum_{\mu_v=0}^{n_v} \binom{n_1-1}{\mu_1-1}\binom{n_2}{\mu_2}\cdots\binom{n_v}{\mu_v} \\ &\qquad\qquad \cdot C_{g,v}(\mu_1,\ldots,\mu_v)\, b^{(n_1+\cdots+n_v)-(\mu_1+\cdots+\mu_v)} c^{\mu_1+\cdots+\mu_v} \end{aligned} \tag{4.4}$$

Then, we plug in the Catalan "recursion" formula of Proposition 2.0.1 to the above equation, simplify the resulting three terms, and re-write them in terms of generalized bc-Motzkin numbers. The key ingredients in this proof are Vandermonde's identity,

$$\sum_{i+j=k}\binom{a}{i}\binom{b}{j} = \binom{a+b}{k}, \tag{4.5}$$

and a particular version of the "Vandermonde-like" identity in Proposition 4.1.2,

$$\sum_{\zeta+\xi=k}\binom{\zeta-1}{\alpha-1}\binom{\xi}{\beta} = \binom{k}{\alpha+\beta}. \tag{4.6}$$

For notational convenience, we let

$$\begin{aligned}\widetilde{M}_{g,v}(n_1,\ldots,n_v;b,c) - b\widetilde{M}_{g,v}(n_1-1,n_2,\ldots,n_v;b,c)\\ = \mathrm{I}_{g,v}(n_1,\ldots,n_v;b,c) + \mathrm{II}_{g,v}(n_1,\ldots,n_v;b,c) + \mathrm{III}_{g,v}(n_1,\ldots,n_v;b,c),\end{aligned} \tag{4.7}$$

where the three terms correspond to the three terms of the Catalan "recursion" formula.

To shorten notation, we will sometimes write

$$\vec{n} = (n_1,\ldots,n_v)$$

and

$$|\vec{n}| = n_1 + \cdots + n_v,$$

and similarly for $\vec{\mu}$.

For the first term, we have

$$\begin{aligned}\mathrm{I}_{g,v}(\vec{n};b,c) = \sum_{j=2}^{v}\sum_{\mu_2=0}^{n_2}\cdots\sum_{\mu_{j-1}=0}^{n_{j-1}}\sum_{\mu_{j+1}=0}^{n_{j+1}}\cdots\sum_{\mu_v=0}^{n_v}\binom{n_2}{\mu_2}\cdots\binom{n_{j-1}}{\mu_{j-1}}\binom{n_{j+1}}{\mu_{j+1}}\cdots\binom{n_v}{\mu_v}\\ \cdot\left[\sum_{\mu_1=0}^{n_1}\sum_{\mu_j=0}^{n_j}\binom{n_1-1}{\mu_1-1}\binom{n_j}{\mu_j}\mu_j C_{g,v-1}(\mu_1+\mu_j-2,\mu_2,\ldots,\hat{\mu_j},\ldots,\mu_v)\, b^{|\vec{n}|-|\vec{\mu}|}c^{|\vec{\mu}|}\right].\end{aligned}$$

Now, since

$$\binom{n_j}{\mu_j}\cdot\mu_j = \binom{n_j-1}{\mu_j-1}\cdot n_j,$$

the term in brackets becomes

$$\begin{aligned}&\sum_{\mu_1=1}^{n_1}\sum_{\mu_j=1}^{n_j}\binom{n_1-1}{\mu_1-1}\binom{n_j-1}{\mu_j-1}n_j\, C_{g,v-1}(\mu_1+\mu_j-2,\mu_2,\ldots,\hat{\mu_j},\ldots,\mu_v)\, b^{|\vec{n}|-|\vec{\mu}|}c^{|\vec{\mu}|}\\ &=\sum_{\mu_1=0}^{n_1-1}\sum_{\mu_j=0}^{n_j-1}\binom{n_1-1}{\mu_1}\binom{n_j-1}{\mu_j}n_j\, C_{g,v-1}(\mu_1+\mu_j,\mu_2,\ldots,\hat{\mu_j},\ldots,\mu_v)\, b^{|\vec{n}|-(2+|\vec{\mu}|)}c^{2+|\vec{\mu}|}\\ &=\sum_{k=0}^{n_1+n_j-2}\sum_{\mu_1+\mu_j=k}\binom{n_1-1}{\mu_1}\binom{n_j-1}{\mu_j}n_j\, C_{g,v-1}(k,\mu_2,\ldots,\hat{\mu_j},\ldots,\mu_v)\\ &\quad\cdot b^{|\vec{n}|-(2+k+\mu_2+\cdots+\widehat{\mu_j}+\cdots+\mu_v)}c^{(2+k+\mu_2+\cdots+\widehat{\mu_j}+\cdots+\mu_v)}\end{aligned}$$

$$= \sum_{k=0}^{n_1+n_j-2} \binom{n_1+n_j-2}{k} n_j\, C_{g,v-1}(k,\mu_2,\ldots,\hat{\mu_j},\ldots,\mu_v)$$

$$\cdot\, c^2 b^{((n_1+n_j-2)+n_2+\cdots+\widehat{n_j}+\cdots+n_v)-(k+\mu_2+\cdots+\widehat{\mu_j}+\cdots+\mu_v)} c^{(k+\mu_2+\cdots+\widehat{\mu_j}+\cdots+\mu_v)}$$

where we have applied Vandermonde's identity (4.5).

Substituting this back into the above expression for $\mathrm{I}_{g,v}(\vec{n};b,c)$ then gives

$$\mathrm{I}_{g,v}(\vec{n};b,c) = c^2 \sum_{j=2}^{v} n_j\, \widetilde{M}_{g,v-1}(n_1+n_j-2,n_2,\ldots,\hat{n_j},\ldots,n_v;b,c).$$

Now, for the second term, we may proceed as follows, using the Vandermonde-like identity of equation (4.6).

$$\mathrm{II}_{g,v}(\vec{n};b,c) = \sum_{\mu_2=0}^{n_2}\cdots\sum_{\mu_v=0}^{n_v}\binom{n_2}{\mu_2}\cdots\binom{n_v}{\mu_v}\Bigg[\sum_{\mu_1=0}^{n_1}\sum_{\alpha+\beta=\mu_1-2}\binom{n_1-1}{\mu_1-1}\Bigg]$$

$$\cdot\, C_{g-1,v+1}(\alpha,\beta,\mu_2,\ldots,\mu_v)\, b^{|\vec{n}|-|\vec{\mu}|}c^{|\vec{\mu}|}$$

$$= \sum_{\mu_2=0}^{n_2}\cdots\sum_{\mu_v=0}^{n_v}\binom{n_2}{\mu_2}\cdots\binom{n_v}{\mu_v}\Bigg[\sum_{\mu_1=0}^{n_1}\sum_{\alpha+\beta=\mu_1-2}\sum_{k=0}^{n_1-2}\binom{k}{\mu_1-2}\Bigg]$$

$$\cdot\, C_{g-1,v+1}(\alpha,\beta,\mu_2,\ldots,\mu_v)\, b^{|\vec{n}|-|\vec{\mu}|}c^{|\vec{\mu}|}$$

$$= \sum_{\mu_2=0}^{n_2}\cdots\sum_{\mu_v=0}^{n_v}\binom{n_2}{\mu_2}\cdots\binom{n_v}{\mu_v}\Bigg[\sum_{\mu_1=0}^{n_1}\sum_{\alpha+\beta=\mu_1-2}\sum_{k=0}^{n_1-2}\sum_{\zeta+\xi=k}\binom{\zeta-1}{\alpha-1}\binom{\xi}{\beta}\Bigg]$$

$$\cdot\, C_{g-1,v+1}(\alpha,\beta,\mu_2,\ldots,\mu_v)\, b^{|\vec{n}|-|\vec{\mu}|}c^{|\vec{\mu}|}$$

$$= \sum_{\mu_2=0}^{n_2}\cdots\sum_{\mu_v=0}^{n_v}\binom{n_2}{\mu_2}\cdots\binom{n_v}{\mu_v}\Bigg[\sum_{k=0}^{n_1-2}\sum_{\zeta+\xi=k}\sum_{\alpha=0}^{\zeta}\sum_{\beta=0}^{\xi}\binom{\zeta-1}{\alpha-1}\binom{\xi}{\beta}\Bigg]$$

$$\cdot\, C_{g-1,v+1}(\alpha,\beta,\mu_2,\ldots,\mu_v)\, b^{|\vec{n}|-((\alpha+\beta+2)+\mu_2+\cdots+\mu_v)}c^{(\alpha+\beta+2)+\mu_2+\cdots+\mu_v}$$

$$= \sum_{k=0}^{n_1-2}\sum_{\zeta+\xi=k}\sum_{\alpha=0}^{\zeta}\sum_{\beta=0}^{\xi}\sum_{\mu_2=0}^{n_2}\cdots\sum_{\mu_v=0}^{n_v}\binom{\zeta-1}{\alpha-1}\binom{\xi}{\beta}\binom{n_2}{\mu_2}\cdots\binom{n_v}{\mu_v}$$

$$\cdot\, C_{g-1,v+1}(\alpha,\beta,\mu_2,\ldots,\mu_v)\, b^{n_1-2-k}b^{((\zeta+\xi)+n_2+\cdots+n_v)-((\alpha+\beta)+\mu_2+\cdots+\mu_v)}$$

$$\cdot\, c^{(\alpha+\beta+2)+\mu_2+\cdots+\mu_v}$$

$$= c^2 \sum_{k=0}^{n_1-2} b^{(n_1-2)-k} \sum_{\zeta+\xi=k} \Big[\widetilde{M}_{g-1,v+1}(\zeta,\xi,n_2,\dots,n_v;b,c)$$

$$- b\widetilde{M}_{g-1,v+1}(\zeta-1,\xi,n_2,\dots,n_v;b,c) \Big]$$

$$= c^2 \Big[\sum_{k=1}^{n_1-2} b^{(n_1-2)-k} \sum_{\zeta+\xi=k} \widetilde{M}_{g-1,v+1}(\zeta,\xi,n_2,\dots,n_v;b,c)$$

$$- \sum_{k=1}^{n_1-2} b^{(n_1-2)-(k-1)} \sum_{\zeta+\xi=k-1} \widetilde{M}_{g-1,v+1}(\zeta,\xi,n_2,\dots,n_v;b,c) \Big]$$

$$= c^2 \Big[\sum_{k=1}^{n_1-2} b^{(n_1-2)-k} \sum_{\zeta+\xi=k} \widetilde{M}_{g-1,v+1}(\zeta,\xi,n_2,\dots,n_v;b,c)$$

$$- \sum_{k=0}^{n_1-3} b^{(n_1-2)-k} \sum_{\zeta+\xi=k} \widetilde{M}_{g-1,v+1}(\zeta,\xi,n_2,\dots,n_v;b,c) \Big]$$

$$= c^2 b^{(n_1-2)-(n_1-2)} \sum_{\zeta+\xi=n_1-2} \widetilde{M}_{g-1,v+1}(\zeta,\xi,n_2,\dots,n_v;b,c)$$

$$= c^2 \sum_{\zeta+\xi=n_1-2} \widetilde{M}_{g-1,v+1}(\zeta,\xi,n_2,\dots,n_v;b,c).$$

Finally, for the third term, we have

$$\mathrm{III}_{g,v}(\vec{n};b,c) = \sum_{g_1+g_2=g,\ I\sqcup J=\{2,\dots,v\}} \sum_{\mu_2=0}^{n_2} \cdots \sum_{\mu_v=0}^{n_v} \binom{n_2}{\mu_2} \cdots \binom{n_v}{\mu_v}$$

$$\cdot \Big[\sum_{\mu_1=0}^{n_1} \sum_{\alpha+\beta=\mu_1-2} \binom{n_1-1}{\mu_1-1} \Big] C_{g_1,|I|+1}(\alpha,\mu_I) C_{g_2,|J|+1}(\beta,\mu_J) b^{|\vec{n}|-|\vec{\mu}|} c^{|\vec{\mu}|}$$

$$= \sum_{g_1+g_2=g,\ I\sqcup J=\{2,\dots,v\}} \sum_{\mu_2=0}^{n_2} \cdots \sum_{\mu_v=0}^{n_v} \binom{n_2}{\mu_2} \cdots \binom{n_v}{\mu_v}$$

$$\cdot \Big[\sum_{\mu_1=0}^{n_1} \sum_{\alpha+\beta=\mu_1-2} \sum_{\zeta+\xi=n_1-2} \binom{\zeta}{\alpha} \binom{\xi}{\beta} \Big] C_{g_1,|I|+1}(\alpha,\mu_I) C_{g_2,|J|+1}(\beta,\mu_J)$$

$$\cdot\, b^{((\zeta+\xi)+n_2+\cdots+n_v)-((\alpha+\beta)+\mu_2+\cdots+\mu_v)} c^{(\alpha+\beta+2)+\mu_2+\cdots+\mu_v}$$

$$= c^2 \sum_{g_1+g_2=g,\ I\sqcup J=\{2,\dots,v\}} \sum_{\mu_2=0}^{n_2} \cdots \sum_{\mu_v=0}^{n_v} \binom{n_2}{\mu_2} \cdots \binom{n_v}{\mu_v}$$

$$\cdot \Big[\sum_{\zeta+\xi=n_1-2} \sum_{\alpha=0}^{\zeta} \sum_{\beta=0}^{\xi} \binom{\zeta}{\alpha} \binom{\xi}{\beta} \Big] C_{g_1,|I|+1}(\alpha,\mu_I) C_{g_2,|J|+1}(\beta,\mu_J)$$

$$
\cdot\, b^{(\zeta+\xi+n_2+\cdots+n_v)-(\alpha+\beta+\mu_2+\cdots+\mu_v)} c^{\alpha+\beta+\mu_2+\cdots+\mu_v}
$$

$$
= c^2 \sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\}} \sum_{\zeta+\xi=n_1-2} \widetilde{M}_{g_1,|I|+1}(\zeta, n_I; b, c)\widetilde{M}_{g_2,|J|+1}(\xi, n_J; b, c)
$$

where we again used the Vandermonde-like identity of equation (4.6).

Putting all this back into equation (4.7) thus completes the proof. $\square$

CHAPTER 5

Differential Recursion for Generalized bc-Motzkin Numbers

We now wish to use the formula in Theorem 4.2.1 to obtain a topological recursion. In analogy with the approach in [**7**], we first define the discrete Laplace transform of the generalized bc-Motzkin numbers, from which we will then obtain a differential recursion formula.

Definition 5.0.1. *We define the discrete Laplace Transform of the generalized bc-Motzkin numbers by*

$$F_{g,v}^{\widetilde{M}(b,c)}(x_1,\ldots,x_v) = \begin{cases} \sum_{n=1}^{\infty} \frac{\widetilde{M}_{0,1}(n;b,c)}{n} x_1^{-n} - \widetilde{M}_{0,1}(0;b,c)\log x_1 & \text{if } (g,v)=(0,1), \\ \sum_{n_1=1}^{\infty} \cdots \sum_{n_v=1}^{\infty} \frac{\widetilde{M}_{g,v}(n_1,n_2,\ldots,n_v;b,c)}{n_1 n_2 \cdots n_v} x_1^{-n_1} \cdots x_v^{-n_v}. \end{cases} \tag{5.1}$$

We would like to obtain a recursion on

$$\frac{\partial}{\partial t_1} F_{g,v}^{\widetilde{M}(b,c)}(t_1,t_2,\ldots,t_v),$$

for a particular choice of change of variable t_i, similar to the Catalan differential recursion given in Proposition 2.0.2.

We will first study the case when $(g,v)=(0,1)$. We differentiate the definition of $F_{0,1}^{\widetilde{M}(b,c)}(x)$ with respect to $x=x_1$ and apply the recursion formula for $(0,1)$-bc-Motzkin numbers from Proposition 4.1.1 to obtain

$$\begin{aligned} \frac{\partial}{\partial x} F_{0,1}^{\widetilde{M}(b,c)}(x) &= -\sum_{n=1}^{\infty} \widetilde{M}_{0,1}(n;b,c)x^{-n-1} - \widetilde{M}_{0,1}(0;b,c)x^{-1} \\ &= -x^{-1} - \sum_{n=1}^{\infty} \Bigg(b\widetilde{M}_{0,1}(n-1;b,c) \\ &\qquad + c^2 \sum_{\zeta+\xi=n-2} \widetilde{M}_{0,1}(\zeta;b,c)\widetilde{M}_{0,1}(\xi;b,c) \Bigg) x^{-n-1} \end{aligned}$$

$$
\begin{aligned}
&= -x^{-1} - b\sum_{k+1=1}^{\infty} \widetilde{M}_{0,1}(k;b,c)\,x^{-(k+1)-1} \\
&\quad - c^2\sum_{\zeta=0}^{\infty}\sum_{\xi=0}^{\infty} \widetilde{M}_{0,1}(\zeta;b,c)\widetilde{M}_{0,1}(\xi;b,c)\,x^{-(\zeta+\xi+2)-1} \\
&= -x^{-1} - bx^{-1}\sum_{k=0}^{\infty} \widetilde{M}_{0,1}(k;b,c)\,x^{-k-1} \\
&\quad - c^2x^{-1}\sum_{\zeta=0}^{\infty}\sum_{\xi=0}^{\infty} \widetilde{M}_{0,1}(\zeta;b,c)\widetilde{M}_{0,1}(\xi;b,c)\,x^{-\zeta-1}x^{-\xi-1} \\
&= -x^{-1} - bx^{-1}\left(-\frac{\partial}{\partial x}F_{0,1}^{\widetilde{M}(b,c)}(x)\right) - c^2x^{-1}\left(-\frac{\partial}{\partial x}F_{0,1}^{\widetilde{M}(b,c)}(x)\right)^2
\end{aligned}
$$

This implies

$$
c^2\left(\frac{\partial}{\partial x}F_{0,1}^{\widetilde{M}(b,c)}(x)\right)^2 + (x-b)\left(\frac{\partial}{\partial x}F_{0,1}^{\widetilde{M}(b,c)}(x)\right) + 1 = 0.
$$

Solving for $\partial/\partial x\, F_{0,1}^{\widetilde{M}(b,c)}(x)$ and applying the quadratic formula then implies

$$
\frac{\partial}{\partial x}F_{0,1}^{\widetilde{M}(b,c)}(x) = \frac{1}{2c}\left[-\left(\frac{x-b}{c}\right) - \sqrt{\left(\frac{x-b}{c}\right)^2 - 4}\,\right],
$$

where we took the negative square root in the quadratic formula.

Thus, we want to make a particular choice of change of variable so that the quantity under the square root is a perfect square, and we can simplify the expression for $\partial/\partial x\, F_{0,1}^{\widetilde{M}(b,c)}(x)$ further. We also want to choose the change of variable in such a way that it reduces to the change of variables for the Catalan case, which was given in equation (2.4), when $b = 0$ and $c = 1$.

With this in mind, we let

$$
\frac{x-b}{c} = 2 + \frac{4}{t^2-1} = 2\frac{t^2+1}{t^2-1}.
$$

So,

$$
\frac{\partial x}{\partial t} = c\frac{-8t}{(t^2-1)^2}.
$$

Then, applying this change of variable, we obtain

$$
\frac{\partial}{\partial t}F_{0,1}^{\widetilde{M}(b,c)}(t) = \frac{\partial x}{\partial t}\frac{\partial}{\partial x}F_{0,1}^{\widetilde{M}(b,c)}(x)
$$

$$= \left(c\frac{-8t}{(t^2-1)^2}\right)\frac{1}{2c}\left[-2\frac{t^2+1}{t^2-1} - \sqrt{\left(2\frac{t^2+1}{t^2-1}\right)^2 - 4}\,\right]$$
$$= \frac{-8t}{(t^2-1)^2}\left[-\frac{t^2+1}{t^2-1} - \sqrt{\frac{(t^2+1)^2-(t^2-1)^2}{(t^2-1)^2}}\,\right]$$
$$= \frac{8t}{(t+1)(t-1)^3}.$$

Thus, we have shown that

$$\frac{\partial}{\partial t}F_{0,1}^{\widetilde{M}(b,c)}(t) = \frac{8t}{(t+1)(t-1)^3}. \tag{5.2}$$

Observe this is precisely the same formula in terms of t as was obtained for $\partial/\partial t F_{0,1}^{C}(t)$ in the Catalan case, and, when $b=0$ and $c=1$, the above change of variable formula does indeed reduce to the change of variable formula (2.4) used in the Catalan case.

More generally, let $t_i = t_i(x_i, b, c)$, for $i \in \{1, 2, \ldots, v\}$, be defined by

$$\frac{x_i - b}{c} = 2 + \frac{4}{t_i^2 - 1}. \tag{5.3}$$

Then, we are led to prove the following new result.

THEOREM 5.0.1. *The discrete Laplace transform $F_{g,v}^{\widetilde{M}(b,c)}(t_1, t_2, \ldots, t_v)$ satisfies the following differential recursion formula, for every $(g,v) \neq (0,1), (0,2)$:*

$$\begin{aligned}
\frac{\partial}{\partial t_1}F_{g,v}^{\widetilde{M}(b,c)}(t_1,t_2,\ldots,t_v) = &-\frac{1}{16}\sum_{j=2}^{v}\left[\frac{t_j}{t_1^2-t_j^2}\left(\frac{(t_1^2-1)^3}{t_1^2}\frac{\partial}{\partial t_1}F_{g,v-1}^{\widetilde{M}(b,c)}(t_1,\ldots,\widehat{t_j},\ldots,t_v)\right.\right.\\
&\left.\left.-\frac{(t_j^2-1)^3}{t_j^2}\frac{\partial}{\partial t_j}F_{g,v-1}^{\widetilde{M}(b,c)}(t_2,\ldots,t_v)\right)\right]\\
&-\frac{1}{16}\sum_{j=2}^{v}\frac{(t_1^2-1)^2}{t_1^2}\left[\frac{\partial}{\partial t_1}F_{g,v-1}^{\widetilde{M}(b,c)}(t_1,\ldots,\widehat{t_j},\ldots,t_v)\right]\\
&-\frac{1}{32}\frac{(t_1^2-1)^3}{t_1^2}\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}F_{g-1,v+1}^{\widetilde{M}(b,c)}(u_1,u_2,t_2,\ldots,t_v)\Big|_{u_1=u_2=t_1}\\
&-\frac{1}{32}\frac{(t_1^2-1)^3}{t_1^2}\sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ stable}\frac{\partial}{\partial t_1}F_{g_1,|I|+1}^{\widetilde{M}(b,c)}(t_1,t_I)\cdot\frac{\partial}{\partial t_1}F_{g_2,|J|+1}^{\widetilde{M}(b,c)}(t_1,t_J)
\end{aligned} \tag{5.4}$$

where the "stable" summation means $2g_1 + |I| - 1 > 0$ and $2g_2 + |J| - 1 > 0$.

And,

$$\frac{\partial}{\partial t_1} F_{0,2}^{\widetilde{M}(b,c)}(t_1, t_2) = \frac{(t_2+1)}{(t_1-1)(t_1+t_2)}. \tag{5.5}$$

Remark 5.0.1. With this choice of change of variable, the formula in Theorem 5.0.1 has no dependence on b and c. The dependence on b and c only appears in the definition of $t_i(x_i, b, c)$. Further, this result has precisely the same form as the Catalan differential recursion formula in Proposition 2.0.2, and the choice of t_i in that case corresponds to $t_i(x_i, 0, 1)$, which is as expected, since the bc-Motzkin numbers reduce to the Catalan numbers in the case when $b = 0$ and $c = 1$.

To prove this result, we may proceed as follows.

Proof. We first differentiate the formula for $F_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v)$ with respect to x_1, then plug in the bc-Motzkin "recursion" formula from Theorem 4.2.1.

Since that formula has four terms, for notational convenience we may write

$$\begin{aligned}\frac{\partial}{\partial x_1} F_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) &= -\sum_{n_1=1}^{\infty} \cdots \sum_{n_v=1}^{\infty} \frac{\widetilde{M}_{g,v}(n_1, n_2, \ldots, n_v; b, c)}{n_2 \cdots n_v} x_1^{-n_1-1} x_2^{-n_2} \cdots x_v^{-n_v} \\ &= \mathrm{I}_{g,v}(x_1, x_2, \ldots, x_v) + \mathrm{II}_{g,v}(x_1, x_2, \ldots, x_v) \\ &\qquad + \mathrm{III}_{g,v}(x_1, x_2, \ldots, x_v) + \mathrm{IV}_{g,v}(x_1, x_2, \ldots, x_v). \end{aligned} \tag{5.6}$$

Then, for the first term, we may compute

$$\begin{aligned}\mathrm{I}_{g,v}(x_1, x_2, \ldots, x_v) &= \sum_{n_1=1}^{\infty} \cdots \sum_{n_v=1}^{\infty} \left[b\widetilde{M}_{g,v}(n_1-1, n_2, \ldots, n_v; b, c)\right] \frac{x_1^{-n_1-1} x_2^{-n_2} \cdots x_v^{-n_v}}{n_2 \cdots n_v} \\ &= b x_1^{-1} \sum_{k=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_v=1}^{\infty} \frac{\widetilde{M}_{g,v}(k, n_2, \ldots, n_v; b, c)}{n_2 \cdots n_v} x_1^{-k-1} x_2^{-n_2} \cdots x_v^{-n_v} \\ &= -b x_1^{-1} \frac{\partial}{\partial x_1} F_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v).\end{aligned}$$

For the second term,

$$\begin{aligned}&\mathrm{II}_{g,v}(x_1, x_2, \ldots, x_v) \\ &= \sum_{n_1=1}^{\infty} \cdots \sum_{n_v=1}^{\infty} \left[c^2 \sum_{j=2}^{v} n_j \, \widetilde{M}_{g,v-1}(n_1+n_j-2, n_2, \ldots, \widehat{n_j}, \ldots, n_v; b; c)\right] \frac{x_1^{-n_1-1} x_2^{-n_2} \cdots x_v^{-n_v}}{n_2 \cdots n_v}\end{aligned}$$

$$= c^2 \sum_{j=2}^{v} \sum_{n_1=1}^{\infty} \cdots \sum_{n_v=1}^{\infty} \frac{\widetilde{M}_{g,v-1}(n_1+n_j-2, n_2, \ldots, \widehat{n_j}, \ldots, n_v; b; c)}{n_2 \cdots \widehat{n_j} \cdots n_v} x_1^{-n_1-1} x_2^{-n_2} \cdots x_v^{-n_v}$$

$$= c^2 \sum_{j=2}^{v} \sum_{k=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \widehat{\sum_{n_j=1}^{\infty}} \cdots \sum_{n_v=1}^{\infty} \sum_{n_1+n_j=k+2} \frac{\widetilde{M}_{g,v-1}(k, n_2, \ldots, \widehat{n_j}, \ldots, n_v; b; c)}{n_2 \cdots \widehat{n_j} \cdots n_v}$$

$$\cdot (x_1^{-n_1-1} x_j^{-n_j}) x_2^{-n_2} \cdots \widehat{x_j^{-n_j}} \cdots x_v^{-n_v}$$

$$= c^2 \sum_{j=2}^{v} \sum_{k=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \widehat{\sum_{n_j=1}^{\infty}} \cdots \sum_{n_v=1}^{\infty} \frac{\widetilde{M}_{g,v-1}(k, n_2, \ldots, \widehat{n_j}, \ldots, n_v; b; c)}{n_2 \cdots \widehat{n_j} \cdots n_v}$$

$$\cdot \left[\sum_{n_1+n_j=k+2} (x_1^{-n_1-1} x_j^{-n_j}) \right] x_2^{-n_2} \cdots \widehat{x_j^{-n_j}} \cdots x_v^{-n_v}$$

$$= c^2 \sum_{j=2}^{v} \sum_{k=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \widehat{\sum_{n_j=1}^{\infty}} \cdots \sum_{n_v=1}^{\infty} \frac{\widetilde{M}_{g,v-1}(k, n_2, \ldots, \widehat{n_j}, \ldots, n_v; b; c)}{n_2 \cdots \widehat{n_j} \cdots n_v}$$

$$\left[\frac{x_1^{-k-1} - x_j^{-k-1}}{x_1(x_j - x_1)} \right] x_2^{-n_2} \cdots \widehat{x_j^{-n_j}} \cdots x_v^{-n_v}$$

$$= c^2 \sum_{j=2}^{v} \frac{1}{x_1(x_j - x_1)} \left[\sum_{k=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \widehat{\sum_{n_j=1}^{\infty}} \cdots \sum_{n_v=1}^{\infty} \frac{\widetilde{M}_{g,v-1}(k, n_2, \ldots, \widehat{n_j}, \ldots, n_v; b; c)}{n_2 \cdots \widehat{n_j} \cdots n_v} \right.$$

$$\left. \cdot \left[x_1^{-k-1} x_2^{-n_2} \cdots \widehat{x_j^{-n_j}} \cdots x_v^{-n_v} - x_2^{-n_2} \cdots x_{j-1}^{-n_{j-1}} x_j^{-k-1} x_{j+1}^{-n_{j+1}} \cdots x_v^{-n_v} \right] \right]$$

$$= c^2 \sum_{j=2}^{v} \frac{1}{x_1(x_j - x_1)} \left[-\frac{\partial}{\partial x_1} F_{g,v-1}^{\widetilde{M}(b,c)}(x_1, \ldots, \widehat{x_j}, \ldots, x_v) + \frac{\partial}{\partial x_j} F_{g,v-1}^{\widetilde{M}(b,c)}(x_2, \ldots, x_v) \right].$$

where we used the fact that

$$\sum_{n_1+n_j=k+2,\, n_1,n_j>0} x_1^{-n_1-1} x_j^{-n_j} = \frac{x_1^{-k-1} - x_j^{-k-1}}{x_1(x_j - x_1)}.$$

For the third term,

$$\mathrm{III}_{g,v}(x_1, x_2, \ldots, x_v)$$

$$= \sum_{n_1=1}^{\infty} \cdots \sum_{n_v=1}^{\infty} \left[c^2 \sum_{\zeta+\xi=n_1-2} \widetilde{M}_{g-1,v+1}(\zeta, \xi, n_2, \ldots, n_v; b, c) \right] \frac{x_1^{-n_1-1} x_2^{-n_2} \cdots x_v^{-n_v}}{n_2 \cdots n_v}$$

$$= c^2 \sum_{\zeta=1}^{\infty} \sum_{\xi=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_v=1}^{\infty} \left[\widetilde{M}_{g-1,v+1}(\zeta, \xi, n_2, \ldots, n_v; b, c) \right] \frac{x_1^{-(\zeta+\xi+2)-1} x_2^{-n_2} \cdots x_v^{-n_v}}{n_2 \cdots n_v}$$

$$= c^2 x_1^{-1} \sum_{\zeta=1}^{\infty} \sum_{\xi=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_v=1}^{\infty} \frac{\widetilde{M}_{g-1,v+1}(\zeta, \xi, n_2, \ldots, n_v; b, c)}{n_2 \cdots n_v} x_1^{-\zeta-1} x_1^{\xi-1} x_2^{-n_2} \cdots x_v^{-n_v}$$

$$= c^2 x_1^{-1} \frac{\partial}{\partial u_1} \frac{\partial}{\partial u_2} F_{g-1,v+1}^{\widetilde{M}(b,c)}(u_1, u_2, x_2, \ldots, x_v) \Big|_{u_1=u_2=x_1}.$$

For the fourth term,

$$\begin{aligned}
&\mathrm{IV}_{g,v}(x_1, x_2, \ldots, x_v) \\
&= \sum_{n_1=1}^{\infty} \cdots \sum_{n_v=1}^{\infty} \Bigg[c^2 \sum_{\zeta+\xi=n_1-2} \sum_{g_1+g_2=g,\ I \sqcup J=\{2,\ldots,v\}} \widetilde{M}_{g_1,|I|+1}(\zeta, n_I; b, c) \widetilde{M}_{g_2,|J|+1}(\xi, n_J; b, c) \Bigg] \\
&\quad \cdot \frac{x_1^{-n_1-1} x_2^{-n_2} \cdots x_v^{-n_v}}{n_2 \cdots n_v} \\
&= c^2 \sum_{g_1+g_2=g,\ I \sqcup J=\{2,\ldots,v\}} \Bigg[\sum_{\zeta=1}^{\infty} \sum_{n_I=1}^{\infty} \sum_{\xi=1}^{\infty} \sum_{n_J=1}^{\infty} \widetilde{M}_{g_1,|I|+1}(\zeta, n_I; b, c) \widetilde{M}_{g_2,|J|+1}(\xi, n_J; b, c) \Bigg] \\
&\quad \cdot \frac{x_1^{-(\zeta+\xi+2)-1} \prod_{i \in I} x_i^{-n_i} \prod_{j \in J} x_j^{-n_j}}{\prod_{i \in I} n_i \prod_{j \in J} n_j} \\
&= c^2 x_1^{-1} \sum_{g_1+g_2=g,\ I \sqcup J=\{2,\ldots,v\}} \frac{\partial}{\partial x_1} F_{g_1,|I|+1}^{\widetilde{M}(b,c)}(x_1, x_I) \cdot \frac{\partial}{\partial x_1} F_{g_2,|J|+1}^{\widetilde{M}(b,c)}(x_1, x_J).
\end{aligned}$$

Upon substituting these four terms back into the original equation (5.6), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial x_1} F_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) \\
&= b x_1^{-1} \frac{\partial}{\partial x_1} F_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) \\
&\quad - c^2 \sum_{j=2}^{v} \frac{1}{x_1(x_j - x_1)} \Bigg[- \frac{\partial}{\partial x_1} F_{g,v-1}^{\widetilde{M}(b,c)}(x_1, \ldots, \widehat{x_j}, \ldots, x_v) + \frac{\partial}{\partial x_j} F_{g,v-1}^{\widetilde{M}(b,c)}(x_2, \ldots, x_v) \Bigg] \\
&\quad - c^2 x_1^{-1} \frac{\partial}{\partial u_1} \frac{\partial}{\partial u_2} F_{g-1,v+1}^{\widetilde{M}(b,c)}(u_1, u_2, x_2, \ldots, x_v) \Big|_{u_1=u_2=x_1} \\
&\quad - c^2 x_1^{-1} \sum_{g_1+g_2=g,\ I \sqcup J=\{2,\ldots,v\}} \frac{\partial}{\partial x_1} F_{g_1,|I|+1}^{\widetilde{M}(b,c)}(x_1, x_I) \cdot \frac{\partial}{\partial x_1} F_{g_2,|J|+1}^{\widetilde{M}(b,c)}(x_1, x_J).
\end{aligned}$$

We will first look at the particular case when $(g, v) = (0, 2)$. Then, this becomes

$$\frac{\partial}{\partial x_1} F_{0,2}^{\widetilde{M}(b,c)}(x_1, x_2) = b x_1^{-1} \frac{\partial}{\partial x_1} F_{0,2}^{\widetilde{M}(b,c)}(x_1, x_2)$$

$$
\begin{aligned}
&- c^2 \frac{1}{x_1(x_2 - x_1)} \left[- \frac{\partial}{\partial x_1} F_{0,1}^{\widetilde{M}(b,c)}(x_1) + \frac{\partial}{\partial x_2} F_{0,1}^{\widetilde{M}(b,c)}(x_2) \right] \\
&- 2c^2 x_1^{-1} \frac{\partial}{\partial x_1} F_{0,1}^{\widetilde{M}(b,c)}(x_1) \cdot \frac{\partial}{\partial x_1} F_{0,2}^{\widetilde{M}(b,c)}(x_1, x_2).
\end{aligned}
$$

This implies

$$
\begin{aligned}
0 = &\left(\frac{x_1 - b}{c^2} + 2 \frac{\partial}{\partial x_1} F_{0,1}^{\widetilde{M}(b,c)}(x_1) \right) \frac{\partial}{\partial x_1} F_{0,2}^{\widetilde{M}(b,c)}(x_1, x_2) \\
&+ \frac{1}{(x_2 - x_1)} \left(- \frac{\partial}{\partial x_1} F_{0,1}^{\widetilde{M}(b,c)}(x_1) + \frac{\partial}{\partial x_2} F_{0,1}^{\widetilde{M}(b,c)}(x_2) \right).
\end{aligned}
$$

The change of variables formula in equation (5.3) then implies

$$
\frac{\partial x_i}{\partial t_i} = c \frac{-8t_i}{(t_i^2 - 1)^2}.
$$

Now, observe that

$$
\begin{aligned}
\frac{1}{(x_j - x_1)} &= \frac{1}{\left(2c \frac{t_j^2 + 1}{t_j^2 - 1} + b \right) - \left(2c \frac{t_1^2 + 1}{t_1^2 - 1} + b \right)} \\
&= \frac{1}{2c} \frac{(t_1^2 - 1)(t_j^2 - 1)}{(t_j^2 + 1)(t_1^2 - 1) - (t_1^2 + 1)(t_j^2 - 1)} \\
&= \frac{1}{4c} \frac{(t_1^2 - 1)(t_j^2 - 1)}{(t_1^2 - t_j^2)}.
\end{aligned}
$$

And, recall from equation (5.2) that, for this choice of t_1,

$$
\frac{\partial}{\partial t_1} F_{0,1}^{\widetilde{M}(b,c)}(t_1) = \frac{8t_1}{(t_1 + 1)(t_1 - 1)^3}.
$$

Hence, changing variables from x_i to t_i and plugging in for $F_{0,1}^{\widetilde{M}(b,c)}(t_1)$ gives

$$
\begin{aligned}
0 &= \left(\frac{2}{c} \frac{t_1^2 + 1}{t_1^2 - 1} + 2 \frac{\partial t_1}{\partial x_1} \frac{\partial}{\partial t_1} F_{0,1}^{\widetilde{M}(b,c)}(t_1) \right) \frac{\partial t_1}{\partial x_1} \frac{\partial}{\partial t_1} F_{0,2}^{\widetilde{M}(b,c)}(t_1, t_2) \\
&\quad + \frac{1}{4c} \frac{(t_1^2 - 1)(t_2^2 - 1)}{(t_1^2 - t_2^2)} \left[- \frac{\partial t_1}{\partial x_1} \frac{\partial}{\partial t_1} F_{0,1}^{\widetilde{M}(b,c)}(t_1) + \frac{\partial t_2}{\partial x_2} \frac{\partial}{\partial t_2} F_{0,1}^{\widetilde{M}(b,c)}(t_2) \right] \\
&= \left(\frac{2}{c} \frac{t_1^2 + 1}{t_1^2 - 1} + 2 \frac{(t_1^2 - 1)^2}{-8ct_1} \frac{8t_1}{(t_1 + 1)(t_1 - 1)^3} \right) \frac{(t_1^2 - 1)^2}{-8ct_1} \frac{\partial}{\partial t_1} F_{0,2}^{\widetilde{M}(b,c)}(t_1, t_2)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4c}\frac{(t_1^2-1)(t_2^2-1)}{(t_1^2-t_2^2)}\Bigg[-\frac{(t_1^2-1)^2}{-8ct_1}\frac{8t_1}{(t_1+1)(t_1-1)^3}+\frac{(t_2^2-1)^2}{-8ct_2}\frac{8t_2}{(t_2+1)(t_2-1)^3}\Bigg]\\
=&\frac{(t_1^2-1)^2}{-4c^2t_1}\left(\frac{(t_1^2+1)}{(t_1-1)(t_1+1)}-\frac{(t_1+1)}{(t_1-1)}\right)\frac{\partial}{\partial t_1}F_{0,2}^{\widetilde{M}(b,c)}(t_1,t_2)\\
&+\frac{1}{4c^2}\frac{(t_1^2-1)(t_2^2-1)}{(t_1^2-t_2^2)}\left[\frac{(t_1^2-1)}{(t_1-1)^2}-\frac{(t_2^2-1)}{(t_2-1)^2}\right]\\
=&\frac{(t_1^2-1)^2}{-4c^2t_1}\left(\frac{-2t_1}{(t_1-1)(t_1+1)}\right)\frac{\partial}{\partial t_1}F_{0,2}^{\widetilde{M}(b,c)}(t_1,t_2)\\
&+\frac{1}{4c^2}\frac{(t_1^2-1)(t_2^2-1)}{(t_1^2-t_2^2)}\left[\frac{(t_1^2-1)}{(t_1-1)^2}-\frac{(t_2^2-1)}{(t_2-1)^2}\right]\\
=&\frac{(t_1^2-1)}{2c^2}\frac{\partial}{\partial t_1}F_{0,2}^{\widetilde{M}(b,c)}(t_1,t_2)+\frac{1}{4c^2}\frac{(t_1^2-1)(t_2^2-1)}{(t_1^2-t_2^2)}\left[\frac{(t_1^2-1)}{(t_1-1)^2}-\frac{(t_2^2-1)}{(t_2-1)^2}\right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{\partial}{\partial t_1}F_{0,2}^{\widetilde{M}(b,c)}(t_1,t_2)&=\frac{-1}{4c^2}\frac{2c^2}{(t_1^2-1)}\frac{(t_1^2-1)(t_2^2-1)}{(t_1^2-t_2^2)}\left[\frac{(t_1^2-1)}{(t_1-1)^2}-\frac{(t_2^2-1)}{(t_2-1)^2}\right]\\
&=\frac{1}{2}\frac{(t_2^2-1)}{(t_1^2-t_2^2)}\left[\frac{(t_2+1)}{(t_2-1)}-\frac{(t_1+1)}{(t_1-1)}\right]\\
&=\frac{1}{2}\frac{(t_2^2-1)}{(t_1^2-t_2^2)}\left[\frac{2(t_1-t_2)}{(t_1-1)(t_2-1)}\right]\\
&=\frac{(t_2+1)}{(t_1-1)(t_1+t_2)}
\end{aligned}
$$

as was claimed.

Now, returning to the general case, we want to write the sum in $\mathrm{IV}_{g,v}(x_1,x_2,\ldots,x_v)$ in such a way that it does not contain any (g,v) or $(g,v-1)$ terms (so that we can combine these terms with the comparable terms occurring elsewhere in the formula). And, after pulling out the (g,v) and $(g,v-1)$ terms from this sum, the remaining terms are precisely the "stable" terms, i.e. when $2g_1+|I|-1>0$ and $2g_2+|J|-1>0$.

Thus, we see that

$$
\begin{aligned}
&\mathrm{IV}_{g,v}(x_1,x_2,\ldots,x_v)\\
&=c^2\Bigg[\sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{stable}}\frac{\partial}{\partial x_1}F_{g_1,|I|+1}^{\widetilde{M}(b,c)}(x_1,x_I)\cdot\frac{\partial}{\partial x_1}F_{g_2,|J|+1}^{\widetilde{M}(b,c)}(x_1,x_J)
\end{aligned}
$$

$$+ 2\frac{\partial}{\partial x_1}F_{0,1}^{\widetilde{M}(b,c)}(x_1)\cdot\frac{\partial}{\partial x_1}F_{g,v}^{\widetilde{M}(b,c)}(x_1,x_2,\ldots,x_v)$$
$$+ 2\sum_{j=2}^{v}\frac{\partial}{\partial x_1}F_{0,2}^{\widetilde{M}(b,c)}(x_1,x_j)\cdot\frac{\partial}{\partial x_1}F_{g,v-1}^{\widetilde{M}(b,c)}(x_1,x_2,\ldots,\widehat{x_j},\ldots,x_v)\Bigg].$$

We now substitute this and the other three terms into equation (5.6) again, rearrange terms so that $\partial/\partial x_1 F_{g,v}^{\widetilde{M}(b,c)}(x_1,x_2,\ldots,x_v)$ appears only on the left side, then change variables to t_i using equation (5.3) and simplify the result.

We obtain

$$\begin{aligned}
&\frac{\partial}{\partial x_1}F_{g,v}^{\widetilde{M}(b,c)}(x_1,x_2,\ldots,x_v)\\
&= -\Bigg\{ - b x_1^{-1}\frac{\partial}{\partial x_1}F_{g,v}^{\widetilde{M}(b,c)}(x_1,\ldots,x_v)\\
&\quad + c^2\sum_{j=2}^{v}\frac{1}{x_1(x_j-x_1)}\Bigg[-\frac{\partial}{\partial x_1}F_{g,v-1}^{\widetilde{M}(b,c)}(x_1,\ldots,\widehat{x_j},\ldots,x_v)\\
&\quad + \frac{\partial}{\partial x_j}F_{g,v-1}^{\widetilde{M}(b,c)}(x_2,\ldots,x_v)\Bigg]\\
&\quad + c^2 x_1^{-1}\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}F_{g-1,v+1}^{\widetilde{M}(b,c)}(u_1,u_2,x_2,\ldots,x_v)\Big|_{u_1=u_2=x_1}\\
&\quad + c^2 x_1^{-1}\Bigg[\sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{stable}}\frac{\partial}{\partial x_1}F_{g_1,|I|+1}^{\widetilde{M}(b,c)}(x_1,x_I)\cdot\frac{\partial}{\partial x_1}F_{g_2,|J|+1}^{\widetilde{M}(b,c)}(x_1,x_J)\\
&\quad + 2\frac{\partial}{\partial x_1}F_{0,1}^{\widetilde{M}(b,c)}(x_1)\cdot\frac{\partial}{\partial x_1}F_{g,v}^{\widetilde{M}(b,c)}(x_1,x_2,\ldots,x_v)\\
&\quad + 2\sum_{j=2}^{v}\frac{\partial}{\partial x_1}F_{0,2}^{\widetilde{M}(b,c)}(x_1,x_j)\cdot\frac{\partial}{\partial x_1}F_{g,v-1}^{\widetilde{M}(b,c)}(x_1,x_2,\ldots,\widehat{x_j},\ldots,x_v)\Bigg]\Bigg\}.
\end{aligned}$$

This implies

$$\begin{aligned}
0 &= \left(\frac{x_1-b}{c^2}\right)\left[\frac{\partial}{\partial x_1}F_{g,v}^{\widetilde{M}(b,c)}(x_1,x_2,\ldots,x_v)\right]\\
&\quad + \sum_{j=2}^{v}\frac{1}{(x_j-x_1)}\left[-\frac{\partial}{\partial x_1}F_{g,v-1}^{\widetilde{M}(b,c)}(x_1,\ldots,\widehat{x_j},\ldots,x_v)+\frac{\partial}{\partial x_j}F_{g,v-1}^{\widetilde{M}(b,c)}(x_2,\ldots,x_v)\right]\\
&\quad + \frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}F_{g-1,v+1}^{\widetilde{M}(b,c)}(u_1,u_2,x_2,\ldots,x_v)\Big|_{u_1=u_2=x_1}
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{stable}} \frac{\partial}{\partial x_1} F^{\widetilde{M}(b,c)}_{g_1,|I|+1}(x_1, x_I) \cdot \frac{\partial}{\partial x_1} F^{\widetilde{M}(b,c)}_{g_2,|J|+1}(x_1, x_J) \\
&+ 2 \frac{\partial}{\partial x_1} F^{\widetilde{M}(b,c)}_{0,1}(x_1) \cdot \frac{\partial}{\partial x_1} F^{\widetilde{M}(b,c)}_{g,v}(x_1, x_2, \ldots, x_v) \\
&+ 2 \sum_{j=2}^{v} \frac{\partial}{\partial x_1} F^{\widetilde{M}(b,c)}_{0,2}(x_1, x_j) \cdot \frac{\partial}{\partial x_1} F^{\widetilde{M}(b,c)}_{g,v-1}(x_1, x_2, \ldots, \widehat{x_j}, \ldots, x_v).
\end{aligned}
$$

And, hence,

$$
\begin{aligned}
0 = &\left(\frac{x_1 - b}{c^2} + 2 \frac{\partial}{\partial x_1} F^{\widetilde{M}(b,c)}_{0,1}(x_1) \right) \left[\frac{\partial}{\partial x_1} F^{\widetilde{M}(b,c)}_{g,v}(x_1, x_2, \ldots, x_v) \right] \\
&+ \sum_{j=2}^{v} \frac{1}{(x_j - x_1)} \left[\frac{\partial}{\partial x_j} F^{\widetilde{M}(b,c)}_{g,v-1}(x_2, \ldots, x_v) \right] \\
&+ \sum_{j=2}^{v} \left(2 \frac{\partial}{\partial x_1} F^{\widetilde{M}(b,c)}_{0,2}(x_1, x_j) - \frac{1}{(x_j - x_1)} \right) \left[\frac{\partial}{\partial x_1} F^{\widetilde{M}(b,c)}_{g,v-1}(x_1, \ldots, \widehat{x_j}, \ldots, x_v) \right] \\
&+ \frac{\partial}{\partial u_1} \frac{\partial}{\partial u_2} F^{\widetilde{M}(b,c)}_{g-1,v+1}(u_1, u_2, x_2, \ldots, x_v) \Big|_{u_1=u_2=x_1} \\
&+ \sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{stable}} \frac{\partial}{\partial x_1} F^{\widetilde{M}(b,c)}_{g_1,|I|+1}(x_1, x_I) \cdot \frac{\partial}{\partial x_1} F^{\widetilde{M}(b,c)}_{g_2,|J|+1}(x_1, x_J)
\end{aligned}
$$

After applying the substitution for x_i in terms of t_i from equation (5.3), this becomes

$$
\begin{aligned}
0 = &\left(\frac{2}{c} \frac{t_1^2 + 1}{t_1^2 - 1} + 2 \frac{\partial t_1}{\partial x_1} \frac{\partial}{\partial t_1} F^{\widetilde{M}(b,c)}_{0,1}(t_1) \right) \left[\frac{\partial t_1}{\partial x_1} \frac{\partial}{\partial t_1} F^{\widetilde{M}(b,c)}_{g,v}(t_1, t_2, \ldots, t_v) \right] \\
&+ \sum_{j=2}^{v} \left(\frac{1}{4c} \frac{(t_1^2 - 1)(t_j^2 - 1)}{(t_1^2 - t_j^2)} \right) \left[\frac{\partial t_j}{\partial x_j} \frac{\partial}{\partial t_j} F^{\widetilde{M}(b,c)}_{g,v-1}(t_2, \ldots, t_v) \right] \\
&+ \sum_{j=2}^{v} \left(2 \frac{\partial t_1}{\partial x_1} \frac{\partial}{\partial t_1} F^{\widetilde{M}(b,c)}_{0,2}(t_1, t_j) - \frac{1}{4c} \frac{(t_1^2 - 1)(t_j^2 - 1)}{(t_1^2 - t_j^2)} \right) \\
&\cdot \left[\frac{\partial t_1}{\partial x_1} \frac{\partial}{\partial t_1} F^{\widetilde{M}(b,c)}_{g,v-1}(t_1, \ldots, \widehat{t_j}, \ldots, t_v) \right] \\
&+ \left(\frac{\partial t_1}{\partial x_1} \right)^2 \frac{\partial}{\partial u_1} \frac{\partial}{\partial u_2} F^{\widetilde{M}(b,c)}_{g-1,v+1}(u_1, u_2, t_2, \ldots, t_v) \Big|_{u_1=u_2=t_1} \\
&+ \sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{stable}} \frac{\partial t_1}{\partial x_1} \frac{\partial}{\partial t_1} F^{\widetilde{M}(b,c)}_{g_1,|I|+1}(t_1, t_I) \cdot \frac{\partial t_1}{\partial x_1} \frac{\partial}{\partial t_1} F^{\widetilde{M}(b,c)}_{g_2,|J|+1}(t_1, t_J) \\
= &\left(\frac{2}{c} \frac{t_1^2 + 1}{t_1^2 - 1} + 2 \frac{(t_1^2 - 1)^2}{-8ct_1} \frac{8t_1}{(t_1 + 1)(t_1 - 1)^3} \right) \left[\frac{(t_1^2 - 1)^2}{-8ct_1} \frac{\partial}{\partial t_1} F^{\widetilde{M}(b,c)}_{g,v}(t_1, t_2, \ldots, t_v) \right]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=2}^{v}\left(\frac{1}{4c}\frac{(t_1^2-1)(t_j^2-1)}{(t_1^2-t_j^2)}\right)\left[\frac{(t_j^2-1)^2}{-8ct_j}\frac{\partial}{\partial t_j}F_{g,v-1}^{\widetilde{M}(b,c)}(t_2,\ldots,t_v)\right]\\
&+\sum_{j=2}^{v}\left(2\frac{(t_1^2-1)^2}{-8ct_1}\frac{(t_j+1)}{(t_1-1)(t_1+t_j)}-\frac{1}{4c}\frac{(t_1^2-1)(t_j^2-1)}{(t_1^2-t_j^2)}\right)\\
&\cdot\left[\frac{(t_1^2-1)^2}{-8ct_1}\frac{\partial}{\partial t_1}F_{g,v-1}^{\widetilde{M}(b,c)}(t_1,\ldots,\widehat{t_j},\ldots,t_v)\right]\\
&+\left(\frac{(t_1^2-1)^2}{-8ct_1}\right)^2\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}F_{g-1,v+1}^{\widetilde{M}(b,c)}(u_1,u_2,t_2,\ldots,t_v)\bigg|_{u_1=u_2=t_1}\\
&+\sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{stable}}\frac{(t_1^2-1)^2}{-8ct_1}\frac{\partial}{\partial t_1}F_{g_1,|I|+1}^{\widetilde{M}(b,c)}(t_1,t_I)\cdot\frac{(t_1^2-1)^2}{-8ct_1}\frac{\partial}{\partial t_1}F_{g_2,|J|+1}^{\widetilde{M}(b,c)}(t_1,t_J)\\
=&\frac{(t_1^2-1)^2}{-4c^2t_1}\left(\frac{t_1^2+1}{t_1^2-1}-\frac{(t_1^2-1)}{(t_1-1)^2}\right)\left[\frac{\partial}{\partial t_1}F_{g,v}^{\widetilde{M}(b,c)}(t_1,t_2,\ldots,t_v)\right]\\
&-\sum_{j=2}^{v}\frac{(t_1^2-1)(t_j^2-1)^3}{32c^2t_j(t_1^2-t_j^2)}\left[\frac{\partial}{\partial t_j}F_{g,v-1}^{\widetilde{M}(b,c)}(t_2,\ldots,t_v)\right]\\
&+\sum_{j=2}^{v}\frac{(t_1^2-1)^3}{32c^2t_1}\left(\frac{(t_1+1)(t_j+1)}{t_1(t_1+t_j)}+\frac{(t_j^2-1)}{(t_1^2-t_j^2)}\right)\left[\frac{\partial}{\partial t_1}F_{g,v-1}^{\widetilde{M}(b,c)}(t_1,\ldots,\widehat{t_j},\ldots,t_v)\right]\\
&+\frac{(t_1^2-1)^4}{64c^2t_1^2}\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}F_{g-1,v+1}^{\widetilde{M}(b,c)}(u_1,u_2,t_2,\ldots,t_v)\bigg|_{u_1=u_2=t_1}\\
&+\frac{(t_1^2-1)^4}{64c^2t_1^2}\sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{stable}}\frac{\partial}{\partial t_1}F_{g_1,|I|+1}^{\widetilde{M}(b,c)}(t_1,t_I)\cdot\frac{\partial}{\partial t_1}F_{g_2,|J|+1}^{\widetilde{M}(b,c)}(t_1,t_J)\\
=&\frac{(t_1^2-1)^2}{-4c^2t_1}\left(\frac{(t_1^2+1)-(t_1+1)^2}{(t_1+1)(t_1-1)}\right)\left[\frac{\partial}{\partial t_1}F_{g,v}^{\widetilde{M}(b,c)}(t_1,t_2,\ldots,t_v)\right]\\
&-\sum_{j=2}^{v}\frac{(t_1^2-1)(t_j^2-1)^3}{32c^2t_j(t_1^2-t_j^2)}\left[\frac{\partial}{\partial t_j}F_{g,v-1}^{\widetilde{M}(b,c)}(t_2,\ldots,t_v)\right]\\
&+\sum_{j=2}^{v}\frac{(t_1^2-1)^3}{32c^2t_1}\left(\frac{t_1+1}{t_1}+\frac{t_j-1}{t_1-t_j}\right)\left(\frac{t_j+1}{t_1+t_j}\right)\left[\frac{\partial}{\partial t_1}F_{g,v-1}^{\widetilde{M}(b,c)}(t_1,\ldots,\widehat{t_j},\ldots,t_v)\right]\\
&+\frac{(t_1^2-1)^4}{64c^2t_1^2}\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}F_{g-1,v+1}^{\widetilde{M}(b,c)}(u_1,u_2,t_2,\ldots,t_v)\bigg|_{u_1=u_2=t_1}\\
&+\frac{(t_1^2-1)^4}{64c^2t_1^2}\sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{stable}}\frac{\partial}{\partial t_1}F_{g_1,|I|+1}^{\widetilde{M}(b,c)}(t_1,t_I)\cdot\frac{\partial}{\partial t_1}F_{g_2,|J|+1}^{\widetilde{M}(b,c)}(t_1,t_J)\\
=&\frac{-2t_1(t_1^2-1)}{-4c^2t_1}\left[\frac{\partial}{\partial t_1}F_{g,v}^{\widetilde{M}(b,c)}(t_1,t_2,\ldots,t_v)\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{j=2}^{v} \frac{(t_1^2-1)(t_j^2-1)^3}{32c^2 t_j (t_1^2-t_j^2)} \left[\frac{\partial}{\partial t_j} F_{g,v-1}^{\widetilde{M}(b,c)}(t_2,\ldots,t_v) \right] \\
&+ \sum_{j=2}^{v} \frac{(t_1^2-1)^3}{32c^2 t_1} \left(\frac{t_1^2-t_j}{t_1(t_1-t_j)} \right) \left(\frac{t_j+1}{t_1+t_j} \right) \left[\frac{\partial}{\partial t_1} F_{g,v-1}^{\widetilde{M}(b,c)}(t_1,\ldots,\widehat{t_j},\ldots,t_v) \right] \\
&+ \frac{(t_1^2-1)^4}{64c^2t_1^2} \frac{\partial}{\partial u_1} \frac{\partial}{\partial u_2} F_{g-1,v+1}^{\widetilde{M}(b,c)}(u_1,u_2,t_2,\ldots,t_v) \Big|_{u_1=u_2=t_1} \\
&+ \frac{(t_1^2-1)^4}{64c^2t_1^2} \sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{stable}} \frac{\partial}{\partial t_1} F_{g_1,|I|+1}^{\widetilde{M}(b,c)}(t_1,t_I) \cdot \frac{\partial}{\partial t_1} F_{g_2,|J|+1}^{\widetilde{M}(b,c)}(t_1,t_J) \\
= &\frac{(t_1^2-1)}{2c^2} \left[\frac{\partial}{\partial t_1} F_{g,v}^{\widetilde{M}(b,c)}(t_1,t_2,\ldots,t_v) \right] \\
&- \sum_{j=2}^{v} \frac{(t_1^2-1)(t_j^2-1)^3}{32c^2 t_j (t_1^2-t_j^2)} \left[\frac{\partial}{\partial t_j} F_{g,v-1}^{\widetilde{M}(b,c)}(t_2,\ldots,t_v) \right] \\
&+ \sum_{j=2}^{v} \frac{(t_1^2-1)^3}{32c^2 t_1^2} \left(1 + \frac{(t_1^2-1)t_j}{t_1^2-t_j^2} \right) \left[\frac{\partial}{\partial t_1} F_{g,v-1}^{\widetilde{M}(b,c)}(t_1,\ldots,\widehat{t_j},\ldots,t_v) \right] \\
&+ \frac{(t_1^2-1)^4}{64c^2t_1^2} \frac{\partial}{\partial u_1} \frac{\partial}{\partial u_2} F_{g-1,v+1}^{\widetilde{M}(b,c)}(u_1,u_2,t_2,\ldots,t_v) \Big|_{u_1=u_2=t_1} \\
&+ \frac{(t_1^2-1)^4}{64c^2t_1^2} \sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{stable}} \frac{\partial}{\partial t_1} F_{g_1,|I|+1}^{\widetilde{M}(b,c)}(t_1,t_I) \cdot \frac{\partial}{\partial t_1} F_{g_2,|J|+1}^{\widetilde{M}(b,c)}(t_1,t_J).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{\partial}{\partial t_1} F_{g,v}^{\widetilde{M}(b,c)}(t_1,t_2,\ldots,t_v) \\
&= -\frac{2c^2}{(t_1^2-1)} \Bigg\{ - \sum_{j=2}^{v} \frac{(t_1^2-1)(t_j^2-1)^3}{32c^2 t_j (t_1^2-t_j^2)} \left[\frac{\partial}{\partial t_j} F_{g,v-1}^{\widetilde{M}(b,c)}(t_2,\ldots,t_v) \right] \\
&\quad + \sum_{j=2}^{v} \frac{(t_1^2-1)^3}{32c^2 t_1^2} \left(1 + \frac{(t_1^2-1)t_j}{t_1^2-t_j^2} \right) \left[\frac{\partial}{\partial t_1} F_{g,v-1}^{\widetilde{M}(b,c)}(t_1,\ldots,\widehat{t_j},\ldots,t_v) \right] \\
&\quad + \frac{(t_1^2-1)^4}{64c^2t_1^2} \frac{\partial}{\partial u_1} \frac{\partial}{\partial u_2} F_{g-1,v+1}^{\widetilde{M}(b,c)}(u_1,u_2,t_2,\ldots,t_v) \Big|_{u_1=u_2=t_1} \\
&\quad + \frac{(t_1^2-1)^4}{64c^2t_1^2} \sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{stable}} \frac{\partial}{\partial t_1} F_{g_1,|I|+1}^{\widetilde{M}(b,c)}(t_1,t_I) \cdot \frac{\partial}{\partial t_1} F_{g_2,|J|+1}^{\widetilde{M}(b,c)}(t_1,t_J) \Bigg\} \\
&= \sum_{j=2}^{v} \frac{(t_j^2-1)^3}{16 t_j (t_1^2-t_j^2)} \left[\frac{\partial}{\partial t_j} F_{g,v-1}^{\widetilde{M}(b,c)}(t_2,\ldots,t_v) \right]
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{j=2}^{v} \left[\frac{(t_1^2-1)^3 t_j}{16 t_1^2 (t_1^2 - t_j^2)} + \frac{(t_1^2-1)^2}{16 t_1^2} \right] \left[\frac{\partial}{\partial t_1} F_{g,v-1}^{\widetilde{M}(b,c)}(t_1, \ldots, \widehat{t_j}, \ldots, t_v) \right] \\
&- \frac{(t_1^2-1)^3}{32 t_1^2} \frac{\partial}{\partial u_1} \frac{\partial}{\partial u_2} F_{g-1,v+1}^{\widetilde{M}(b,c)}(u_1, u_2, t_2, \ldots, t_v) \Big|_{u_1=u_2=t_1} \\
&- \frac{(t_1^2-1)^3}{32 t_1^2} \sum_{g_1+g_2=g,\ I \sqcup J = \{2,\ldots,v\},\ \text{stable}} \frac{\partial}{\partial t_1} F_{g_1,|I|+1}^{\widetilde{M}(b,c)}(t_1, t_I) \cdot \frac{\partial}{\partial t_1} F_{g_2,|J|+1}^{\widetilde{M}(b,c)}(t_1, t_J).
\end{aligned}
$$

This completes the proof of the above theorem. □

From this differential recursion formula, we can obtain all $F_{g,v}^{\widetilde{M}(b,c)}(t_1, \ldots, t_v)$, where $(g, v) \neq (0, 1), (0, 2)$, by integrating the right side of the equation in Theorem 5.0.1 from -1 to t_1 with respect to the variable t_1. We will now look at two examples.

EXAMPLE 5.0.1. For the case when $(g, v) = (1, 1)$, we see that

$$
\begin{aligned}
\frac{\partial}{\partial t} F_{1,1}^{\widetilde{M}(b,c)}(t) &= -\frac{1}{32} \frac{(t^2-1)^3}{t^2} \frac{\partial}{\partial u_1} \frac{\partial}{\partial u_2} F_{0,2}^{\widetilde{M}(b,c)}(u_1, u_2) \Big|_{u_1=u_2=t} \\
&= -\frac{1}{128} \frac{(t^2-1)^3}{t^4},
\end{aligned}
$$

so

$$
\begin{aligned}
F_{1,1}^{\widetilde{M}(b,c)}(t) &= -\frac{1}{128} \int_{-1}^{t} \frac{(\tau^2-1)^3}{\tau^4} \, d\tau \\
&= -\frac{1}{384} \frac{(1+t)^4}{t^2} \left(t - 4 + \frac{1}{t} \right).
\end{aligned}
\tag{5.7}
$$

EXAMPLE 5.0.2. When $(g, v) = (0, 3)$, we see that

$$
F_{0,3}^{\widetilde{M}(b,c)}(t_1, t_2, t_3) = -\frac{1}{16} (t_1+1)(t_2+1)(t_3+1) \left(1 + \frac{1}{t_1 t_2 t_3} \right). \tag{5.8}
$$

REMARK 5.0.2. Since $F_{1,1}^{\widetilde{M}(b,c)}(t)$ and $F_{0,3}^{\widetilde{M}(b,c)}(t_1, t_2, t_3)$ are both Laurent polynomials, the differential recursion formula in Theorem 5.0.1 implies that all higher order discrete Laplace transforms $F_{g,v}^{\widetilde{M}(b,c)}(t_1, t_2, \ldots, t_v)$ will also be Laurent polynomials. This can be seen by observing that $F_{g,v}^{\widetilde{M}(b,c)}$, for $(g, v) \neq (0, 1), (0, 2), (0, 3)$, and $(1, 1)$ depends only on $F_{1,1}^{\widetilde{M}(b,c)}, F_{0,3}^{\widetilde{M}(b,c)}$, and higher order $F_{g,v}^{\widetilde{M}(b,c)}$, which, recursively, must be Laurent polynomials, because $F_{1,1}^{\widetilde{M}(b,c)}$ and $F_{0,3}^{\widetilde{M}(b,c)}$ are Laurent polynomials.

It follows from Theorem 2.0.2 and Theorem 5.0.1 that, since the differential recursion formulas for the bc-Motzkin numbers and the Catalan numbers are identical (up to the change of variable) and have the same initial conditions, their discrete Laplace transforms must be the same. This gives the following Corollary.

Corollary 5.0.1. *We have*

$$F_{g,v}^{\widetilde{M}(b,c)}(t_1, t_2, \ldots, t_v) = F_{g,v}^{C}(t_1, t_2, \ldots, t_v), \tag{5.9}$$

where on the left side the t_i are defined by equation (5.3) *and on the right side the t_i are defined by equation* (2.4).

Therefore, we see that the following results which are known to hold for generalized Catalan numbers are also true for the case of generalized bc-Motzkin numbers. (See also [**3**], [**7**], [**9**], and [**19**].)

For all (g, v) with $2g - 2 + v > 0$, the discrete Laplace transform $F_{g,v}^{\widetilde{M}(b,c)}(t_1, t_2, \ldots, t_v)$ satisfies the following corollaries of Theorem 5.0.1, where the t_i are as defined above.

Corollary 5.0.2. $F_{g,v}^{\widetilde{M}(b,c)}(t_1, \ldots, t_v)$ *is a Laurent polynomial in the t_i-variables, of degree* $3(2g - 2 + v)$*. And,*

$$F_{g,v}^{\widetilde{M}(b,c)}(1/t_1, 1/t_2, \ldots, 1/t_v) = F_{g,v}^{\widetilde{M}(b,c)}(t_1, t_2, \ldots, t_v).$$

Corollary 5.0.3. *The special values at $t_i = -1$ are given by*

$$F_{g,v}^{\widetilde{M}(b,c)}(t_1, t_2, \ldots, t_v)|_{t_i=-1} = 0,$$

for each i.

The diagonal value at $t_i = 1$ gives the orbifold Euler characteristic of the moduli space $\mathcal{M}_{g,n}$,

$$F_{g,v}^{\widetilde{M}(b,c)}(1, 1, \ldots, 1) = (-1)^n \chi(\mathcal{M}_{g,n}).$$

Corollary 5.0.4. *The restriction of the Laurent polynomial* $F_{g,v}^{\widetilde{M}(b,c)}(t_1, \ldots, t_v)$ *to its highest degree terms gives a homogeneous polynomial defined by*

$$F_{g,v}^{\widetilde{M}(b,c),highest}(t_1, \ldots, t_v) = \frac{(-1)^v}{2^{2g-2+v}} \sum_{d_1+\cdots+d_v=3g-3+v} \langle \tau_{d_1} \cdots \tau_{d_v} \rangle_{g,n} \prod_{i=1}^{v} |2d_i - 1|!! \left(\frac{t_i}{2} \right)^{2d_i+1},$$

where the $\langle \tau_{d_1} \cdots \tau_{d_v} \rangle_{g,n}$ *represent the intersection numbers on the moduli space of stable curves.*

CHAPTER 6

Topological Recursion for Generalized bc-Motzkin Numbers

Just as for the Catalan case, as discussed in Chapter 2 of this dissertation and in [**7**], the differential recursion formula for generalized bc-Motzkin numbers in Theorem 5.0.1 leads directly to the following topological recursion for generalized bc-Motzkin numbers. Note that, since the bc-Motzkin differential recursion formula has the same form, with the same initial conditions, as the differential recursion formula for Catalan numbers which was given in Proposition 2.0.2 (up to the slightly different change of variables from x_i to t_i), the topological recursion for generalized bc-Motzkin numbers and its proof are identical to those in the Catalan case, as given in Proposition 2.0.3.

Remark 6.0.1. Just as with the differential recursion formula, the dependence on b and c only appears in this change of variable from the x_i to the t_i. This tells us that there does indeed exist a topological recursion for these generalized bc-Motzkin numbers, and furthermore it has precisely the same form as the result which was obtained in [**7**] for the generalized Catalan numbers.

Thus, we have the following new theorem for the generalized bc-Motzkin numbers.

Theorem 6.0.1. *Define symmetric v-linear differential forms on $(\mathbb{P}^1)^v$ for $2g-2+v>0$ by*

$$W_{g,v}^{\widetilde{M}(b,c)}(t_1,t_2,\ldots,t_v) = d_{t_1}\cdots d_{t_v} F_{g,v}^{\widetilde{M}(b,c)}(t_1,t_2,\ldots,t_v), \tag{6.1}$$

and for $(g,v)=(0,2)$ by

$$W_{0,2}^{\widetilde{M}(b,c)}(t_1,t_2) = \frac{dt_1\,dt_2}{(t_1-t_2)^2}. \tag{6.2}$$

Then, these differential forms satisfy the following integral recursion equation:

$$W_{g,v}^{\widetilde{M}(b,c)}(t_1,t_2,\ldots,t_v) = -\frac{1}{64}\frac{1}{2\pi i}\int_\gamma \left(\frac{1}{t+t_1}+\frac{1}{t-t_1}\right)\frac{(t^2-1)^3}{t^2}\,\frac{1}{dt}\,dt_1$$
$$\cdot\Bigg[\sum_{j=2}^{v}\Bigg(W_{0,2}^{\widetilde{M}(b,c)}(t,t_j)W_{g,v-1}^{\widetilde{M}(b,c)}(-t,t_2,\ldots,\widehat{t_j},\ldots,t_v)$$

$$+ W_{0,2}^{\widetilde{M}(b,c)}(-t,t_j)W_{g,v-1}^{\widetilde{M}(b,c)}(t,t_2,\dots,\widehat{t_j},\dots,t_v)\Big)$$

$$+ W_{g-1,v+1}^{\widetilde{M}(b,c)}(t,-t,t_2,\dots,t_v)$$

$$+ \sum_{g_1+g_2=g,\ I\sqcup J=\{2,\dots,v\},\ stable} W_{g_1,|I|+1}^{\widetilde{M}(b,c)}(t,t_I)W_{g_2,|J|+1}^{\widetilde{M}(b,c)}(-t,t_J)\Big] \quad (6.3)$$

The "stable" summation means $2g_1 + |I| - 1 > 0$ *and* $2g_2 + |J| - 1 > 0$.

The curve γ *is as given in Figure 1.1.*

As in the Catalan case, given in Proposition 2.0.3, these differential forms are called the Eynard-Orantin differential forms, and the recursion is called the topological recursion for the generalized bc-Motzkin numbers.

REMARK 6.0.2. Observe that using Equation 5.5 above for $\partial/\partial t_1 F_{0,2}^{\widetilde{M}(b,c)}(t_1,t_2)$ gives

$$\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}F_{0,2}^{\widetilde{M}(b,c)}(t_1,t_2)\,dt_1\,dt_2 = \frac{dt_1\,dt_2}{(t_1+t_2)^2} = \frac{dt_1\,dt_2}{(t_1-t_2)^2} - (\tilde{\pi}\times\tilde{\pi})^*\frac{dx_1\,dx_2}{(x_1-x_2)^2}, \quad (6.4)$$

where $\tilde{\pi} : \mathbb{P}^1 \to \mathbb{P}^1$ is the variable transformation

$$x_i = 2c\left(\frac{t_i^2+1}{t_i^2-1}\right) + b \quad (6.5)$$

which was defined in (5.3).

We may prove Theorem 6.0.1 as follows.

PROOF. For notational convenience, we define the functions $w_{g,v}^{\widetilde{M}(b,c)}$ by

$$W_{g,v}^{\widetilde{M}(b,c)}(t_1,t_2,\dots,t_v) = w_{g,v}^{\widetilde{M}(b,c)}(t_1,t_2,\dots,t_v)\,dt_1\,dt_2\cdots dt_v. \quad (6.6)$$

We will use the following observations:

(1) When (g,v) is stable, the $w_{g,v}^{\widetilde{M}(b,c)}(t_1,t_2,\dots,t_v)$ are symmetric in the variables t_i. And, they are Laurent polynomials, so the only singularities can be when one of the t_i is zero.
(2) Further, $W_{g,v}^{\widetilde{M}(b,c)}$ is an odd differential form, so $w_{g,v}^{\widetilde{M}(b,c)}(t_1,t_2,\dots,t_v)$ is an even function.

We first apply the definition of the Eynard-Orantin differential forms to the differential recursion formula in Theorem 5.0.1. Since there are four terms in this formula, we may write

$$W_{g,v}^{\widetilde{M}(b,c)}(t_1, t_2, \ldots, t_v) = \frac{\partial}{\partial t_2} \cdots \frac{\partial}{\partial t_v} \left[\frac{\partial}{\partial t_1} F_{g,v}^{\widetilde{M}(b,c)}(t_1, t_2, \ldots, t_v) \right] dt_1\, dt_2 \cdots dt_v$$

$$= \mathrm{I}_{g,v}(t_1, t_2, \ldots, t_v) + \mathrm{II}_{g,v}(t_1, t_2, \ldots, t_v) + \mathrm{III}_{g,v}(t_1, t_2, \ldots, t_v) + \mathrm{IV}_{g,v}(t_1, t_2, \ldots, t_v). \quad (6.7)$$

Then, we will show that the result is equal to the formula given in equation (6.3) of Theorem 6.0.1.

For the first term, we see that

$$\begin{aligned}
&\mathrm{I}_{g,v}(t_1, t_2, \ldots, t_v) \\
&= \frac{\partial}{\partial t_2} \cdots \frac{\partial}{\partial t_v} \Bigg[-\frac{1}{16} \sum_{j=2}^{v} \Bigg[\frac{t_j}{t_1^2 - t_j^2} \Bigg(\frac{(t_1^2-1)^3}{t_1^2} \frac{\partial}{\partial t_1} F_{g,v-1}^{\widetilde{M}(b,c)}(t_1, \ldots, \widehat{t_j}, \ldots, t_v) \\
&\quad - \frac{(t_j^2-1)^3}{t_j^2} \frac{\partial}{\partial t_j} F_{g,v-1}^{\widetilde{M}(b,c)}(t_2, \ldots, t_v) \Bigg) \Bigg] dt_1 dt_2 \cdots dt_v \\
&= -\frac{1}{16} \frac{\partial}{\partial t_j} \sum_{j=2}^{v} \frac{t_j}{t_1^2 - t_j^2} \Bigg(\frac{(t_1^2-1)^3}{t_1^2} \frac{\partial}{\partial t_1} \cdots \widehat{\frac{\partial}{\partial t_j}} \cdots \frac{\partial}{\partial t_v} F_{g,v-1}^{\widetilde{M}(b,c)}(t_1, \ldots, \widehat{t_j}, \ldots, t_v) \\
&\quad - \frac{(t_j^2-1)^3}{t_j^2} \frac{\partial}{\partial t_2} \cdots \frac{\partial}{\partial t_v} F_{g,v-1}^{\widetilde{M}(b,c)}(t_2, \ldots, t_v) \Bigg) dt_1 dt_2 \cdots dt_v \\
&= -\frac{1}{16} \frac{\partial}{\partial t_j} \sum_{j=2}^{v} \frac{t_j}{t_1^2 - t_j^2} \Bigg[\frac{(t_1^2-1)^3}{t_1^2} W_{g,v-1}^{\widetilde{M}(b,c)}(t_1, \ldots, \widehat{t_j}, \ldots, t_v)\, dt_j \\
&\quad - \frac{(t_j^2-1)^3}{t_j^2} W_{g,v-1}^{\widetilde{M}(b,c)}(t_2, \ldots, t_v)\, dt_1 \Bigg] \\
&= \frac{1}{16} \sum_{j=2}^{v} \Bigg[\frac{\partial}{\partial t_j} \Bigg(\frac{t_j}{t_1^2 - t_j^2} \frac{(t_j^2-1)^3}{t_j^2} W_{g,v-1}^{\widetilde{M}(b,c)}(t_2, \ldots, t_v) \Bigg) dt_1 \\
&\quad - \frac{(t_1^2 - t_j^2)(1) - t_j(-2t_j)}{(t_1^2 - t_j^2)^2} \frac{(t_1^2-1)^3}{t_1^2} W_{g,v-1}^{\widetilde{M}(b,c)}(t_1, \ldots, \widehat{t_j}, \ldots, t_v)\, dt_j \Bigg] \\
&= \frac{1}{16} \sum_{j=2}^{v} \Bigg[\frac{\partial}{\partial t_j} \Bigg(\frac{t_j}{t_1^2 - t_j^2} \frac{(t_j^2-1)^3}{t_j^2} W_{g,v-1}^{\widetilde{M}(b,c)}(t_2, \ldots, t_v) \Bigg) dt_1 \\
&\quad - \frac{t_1^2 + t_j^2}{(t_1^2 - t_j^2)^2} \frac{(t_1^2-1)^3}{t_1^2} W_{g,v-1}^{\widetilde{M}(b,c)}(t_1, \ldots, \widehat{t_j}, \ldots, t_v)\, dt_j \Bigg].
\end{aligned}$$

And, from the topological recursion formula, we have

$$-\frac{1}{64} \frac{1}{2\pi i} \int_{\gamma} \left(\frac{1}{t + t_1} + \frac{1}{t - t_1} \right) \frac{(t^2-1)^3}{t^2} \cdot \frac{1}{dt} \cdot dt_1$$

$$
\begin{aligned}
&\times \sum_{j=2}^{v} \Bigg(W_{0,2}^{\widetilde{M}(b,c)}(t,t_j) W_{g,v-1}^{\widetilde{M}(b,c)}(-t,t_2,\dots,\widehat{t_j},\dots,t_v) \\
&+ W_{0,2}^{\widetilde{M}(b,c)}(-t,t_j) W_{g,v-1}^{\widetilde{M}(b,c)}(t,t_2,\dots,\widehat{t_j},\dots,t_v) \Bigg) \\
&= -\frac{1}{64}\frac{1}{2\pi i}\int_\gamma \left(\frac{2(t^2-1)^3/t}{(t+t_1)(t-t_1)} \right) \cdot \frac{1}{dt} \cdot dt_1 \\
&\times \sum_{j=2}^{v} \Bigg(\frac{dt\cdot dt_j}{(t-t_j)^2} w_{g,v-1}^{\widetilde{M}(b,c)}(-t,t_2,\dots,\widehat{t_j},\dots,t_v)\ (-dt)\cdot dt_2 \cdots \widehat{dt_j} \cdots dt_v \\
&+ \frac{-dt\cdot dt_j}{(-t-t_j)^2} w_{g,v-1}^{\widetilde{M}(b,c)}(t,t_2,\dots,\widehat{t_j},\dots,t_v)\ dt\cdot dt_2 \cdots \widehat{dt_j} \cdots dt_v \Bigg) \\
&= -\frac{1}{64}\frac{1}{2\pi i}\int_\gamma \left(\frac{2(t^2-1)^3/t}{(t+t_1)(t-t_1)} \right) \cdot \frac{1}{dt} \cdot dt_1 \\
&\times \sum_{j=2}^{v} \Bigg(-\frac{1}{(t-t_j)^2} w_{g,v-1}^{\widetilde{M}(b,c)}(t,t_2,\dots,\widehat{t_j},\dots,t_v) \\
&- \frac{1}{(t+t_j)^2} w_{g,v-1}^{\widetilde{M}(b,c)}(t,t_2,\dots,\widehat{t_j},\dots,t_v) \Bigg) dt\cdot dt\cdot dt_2\cdots dt_v \\
&= \frac{1}{64}\sum_{j=2}^{v} \Bigg[\frac{1}{2\pi i}\int_\gamma \left(\frac{2(t^2-1)^3/t}{(t+t_1)(t-t_1)} \right)\left(\frac{1}{(t-t_j)^2} + \frac{1}{(t+t_j)^2} \right) \\
&\cdot w_{g,v-1}^{\widetilde{M}(b,c)}(t,t_2,\dots,\widehat{t_j},\dots,t_v)\ dt \Bigg] dt_1\cdot dt_2\cdots dt_v \\
&= -\frac{1}{16}\sum_{j=2}^{v} \Bigg[\Big(\operatorname{Res}_{t=t_1} + \operatorname{Res}_{t=-t_1} + \operatorname{Res}_{t=t_j} + \operatorname{Res}_{t=-t_j} \Big) \left(\frac{(t^2+t_j^2)(t^2-1)^3/t}{(t+t_1)(t-t_1)(t-t_j)^2(t+t_j)^2} \right) \\
&\times w_{g,v-1}^{\widetilde{M}(b,c)}(t,t_2,\dots,\widehat{t_j},\dots,t_v) \Bigg] dt_1\cdot dt_2\cdots dt_v \\
&= -\frac{1}{16}\sum_{j=2}^{v} \Bigg\{ \Bigg(\frac{(t_1^2+t_j^2)(t_1^2-1)^3/t_1}{(t_1+t_1)(t_1-t_j)^2(t_1+t_j)^2} \\
&+ \frac{(t_1^2+t_j^2)(t_1^2-1)^3/(-t_1)}{(-t_1-t_1)(-t_1-t_j)^2(-t_1+t_j)^2} \Bigg) w_{g,v-1}^{\widetilde{M}(b,c)}(t_1,t_2,\dots,\widehat{t_j},\dots,t_v) \\
&+ \Bigg[\frac{\partial}{\partial t}\left(\frac{(t^2+t_j^2)(t^2-1)^3/t}{(t+t_1)(t-t_1)(t+t_j)^2} \times w_{g,v-1}^{\widetilde{M}(b,c)}(t,t_2,\dots,\widehat{t_j},\dots,t_v) \right)\Bigg|_{t=t_j} \\
&+ \frac{\partial}{\partial t}\left(\frac{(t^2+t_j^2)(t^2-1)^3/t}{(t+t_1)(t-t_1)(t-t_j)^2} \times w_{g,v-1}^{\widetilde{M}(b,c)}(t,t_2,\dots,\widehat{t_j},\dots,t_v) \right)\Bigg|_{t=-t_j} \Bigg] \Bigg\} dt_1\cdot dt_2\cdots dt_v
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{16}\sum_{j=2}^{v}\Bigg\{2\bigg(\frac{(t_1^2+t_j^2)(t_1^2-1)^3}{2t_1^2(t_1-t_j)^2(t_1+t_j)^2}\bigg)w_{g,v-1}^{\widetilde{M}(b,c)}(t_1,t_2,\ldots,\widehat{t_j},\ldots,t_v)\\
&\quad+\frac{\partial}{\partial t_j}\bigg(\frac{(t_j^2-1)^3}{t_j(t_j^2-t_1^2)}w_{g,v-1}^{\widetilde{M}(b,c)}(t_2,\ldots,t_v)\bigg)\Bigg\}dt_1\cdot dt_2\cdots dt_v\\
&= -\frac{1}{16}\sum_{j=2}^{v}\bigg[\frac{(t_1^2+t_j^2)(t_1^2-1)^3}{t_1^2(t_1^2-t_j^2)^2}W_{g,v-1}^{\widetilde{M}(b,c)}(t_1,t_2,\ldots,\widehat{t_j},\ldots,t_v)\,dt_j\\
&\quad+\frac{\partial}{\partial t_j}\bigg(\frac{(t_j^2-1)^3}{t_j(t_j^2-t_1^2)}W_{g,v-1}^{\widetilde{M}(b,c)}(t_2,\ldots,t_v)\bigg)\,dt_1\bigg]\\
&= \frac{1}{16}\sum_{j=2}^{v}\bigg[\frac{\partial}{\partial t_j}\bigg(\frac{(t_j^2-1)^3}{t_j(t_1^2-t_j^2)}W_{g,v-1}^{\widetilde{M}(b,c)}(t_2,\ldots,t_v)\bigg)\,dt_1\\
&\quad-\frac{(t_1^2+t_j^2)(t_1^2-1)^3}{t_1^2(t_1^2-t_j^2)^2}W_{g,v-1}^{\widetilde{M}(b,c)}(t_1,t_2,\ldots,\widehat{t_j},\ldots,t_v)\,dt_j\bigg]\\
&= \mathrm{I}_{g,v}(t_1,t_2,\ldots,t_v),
\end{aligned}
$$

where we used that

$$
\begin{aligned}
&\frac{\partial}{\partial t}\bigg(\frac{(t^2+t_j^2)(t^2-1)^3/t}{(t+t_1)(t-t_1)(t+t_j)^2}\times w_{g,v-1}^{\widetilde{M}(b,c)}(t,t_2,\ldots,\widehat{t_j},\ldots,t_v)\bigg)\bigg|_{t=t_j}\\
&= \bigg[\frac{(t^2-1)^3}{t(t+t_1)(t-t_1)}w_{g,v-1}^{\widetilde{M}(b,c)}(t,t_2,\ldots,\widehat{t_j},\ldots,t_v)\bigg(\frac{\partial}{\partial t}\frac{(t^2+t_j^2)}{(t+t_j)^2}\bigg)\\
&\quad+\frac{(t^2+t_j^2)}{(t+t_j)^2}\frac{\partial}{\partial t}\bigg(\frac{(t^2-1)^3}{t(t+t_1)(t-t_1)}w_{g,v-1}^{\widetilde{M}(b,c)}(t,t_2,\ldots,\widehat{t_j},\ldots,t_v)\bigg)\bigg]_{t=t_j}\\
&= \frac{(t_j^2-1)^3}{t_j(t_j+t_1)(t_j-t_1)}w_{g,v-1}^{\widetilde{M}(b,c)}(t_2,\ldots,t_v)\bigg(\frac{(t_j+t_j)^2(2t_j)-(t_j^2+t_j^2)2(t_j+t_j)}{(t_j+t_j)^4}\bigg)\\
&\quad+\frac{(t_j^2+t_j^2)}{(t_j+t_j)^2}\frac{\partial}{\partial t_j}\bigg(\frac{(t_j^2-1)^3}{t_j(t_j+t_1)(t_j-t_1)}w_{g,v-1}^{\widetilde{M}(b,c)}(t_2,\ldots,t_v)\bigg)\\
&= \frac{(t_j^2-1)^3}{t_j(t_j+t_1)(t_j-t_1)}w_{g,v-1}^{\widetilde{M}(b,c)}(t_2,\ldots,t_v)\bigg(\frac{8t_j^3-8t_j^3}{(2t_j)^4}\bigg)\\
&\quad+\frac{2t_j^2}{(2t_j)^2}\frac{\partial}{\partial t_j}\bigg(\frac{(t_j^2-1)^3}{t_j(t_j+t_1)(t_j-t_1)}w_{g,v-1}^{\widetilde{M}(b,c)}(t_2,\ldots,t_v)\bigg)\\
&= \frac{1}{2}\frac{\partial}{\partial t_j}\bigg(\frac{(t_j^2-1)^3}{t_j(t_j^2-t_1^2)}w_{g,v-1}^{\widetilde{M}(b,c)}(t_2,\ldots,t_v)\bigg)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{(t^2+t_j^2)(t^2-1)^3/t}{(t+t_1)(t-t_1)(t-t_j)^2}\times w_{g,v-1}^{\widetilde{M}(b,c)}(t,t_2,\dots,\widehat{t_j},\dots,t_v)\right)\Big|_{t=-t_j}\\
&=\Bigg[\frac{(t^2-1)^3}{t(t+t_1)(t-t_1)}w_{g,v-1}^{\widetilde{M}(b,c)}(t,t_2,\dots,\widehat{t_j},\dots,t_v)\left(\frac{\partial}{\partial t}\frac{(t^2+t_j^2)}{(t-t_j)^2}\right)\\
&\quad+\frac{(t^2+t_j^2)}{(t-t_j)^2}\frac{\partial}{\partial t}\left(\frac{(t^2-1)^3}{t(t+t_1)(t-t_1)}w_{g,v-1}^{\widetilde{M}(b,c)}(t,t_2,\dots,\widehat{t_j},\dots,t_v)\right)\Bigg]_{t=-t_j}\\
&=\frac{(t_j^2-1)^3}{-t_j(-t_j+t_1)(-t_j-t_1)}w_{g,v-1}^{\widetilde{M}(b,c)}(t_2,\dots,t_v)\left(\frac{(-t_j-t_j)^2(-2t_j)-(t_j^2+t_j^2)2(-t_j-t_j)}{(-t_j-t_j)^4}\right)\\
&\quad+\frac{(t_j^2+t_j^2)}{(-t_j-t_j)^2}\left(-\frac{\partial}{\partial t_j}\right)\left(\frac{(t_j^2-1)^3}{-t_j(-t_j+t_1)(-t_j-t_1)}w_{g,v-1}^{\widetilde{M}(b,c)}(t_2,\dots,t_v)\right)\\
&=\frac{(t_j^2-1)^3}{-t_j(-t_j+t_1)(-t_j-t_1)}w_{g,v-1}^{\widetilde{M}(b,c)}(t_2,\dots,t_v)\left(\frac{-8t_j^3+8t_j^3}{(2t_j)^4}\right)\\
&\quad+\frac{2t_j^2}{(2t_j)^2}\left(-\frac{\partial}{\partial t_j}\right)\left(\frac{(t_j^2-1)^3}{t_j(-t_j+t_1)(t_j+t_1)}w_{g,v-1}^{\widetilde{M}(b,c)}(t_2,\dots,t_v)\right)\\
&=\frac{1}{2}\frac{\partial}{\partial t_j}\left(\frac{(t_j^2-1)^3}{t_j(t_j^2-t_1^2)}w_{g,v-1}^{\widetilde{M}(b,c)}(t_2,\dots,t_v)\right).
\end{aligned}
$$

Here, we applied the Cauchy Residue Theorem to evaluate this integral around the contour γ given in Figure 1.1.

The second term in the differential recursion formula of Theorem 5.0.1 becomes zero when we differentiate with respect to t_j, so

$$\mathrm{II}_{g,v}(t_1,t_2,\dots,t_v)=0.$$

Now, we may compute

$$
\begin{aligned}
&-\frac{1}{64}\frac{1}{2\pi i}\int_\gamma\left(\frac{1}{t+t_1}+\frac{1}{t-t_1}\right)\frac{(t^2-1)^3}{t^2}\,dt\\
&=\frac{1}{64}\left[\mathrm{Res}_{t=t_1}\frac{2(t^2-1)^3/t}{(t+t_1)(t-t_1)}+\mathrm{Res}_{t=-t_1}\frac{2(t^2-1)^3/t}{(t+t_1)(t-t_1)}\right]\\
&=\frac{1}{64}\left[\frac{2(t_1^2-1)^3/t_1}{2t_1}+\frac{2(t_1^2-1)^3/(-t_1)}{-2t_1}\right]\\
&=\frac{1}{32}\frac{(t_1^2-1)^3}{t_1^2}.
\end{aligned}
$$

Thus, for the third term in the differential recursion, we have

$$
\begin{aligned}
&\mathrm{III}_{g,v}(t_1, t_2, \ldots, t_v) \\
&= \frac{\partial}{\partial t_2} \cdots \frac{\partial}{\partial t_v} \left[-\frac{1}{32} \frac{(t_1^2-1)^3}{t_1^2} \frac{\partial}{\partial u_1} \frac{\partial}{\partial u_2} F_{g-1,v+1}^{\widetilde{M}(b,c)}(u_1, u_2, t_2, \ldots, t_v) \Big|_{u_1=u_2=t_1} \right] dt_1 dt_2 \cdots dt_v \\
&= -\frac{1}{32} \frac{(t_1^2-1)^3}{t_1^2} W_{g-1,v+1}^{\widetilde{M}(b,c)}(u_1, u_2, t_2, \ldots, t_v) \frac{1}{du_1} \frac{1}{du_2} \Big|_{u_1=u_2=t_1} dt_1 \\
&= -\frac{1}{32} \frac{(t_1^2-1)^3}{t_1^2} W_{g-1,v+1}^{\widetilde{M}(b,c)}(t_1, t_1, t_2, \ldots, t_v) \frac{1}{dt_1}.
\end{aligned}
$$

And, from the topological recursion formula, we have

$$
\begin{aligned}
&-\frac{1}{64} \frac{1}{2\pi i} \int_\gamma \left(\frac{1}{t+t_1} + \frac{1}{t-t_1} \right) \frac{(t^2-1)^3}{t^2} \cdot \frac{1}{dt} \cdot dt_1 \times W_{g-1,v+1}^{\widetilde{M}(b,c)}(t, -t, t_2, \ldots, t_v) \\
&= -\frac{1}{64} \frac{1}{2\pi i} \int_\gamma \left(\frac{1}{t+t_1} + \frac{1}{t-t_1} \right) \frac{(t^2-1)^3}{t^2} \cdot \frac{1}{dt} \cdot dt_1 \\
&\quad \times w_{g-1,v+1}^{\widetilde{M}(b,c)}(t, -t, t_2, \ldots, t_v)\, dt \cdot (-dt) \cdot dt_2 \cdots dt_v \\
&= \frac{1}{64} \left[\frac{1}{2\pi i} \int_\gamma \left(\frac{1}{t+t_1} + \frac{1}{t-t_1} \right) \frac{(t^2-1)^3}{t^2} \times w_{g-1,v+1}^{\widetilde{M}(b,c)}(t, t, t_2, \ldots, t_v)\, dt \right] dt_1 \cdot dt_2 \cdots dt_v \\
&= -\frac{1}{32} \frac{(t_1^2-1)^3}{t_1^2} w_{g-1,v+1}^{\widetilde{M}(b,c)}(t_1, t_1, t_2, \ldots, t_v)\, dt_1 \cdot dt_2 \cdots dt_v \\
&= -\frac{1}{32} \frac{(t_1^2-1)^3}{t_1^2} W_{g-1,v+1}^{\widetilde{M}(b,c)}(t_1, t_1, t_2, \ldots, t_v)\, \frac{1}{dt_1} \\
&= \mathrm{III}_{g,v}^{\widetilde{M}(b,c)}(t_1, t_2, \ldots, t_v).
\end{aligned}
$$

Again, we applied the Cauchy Residue Theorem to evaluate this integral around γ.

Finally, from the fourth term in the differential recursion formula, we have

$$
\begin{aligned}
&\mathrm{IV}_{g,v}(t_1, t_2, \ldots, t_v) \\
&= \frac{\partial}{\partial t_2} \cdots \frac{\partial}{\partial t_v} \left[-\frac{1}{32} \frac{(t_1^2-1)^3}{t_1^2} \right. \\
&\quad \left. \cdot \sum_{g_1+g_2=g,\ I \sqcup J=\{2,\ldots,v\},\ \text{stable}} \frac{\partial}{\partial t_1} F_{g_1,|I|+1}^{\widetilde{M}(b,c)}(t_1, t_I) \cdot \frac{\partial}{\partial t_1} F_{g_2,|J|+1}^{\widetilde{M}(b,c)}(t_1, t_J) \right] dt_1 dt_2 \cdots dt_v
\end{aligned}
$$

$$= -\frac{1}{32}\frac{(t_1^2-1)^3}{t_1^2} \sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{stable}} \frac{\partial}{\partial t_1}\frac{\partial}{\partial t_I} F^{\widetilde{M}(b,c)}_{g_1,|I|+1}(t_1,t_I)\ dt_1 dt_I$$
$$\cdot \frac{\partial}{\partial t_1}\frac{\partial}{\partial t_J} F^{\widetilde{M}(b,c)}_{g_2,|J|+1}(t_1,t_J) dt_1 dt_J \frac{1}{dt_1}$$
$$= -\frac{1}{32}\frac{(t_1^2-1)^3}{t_1^2} \sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{stable}} W^{\widetilde{M}(b,c)}_{g_1,|I|+1}(t_1,t_I)\cdot W^{\widetilde{M}(b,c)}_{g_2,|J|+1}(t_1,t_J)\cdot \frac{1}{dt_1}.$$

And, from the topological recursion formula, we have

$$-\frac{1}{64}\frac{1}{2\pi i}\int_\gamma \left(\frac{1}{t+t_1}+\frac{1}{t-t_1}\right)\frac{(t^2-1)^3}{t^2}\cdot\frac{1}{dt}\cdot dt_1$$
$$\times \left[\sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{stable}} \frac{\partial}{\partial t_1} W^{\widetilde{M}(b,c)}_{g_1,|I|+1}(t,t_I) W^{\widetilde{M}(b,c)}_{g_2,|J|+1}(-t,t_J)\right]$$
$$= -\frac{1}{64}\frac{1}{2\pi i}\int_\gamma \left(\frac{1}{t+t_1}+\frac{1}{t-t_1}\right)\frac{(t^2-1)^3}{t^2}\cdot\frac{1}{dt}\cdot dt_1$$
$$\times \sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{stable}} \left[w^D_{g_1,|I|+1}(t,t_I)\ dt\cdot\prod_{i\in I} dt_i\right]\left[w^D_{g_2,|J|+1}(-t,t_J)\ (-dt)\cdot\prod_{j\in J} dt_j\right]$$
$$= \frac{1}{64}\frac{1}{2\pi i}\int_\gamma \left(\frac{1}{t+t_1}+\frac{1}{t-t_1}\right)\frac{(t^2-1)^3}{t^2}$$
$$\times \sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{stable}} w^D_{g_1,|I|+1}(t,t_I) w^D_{g_2,|J|+1}(t,t_J)\ dt\cdot dt_1\cdot dt_2\cdots dt_v$$
$$= -\frac{1}{32}\frac{(t_1^2-1)^3}{t_1^2} \sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{stable}} w^D_{g_1,|I|+1}(t_1,t_I) w^D_{g_2,|J|+1}(t_1,t_J)\ dt_1\cdot dt_2\cdots dt_v$$
$$= -\frac{1}{32}\frac{(t_1^2-1)^3}{t_1^2} \sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{stable}} W^D_{g_1,|I|+1}(t_1,t_I) W^D_{g_2,|J|+1}(t_1,t_J)\ \frac{1}{dt_1}$$
$$= \mathrm{IV}_{g,v}(t_1,t_2,\ldots,t_v).$$

Yet again, we applied the Cauchy Residue Theorem to evaluate this integral around γ.

This completes the proof of Theorem 6.0.1. □

From this topological recursion in Theorem 6.0.1, and the initial case of $W^{\widetilde{M}(b,c)}_{0,2}$ given in that theorem, we can recursively compute all of the Eynard-Orantin differential forms $W^{\widetilde{M}(b,c)}_{g,v}$.

Let us now look at some examples.

Example 6.0.1. When $(g,v)=(1,1)$, we have

$$\begin{aligned} W_{1,1}^{\widetilde{M}(b,c)}(t_1) &= -\frac{1}{64}\frac{1}{2\pi i}\int_\gamma \left(\frac{1}{t+t_1}+\frac{1}{t-t_1}\right)\frac{(t^2-1)^3}{t^2}\,\frac{1}{dt}\,dt_1 \times W_{0,2}^{\widetilde{M}(b,c)}(t,-t) \\ &= \frac{1}{128}\frac{1}{2\pi i}\left[\int_\gamma \frac{(t^2-1)^3/t^3}{(t+t_1)(t-t_1)}\,dt\right]dt_1 \\ &= \frac{1}{128}\frac{1}{2\pi i}\left[-2\pi i\mathrm{Res}_{t=t_1}\frac{(t^2-1)^3/t^3}{(t+t_1)(t-t_1)} - 2\pi i\mathrm{Res}_{t=-t_1}\frac{(t^2-1)^3/t^3}{(t+t_1)(t-t_1)}\right]dt_1 \\ &= -\frac{1}{128}\frac{(t_1^2-1)^3}{t_1^4}\,dt_1. \end{aligned} \tag{6.8}$$

Example 6.0.2. Similarly, when $(g,v)=(0,3)$, we may compute

$$W_{0,3}^{\widetilde{M}(b,c)}(t_1,t_2,t_3) = -\frac{1}{16}\left(\frac{1}{t_1t_2t_3}-1\right). \tag{6.9}$$

CHAPTER 7

A Recursion Formula for the Generating Function of Generalized bc-Motzkin Numbers

In this chapter, we wish to study the generating function for the generalized bc-Motzkin numbers. In Chapter 8 these generating functions will be used to give a closed form formula for generalized Catalan numbers for some cases of small (g, v), and in Chapter 9 they will be used to give some identities for the generalized Catalan numbers for some cases of small (g, v).

We will first make the following definition.

Definition 7.0.1. *The generating function for the* $(0, 1)$ *bc-Motzkin numbers is given by*

$$G_{0,1}^{\widetilde{M}(b,c)}(x) = \sum_{n=0}^{\infty} \widetilde{M}_{0,1}(n; b, c)\, x^n. \tag{7.1}$$

Now, recall from Proposition 4.1.1 that the bc-Motzkin numbers $M_{0,1}(n; b, c)$ satisfy

$$\widetilde{M}_{0,1}(n; b, c) - b\,\widetilde{M}_{0,1}(n-1; b, c) = c^2 \sum_{\alpha+\beta=n-2} \widetilde{M}_{0,1}(\alpha; b, c)\widetilde{M}_{0,1}(\beta; b, c). \tag{7.2}$$

We may use this result to obtain the generating function for $M_{0,1}(n; b, c)$.

Proposition 7.0.1. *The generating function* $G_{0,1}^{\widetilde{M}(b,c)}(x)$ *satisfies*

$$G_{0,1}^{\widetilde{M}(b,c)}(x) = \frac{1 - bx - \sqrt{1 - 2bx + (b^2 - 4c^2)x^2}}{2c^2x^2}. \tag{7.3}$$

Proof. Since the recursion formula in equation (7.2) holds for $n \geq 1$, we have

$$\begin{aligned} G_{0,1}^{\widetilde{M}(b,c)}(x) &= \widetilde{M}_{0,1}(0; b, c) + \sum_{n=1}^{\infty} \widetilde{M}_{0,1}(n; b, c)\, x^n \\ &= 1 + \sum_{n=1}^{\infty} \left[b\,\widetilde{M}_{0,1}(n-1; b, c) + c^2 \sum_{\alpha+\beta=n-2} \widetilde{M}_{0,1}(\alpha; b, c)\widetilde{M}_{0,1}(\beta; b, c) \right] x^n \end{aligned}$$

$$= 1 + b \sum_{k+1=1}^{\infty} \widetilde{M}_{0,1}(k; b, c)\, x^{k+1} + c^2 \sum_{n=1}^{\infty} \sum_{\alpha+\beta=n-2} \widetilde{M}_{0,1}(\alpha; b, c)\widetilde{M}_{0,1}(\beta; b, c)\, x^{\alpha+\beta+2}$$

$$= 1 + bx\, G_{0,1}^{\widetilde{M}(b,c)}(x) + c^2 x^2 \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty} \widetilde{M}_{0,1}(\alpha; b, c)\widetilde{M}_{0,1}(\beta; b, c)\, x^{\alpha+\beta}$$

$$= 1 + bx\, G_{0,1}^{\widetilde{M}(b,c)}(x) + c^2 x^2 [G_{0,1}^{\widetilde{M}(b,c)}(x)]^2$$

Thus,

$$c^2 x^2 [G_{0,1}^{\widetilde{M}(b,c)}(x)]^2 + (bx - 1) G_{0,1}^{\widetilde{M}(b,c)}(x) + 1 = 0.$$

Applying the quadratic formula tells us that

$$G_{0,1}^{\widetilde{M}(b,c)}(x) = \frac{-(bx-1) \pm \sqrt{(bx-1)^2 - 4c^2x^2}}{2c^2x^2}.$$

Then, taking the negative square root and simplifying the result gives

$$G_{0,1}^{\widetilde{M}(b,c)}(x) = \frac{1 - bx - \sqrt{1 - 2bx + (b^2 - 4c^2)x^2}}{2c^2x^2}$$

as was claimed. □

Remark 7.0.1. As expected, setting $b = 0$ and $c = 1$ in Proposition 7.0.1 gives the Catalan generating function,

$$G_{0,1}^{C}(x) = \frac{1 - \sqrt{1 - 4x^2}}{2x^2}. \tag{7.4}$$

We now wish to use the recursion formula for generalized bc-Motzkin numbers which was given in Theorem 4.2.1 to obtain a recursion formula for the generating function for these generalized bc-Motzkin numbers.

We will first make the following definition.

Definition 7.0.2. *We define the generating function for the generalized bc-Motzkin numbers by*

$$G_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \cdots \sum_{n_v=0}^{\infty} \widetilde{M}_{g,v}(n_1, n_2, \ldots, n_v; b, c)\, x_1^{n_1} x_2^{n_2} \cdots x_v^{n_v}. \tag{7.5}$$

We may use this definition, along with the recursion formula in Theorem 4.2.1, to prove the following result.

Theorem 7.0.1. *For $(g,v) \neq (0,1),(0,2)$, the generalized bc-Motzkin number generating function $G^{\widetilde{M}(b,c)}_{g,v}(x_1,x_2,\dots,x_v)$ satisfies*

$$\begin{aligned} G^{\widetilde{M}(b,c)}_{g,v}(x_1,x_2,\dots,x_v) = [1-2bx_1+(b^2-4c^2)x_1^2]^{-1/2}\Bigg[&\mathcal{II}^{\widetilde{M}(b,c)}_{g,v}(x_1,x_2,\dots,x_v) \\ &+ 2c^2x_1^2\sum_{j=2}^{v} G^{\widetilde{M}(b,c)}_{g,v-1}(x_1,x_2,\dots,\hat{x_j},\dots,x_v)G^{\widetilde{M}(b,c)}_{0,2}(x_1,x_j) \\ &+ c^2x_1^2\, G^{\widetilde{M}(b,c)}_{g-1,v+1}(x_1,x_1,x_2,\dots,x_v) \\ &+ c^2x_1^2 \sum_{g_1+g_2=g,\ I\sqcup J=\{2,\dots,v\},\ stable} G^{\widetilde{M}(b,c)}_{g_1,|I|+1}(x_1,x_I)G^{\widetilde{M}(b,c)}_{g_2,|J|+1}(x_1,x_J)\Bigg] \end{aligned} \quad (7.6)$$

By "stable," we mean $2g_1+|I|-1>0$ and $2g_2+|J|-1>0$.

Here,

$$\begin{aligned} &\mathcal{II}^{\widetilde{M}(b,c)}_{g,v}(x_1,x_2,\dots,x_v) \\ &= c^2\sum_{j=2}^{v}\Bigg\{\frac{x_1x_j}{(x_1-x_j)^2}\left[x_1^2\cdot G^{\widetilde{M}(b,c)}_{g,v-1}(x_1,x_2,\dots,\hat{x_j},\dots,x_v) - x_j^2\cdot G^{\widetilde{M}(b,c)}_{g,v-1}(x_j,x_2,\dots,\hat{x_j},\dots,x_v)\right] \\ &\qquad - \frac{x_1x_j}{x_1-x_j}\cdot\frac{\partial}{\partial x_j}\left[x_j^2\cdot G^{\widetilde{M}(b,c)}_{g,v-1}(x_j,x_2,\dots,\hat{x_j},\dots,x_v)\right]\Bigg\} \end{aligned} \quad (7.7)$$

and if $x_j = x_1$ for $j \neq 1$, we take the limit of the above expression for $\mathcal{II}^{\widetilde{M}(b,c)}_{g,v}(x_1,x_2,\dots,x_v)$ as $x_j \to x_1$.

Further, $G^{\widetilde{M}(b,c)}_{0,1}(x)$ is as given in Proposition 7.0.1, and

$$G^{\widetilde{M}(b,c)}_{0,2}(x_1,x_2) = [1-2bx_1+(b^2-4c^2)x_1^2]^{-1/2}\left[\mathcal{II}^{\widetilde{M}(b,c)}_{0,2}(x_1,x_2)\right]. \quad (7.8)$$

Remark 7.0.2. This theorem can be proved directly from Theorem 5.0.1 for the discrete Laplace transform of the generalized bc-Motzkin numbers, since

$$G^{\widetilde{M}(b,c)}_{g,v}(x_1^{-1},x_2^{-1},\dots,x_v^{-1}) = (-1)^v x_1x_2\cdots x_v\left[\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\cdots\frac{\partial}{\partial x_v}F^{\widetilde{M}(b,c)}_{g,v}(x_1,x_2,\dots,x_v)\right]. \quad (7.9)$$

However, we present here a stand-alone proof of Theorem 7.0.1, which is interesting in its own right for the techniques applied.

PROOF. Assume $(g,v) \neq (0,1)$. Then, since the recursion formula for the generalized bc-Motzkin numbers holds for $n_1 \geq 1$, we may compute:

$$\begin{aligned}
&G_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) \\
&= \sum_{n_2=0}^{\infty} \cdots \sum_{n_v=0}^{\infty} \widetilde{M}_{g,v}(0, n_2, \ldots, n_v; b, c)\, x_1^{n_1} x_2^{n_2} \cdots x_v^{n_v} \\
&\quad + \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_v=0}^{\infty} \widetilde{M}_{g,v}(n_1, n_2, \ldots, n_v; b, c)\, x_1^{n_1} x_2^{n_2} \cdots x_v^{n_v} \\
&= 0 + \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_v=0}^{\infty} \Bigg[b\widetilde{M}_{g,v}(n_1 - 1, n_2, \ldots, n_v; b, c) \\
&\quad + c^2 \Bigg(\sum_{j=2}^{v} n_j\, \widetilde{M}_{g,v-1}(n_1 + n_j - 2, n_2, \ldots, \hat{n_j}, \ldots, n_v; b, c) \\
&\quad + \sum_{\zeta+\xi=n_1-2} \widetilde{M}_{g-1,v+1}(\zeta, \xi, n_2, \ldots, n_v; b, c) \\
&\quad + \sum_{g_1+g_2=g} \sum_{I \sqcup J=\{2,\ldots,v\}} \sum_{\zeta+\xi=n_1-2} \widetilde{M}_{g_1,|I|+1}(\zeta, n_I; b, c) \widetilde{M}_{g_2,|J|+1}(\xi, n_J; b, c) \Bigg) \Bigg] \cdot x_1^{n_1} x_2^{n_2} \cdots x_v^{n_v}
\end{aligned}$$

For notational convenience, we will write this as four terms,

$$\begin{aligned}
G_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) &= \mathcal{I}_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) + \mathcal{II}_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) \\
&\quad + \mathcal{III}_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) + \mathcal{IV}_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) \qquad (7.10)
\end{aligned}$$

Now, for the first term, we have:

$$\begin{aligned}
&\mathcal{I}_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) \\
&:= \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_v=0}^{\infty} \left[b\widetilde{M}_{g,v}(n_1 - 1, n_2, \ldots, n_v; b, c) \right] x_1^{n_1} x_2^{n_2} \cdots x_v^{n_v} \\
&= b \sum_{k+1=1}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_v=0}^{\infty} \widetilde{M}_{g,v}(k, n_2, \ldots, n_v; b, c)\, x_1^{k+1} x_2^{n_2} \cdots x_v^{n_v} \\
&= bx_1 \sum_{k=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_v=0}^{\infty} \widetilde{M}_{g,v}(k, n_2, \ldots, n_v; b, c)\, x_1^{k} x_2^{n_2} \cdots x_v^{n_v} \\
&= bx_1\, G_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v)
\end{aligned}$$

For the third term, we have:

$$
\begin{aligned}
&\mathcal{III}_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) \\
&:= \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_v=0}^{\infty} \left[c^2 \sum_{\zeta+\xi=n_1-2} \widetilde{M}_{g-1,v+1}(\zeta, \xi, n_2, \ldots, n_v; b, c) \right] x_1^{n_1} x_2^{n_2} \cdots x_v^{n_v} \\
&= c^2 \sum_{\zeta=0}^{\infty} \sum_{\xi=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_v=0}^{\infty} \widetilde{M}_{g-1,v+1}(\zeta, \xi, n_2, \ldots, n_v; b, c)\, x_1^{\zeta+\xi+2} x_2^{n_2} \cdots x_v^{n_v} \\
&= c^2 x_1^2\, G_{g-1,v+1}^{\widetilde{M}(b,c)}(x_1, x_1, x_2, \ldots, x_v)
\end{aligned}
$$

And, for the fourth term, we have:

$$
\begin{aligned}
&\mathcal{IV}_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) \\
&:= \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_v=0}^{\infty} \left[c^2 \sum_{g_1+g_2=g} \sum_{I \sqcup J = \{2,\ldots,v\}} \sum_{\zeta+\xi=n_1-2} \widetilde{M}_{g_1,|I|+1}(\zeta, n_I; b, c) \widetilde{M}_{g_2,|J|+1}(\xi, n_J; b, c) \right] \\
&\qquad \cdot x_1^{n_1} x_2^{n_2} \cdots x_v^{n_v} \\
&= c^2 \sum_{g_1+g_2=g} \sum_{I \sqcup J = \{2,\ldots,v\}} \left[\sum_{\zeta=0}^{\infty} \sum_{\xi=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_v=0}^{\infty} \widetilde{M}_{g_1,|I|+1}(\zeta, n_I; b, c) \widetilde{M}_{g_2,|J|+1}(\xi, n_J; b, c) \right] \\
&\qquad \cdot x_1^{\zeta+\xi+2} x_2^{n_2} \cdots x_v^{n_v} \\
&= c^2 x_1^2 \sum_{g_1+g_2=g} \sum_{I \sqcup J = \{2,\ldots,v\}} G_{g_1,|I|+1}^{\widetilde{M}(b,c)}(x_1, x_I) G_{g_2,|J|+1}^{\widetilde{M}(b,c)}(x_1, x_J)
\end{aligned}
$$

Putting all this together, we thus see that

$$
\begin{aligned}
G_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) &= b x_1\, G_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) \\
&\quad + \mathcal{II}_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) \\
&\quad + c^2 x_1^2\, G_{g-1,v+1}^{\widetilde{M}(b,c)}(x_1, x_1, x_2, \ldots, x_v) \\
&\quad + c^2 x_1^2 \sum_{g_1+g_2=g} \sum_{I \sqcup J = \{2,\ldots,v\}} G_{g_1,|I|+1}^{\widetilde{M}(b,c)}(x_1, x_I) G_{g_2,|J|+1}^{\widetilde{M}(b,c)}(x_1, x_J)
\end{aligned}
\tag{7.11}
$$

where

$$\mathcal{II}_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v)$$

$$:= \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_v=0}^{\infty} \left[c^2 \sum_{j=2}^{v} n_j \, \widetilde{M}_{g,v-1}(n_1 + n_j - 2, n_2, \ldots, \hat{n_j}, \ldots, n_v; b, c) \right] \cdot x_1^{n_1} x_2^{n_2} \cdots x_v^{n_v} \quad (7.12)$$

Now, when $(g, v) = (0, 2)$, equation (7.11) becomes

$$\begin{aligned} G_{0,2}^{\widetilde{M}(b,c)}(x_1, x_2) &= bx_1 \, G_{0,2}^{\widetilde{M}(b,c)}(x_1, x_2) \\ &+ \mathcal{II}_{0,2}^{\widetilde{M}(b,c)}(x_1, x_2) \\ &+ 2c^2 x_1^2 G_{0,1}^{\widetilde{M}(b,c)}(x_1) G_{0,2}^{\widetilde{M}(b,c)}(x_1, x_2) \end{aligned}$$

Rearranging terms so that $G_{0,2}^{\widetilde{M}(b,c)}(x_1, x_2)$ only appears on the left side, and substituting in for $G_{0,1}^{\widetilde{M}(b,c)}(x_1)$, gives

$$\begin{aligned} G_{0,2}^{\widetilde{M}(b,c)}(x_1, x_2) &= [1 - bx_1 - 2c^2 x_1^2 G_{0,1}^{\widetilde{M}(b,c)}(x_1)]^{-1} \mathcal{II}_{0,2}^{\widetilde{M}(b,c)}(x_1, x_2) \\ &= [1 - 2bx_1 + (b^2 - 4c^2) x_1^2]^{-1/2} \left[\mathcal{II}_{0,2}^{\widetilde{M}(b,c)}(x_1, x_2) \right] \end{aligned}$$

as was claimed.

We would like to obtain a formula where the (g, v)-term $G_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v)$ appears only on the left side. Thus, for $(g, v) \neq (0, 1), (0, 2)$, we see that:

$$\begin{aligned} G_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) &= bx_1 \, G_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) \\ &+ \mathcal{II}_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) \\ &+ c^2 x_1^2 \, G_{g-1,v+1}^{\widetilde{M}(b,c)}(x_1, x_1, x_2, \ldots, x_v) \\ &+ c^2 x_1^2 \Bigg[2G_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) G_{0,1}^{\widetilde{M}(b,c)}(x_1) \\ &\quad + 2 \sum_{j=2}^{v} G_{g,v-1}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, \hat{x_j}, \ldots, x_v) G_{0,2}^{\widetilde{M}(b,c)}(x_1, x_j) \\ &\quad + \sum_{g_1+g_2=g, \; I \sqcup J = \{2,\ldots,v\}, \text{ stable}} G_{g_1,|I|+1}^{\widetilde{M}(b,c)}(x_1, x_I) G_{g_2,|J|+1}^{\widetilde{M}(b,c)}(x_1, x_J) \Bigg] \end{aligned}$$

This implies:

$$
\begin{aligned}
&G^{\widetilde{M}(b,c)}_{g,v}(x_1,x_2,\dots,x_v)[1-bx_1-2c^2x_1^2G^{\widetilde{M}(b,c)}_{0,1}(x_1)]\\
&=\mathcal{II}^{\widetilde{M}(b,c)}_{g,v}(x_1,x_2,\dots,x_v)+2c^2x_1^2\sum_{j=2}^{v}G^{\widetilde{M}(b,c)}_{g,v-1}(x_1,x_2,\dots,\hat{x_j},\dots,x_v)G^{\widetilde{M}(b,c)}_{0,2}(x_1,x_j)\\
&\quad+c^2x_1^2\,G^{\widetilde{M}(b,c)}_{g-1,v+1}(x_1,x_1,x_2,\dots,x_v)\\
&\quad+c^2x_1^2\sum_{g_1+g_2=g,\ I\sqcup J=\{2,\dots,v\},\ \text{stable}}G^{\widetilde{M}(b,c)}_{g_1,|I|+1}(x_1,x_I)G^{\widetilde{M}(b,c)}_{g_2,|J|+1}(x_1,x_J)
\end{aligned}
$$

Hence, this becomes

$$
\begin{aligned}
G^{\widetilde{M}(b,c)}_{g,v}(x_1,x_2,\dots,x_v)=[1-2bx+(b^2-4c^2)x^2]^{-1/2}\Big[&\mathcal{II}^{\widetilde{M}(b,c)}_{g,v}(x_1,x_2,\dots,x_v)\\
&+2c^2x_1^2\sum_{j=2}^{v}G^{\widetilde{M}(b,c)}_{g,v-1}(x_1,x_2,\dots,\hat{x_j},\dots,x_v)G^{\widetilde{M}(b,c)}_{0,2}(x_1,x_j)\\
&+c^2x_1^2\,G^{\widetilde{M}(b,c)}_{g-1,v+1}(x_1,x_1,x_2,\dots,x_v)\\
&+c^2x_1^2\sum_{g_1+g_2=g,\ I\sqcup J=\{2,\dots,v\},\ \text{stable}}G^{\widetilde{M}(b,c)}_{g_1,|I|+1}(x_1,x_I)G^{\widetilde{M}(b,c)}_{g_2,|J|+1}(x_1,x_J)\Big]
\end{aligned}
$$

which is equation (7.6) in the above theorem.

Our next step is to obtain the formula (7.7) for $\mathcal{II}^{\widetilde{M}(b,c)}_{g,v}(x_1,x_2,\dots,x_v)$ as stated in the theorem. First, observe that:

$$
\begin{aligned}
&\mathcal{II}^{\widetilde{M}(b,c)}_{g,v}(x_1,x_2,\dots,x_v)\\
&=\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_v=0}^{\infty}\left[c^2\sum_{j=2}^{v}n_j\,\widetilde{M}_{g,v-1}(n_1+n_j-2,n_2,\dots,\hat{n_j},\dots,n_v;b,c)\right]x_1^{n_1}x_2^{n_2}\cdots x_v^{n_v}\\
&=c^2\sum_{j=2}^{v}\sum_{n_2=0}^{\infty}\cdots\widehat{\sum_{n_j=0}^{\infty}}\cdots\sum_{n_v=0}^{\infty}\left[\sum_{n_1=1}^{\infty}\sum_{n_j=0}^{\infty}n_j\,\widetilde{M}_{g,v-1}(n_1+n_j-2,n_2,\dots,\hat{n_j},\dots,n_v;b,c)x_1^{n_1}x_j^{n_j}\right]\\
&\quad\cdot x_2^{n_1}\cdots\widehat{x_j^{n_j}}\cdots x_v^{n_v}\\
&=c^2\sum_{j=2}^{v}\sum_{n_2=0}^{\infty}\cdots\widehat{\sum_{n_j=0}^{\infty}}\cdots\sum_{n_v=0}^{\infty}\left[\sum_{k=0}^{\infty}\sum_{n_1+n_j=k+2}\left(n_j\,x_1^{n_1}x_j^{n_j}\right)\widetilde{M}_{g,v-1}(k,n_2,\dots,\hat{n_j},\dots,n_v;b,c)\right]\\
&\quad\cdot x_2^{n_1}\cdots\widehat{x_j^{n_j}}\cdots x_v^{n_v}
\end{aligned}
$$

$$- c^2 \sum_{j=2}^{v} \sum_{n_2=0}^{\infty} \cdots \widehat{\sum_{n_j=0}^{\infty}} \cdots \sum_{n_v=0}^{\infty} \left[\sum_{n_j=0}^{\infty} n_j \, \widetilde{M}_{g,v-1}(n_j - 2, n_2, \ldots, \hat{n_j}, \ldots, n_v; b, c) x_j^{n_j} \right]$$
$$\cdot x_2^{n_1} \cdots \widehat{x_j^{n_j}} \cdots x_v^{n_v}$$

Now, assuming $x_j \neq x_1$ for $j \neq 1$, we may compute:

$$\begin{aligned} \sum_{n_1+n_j=k+2} \left(n_j \, x_1^{n_1} x_j^{n_j}\right) &= \sum_{n_j=0}^{k+2} \left(n_j \, x_1^{(k+2)-n_j} x_j^{n_j}\right) \\ &= x_1^{k+1} x_j \sum_{n_j=0}^{k+2} n_j \, (x_j/x_1)^{n_j - 1} \end{aligned} \tag{7.13}$$

Recall that

$$\sum_{m=0}^{\ell} x^m = \frac{x^{\ell+1} - 1}{x - 1}. \tag{7.14}$$

So,

$$\begin{aligned} \sum_{m=0}^{\ell} m x^{m-1} &= \frac{d}{dx} \sum_{m=0}^{\ell} x^m \\ &= \frac{d}{dx} \frac{x^{\ell+1} - 1}{x - 1} \\ &= \frac{(x-1)((\ell+1)x^{\ell}) - (x^{\ell+1} - 1)}{(x-1)^2} \\ &= \frac{(\ell+1)x^{\ell+1} - (\ell+1)x^{\ell} - x^{\ell+1} + 1}{(x-1)^2} \\ &= \frac{\ell x^{\ell+1} - (\ell+1)x^{\ell} + 1}{(x-1)^2} \\ &= \frac{\ell x^{\ell}(x-1) + x^{\ell} + 1}{(x-1)^2} \end{aligned} \tag{7.15}$$

Thus, plugging the result of equation (7.13) into (7.15), we see that

$$\begin{aligned} \sum_{n_1+n_j=k+2} \left(n_j \, x_1^{n_1} x_j^{n_j}\right) &= x_1^{k+1} x_j \left[\frac{(k+2)(x_j/x_1)^{k+2+1} - (k+2+1)(x_j/x_1)^{k+2} + 1}{((x_j/x_1) - 1)^2} \right] \\ &= x_j \left[\frac{x_1^{k+3}[(k+2)(x_j/x_1)^{k+3} - (k+3)(x_j/x_1)^{k+2} + 1]}{x_1^2((x_j/x_1) - 1)^2} \right] \\ &= x_j \left[\frac{(k+2)x_j^{k+3} - (k+3)x_1 x_j^{k+2} + x_1^{k+3}}{(x_j - x_1)^2} \right] \end{aligned}$$

$$= \frac{(k+2)x_j^{k+4} - (k+3)x_1 x_j^{k+3} + x_1^{k+3} x_j}{(x_1 - x_j)^2}$$

$$= \frac{(k+2)x_j^{k+4} - (k+2)x_1 x_j^{k+3}}{(x_1 - x_j)^2} - \frac{x_1 x_j^3}{(x_1 - x_j)^2}\, x_j^k + \frac{x_1^3 x_j}{(x_1 - x_j)^2}\, x_1^k$$

$$= \frac{x_j^2 (x_j - x_1)}{(x_1 - x_j)^2}\,(k+2)x_j^{k+1} - \frac{x_1 x_j^3}{(x_1 - x_j)^2}\, x_j^k + \frac{x_1^3 x_j}{(x_1 - x_j)^2}\, x_1^k$$

$$= \frac{x_1^3 x_j}{(x_1 - x_j)^2}\, x_1^k - \frac{x_1 x_j^3}{(x_1 - x_j)^2}\, x_j^k - \frac{x_j^2}{x_1 - x_j}\,(k+2)x_j^{k+1}$$

Next, observe that

$$\frac{\partial}{\partial x_j}\left[x_j^2 \cdot G_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v)\right]$$
$$= \sum_{n_1=0}^{\infty} \cdots \sum_{n_v=0}^{\infty} (n_j + 2)\, \widetilde{M}_{g,v}(n_1, n_2, \ldots, n_v; b, c)\, x_1^{n_1} \cdots x_j^{n_j+1} \cdots x_v^{n_v}$$

And,

$$c^2 \sum_{j=2}^{v} \sum_{n_2=0}^{\infty} \cdots \widehat{\sum_{n_j=0}^{\infty}} \cdots \sum_{n_v=0}^{\infty} \left[\sum_{n_j=0}^{\infty} n_j\, \widetilde{M}_{g,v-1}(n_j - 2, n_2, \ldots, \hat{n_j}, \ldots, n_v; b, c) x_j^{n_j}\right]$$
$$\cdot\, x_2^{n_1} \cdots \widehat{x_j^{n_j}} \cdots x_v^{n_v}$$

$$= c^2 \sum_{j=2}^{v} \sum_{n_2=0}^{\infty} \cdots \widehat{\sum_{n_j=0}^{\infty}} \cdots \sum_{n_v=0}^{\infty} \left[\sum_{k+2=2}^{\infty} (k+2)\, \widetilde{M}_{g,v-1}(k, n_2, \ldots, \hat{n_j}, \ldots, n_v; b, c) x_j^{k+2}\right]$$
$$\cdot\, x_2^{n_1} \cdots \widehat{x_j^{n_j}} \cdots x_v^{n_v}$$

$$= c^2 x_j \sum_{j=2}^{v} \frac{\partial}{\partial x_j}\left[x_j^2 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_j, x_2, \ldots, \hat{x_j}, \ldots, x_v)\right]$$

Putting all this back into our above work for the term $\mathcal{II}_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v)$ then gives:

$$\mathcal{II}_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v)$$
$$= c^2 \sum_{j=2}^{v} \left\{ \frac{x_1^3 x_j}{(x_1 - x_j)^2} \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, \hat{x_j}, \ldots, x_v) \right.$$
$$- \frac{x_1 x_j^3}{(x_1 - x_j)^2} \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_j, x_2, \ldots, \hat{x_j}, \ldots, x_v)$$

$$
\begin{aligned}
&- \frac{x_j^2}{x_1 - x_j} \cdot \frac{\partial}{\partial x_j}\left[x_j^2 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_j, x_2, \ldots, \hat{x}_j, \ldots, x_v)\right] \\
&- x_j \cdot \frac{\partial}{\partial x_j}\left[x_j^2 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_j, x_2, \ldots, \hat{x}_j, \ldots, x_v)\right]\Bigg\} \\
= c^2 \sum_{j=2}^{v} \Bigg\{ &\frac{x_1 x_j}{(x_1 - x_j)^2}\left[x_1^2 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, \hat{x}_j, \ldots, x_v)\right. \\
&\left. - x_j^2 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_j, x_2, \ldots, \hat{x}_j, \ldots, x_v)\right] \\
&- \frac{x_1 x_j}{x_1 - x_j} \cdot \frac{\partial}{\partial x_j}\left[x_j^2 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_j, x_2, \ldots, \hat{x}_j, \ldots, x_v)\right]\Bigg\}
\end{aligned}
$$

as was claimed.

Observe that this holds as long as none of the x_j are equal to x_1 when $j \neq 1$. If $x_j = x_1$ for $j \neq 1$, then the jth term in the definition of $\mathcal{II}_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v)$ becomes:

$$
\begin{aligned}
&c^2 \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_v=0}^{\infty} n_j \widetilde{M}_{g,v-1}(n_1 + n_j - 2, n_2, \ldots, \hat{n}_j, \ldots, n_v; b, c)\, x_1^{n_1} x_2^{n_2} \cdots x_v^{n_v}\Big|_{x_j = x_1} \\
&= c^2 \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_v=0}^{\infty} n_j \widetilde{M}_{g,v-1}(n_1 + n_j - 2, n_2, \ldots, \hat{n}_j, \ldots, n_v; b, c)\, x_1^{n_1 + n_j} \\
&\quad \cdot x_2^{n_2} \cdots \widehat{x_j^{n_j}} \cdots x_v^{n_v} \\
&= c^2 \Bigg[\sum_{k=0}^{\infty} \sum_{n_1 + n_j = k+2} \sum_{n_2=0}^{\infty} \cdots \widehat{\sum_{n_j=0}^{\infty}} \cdots \sum_{n_v=0}^{\infty} n_j \widetilde{M}_{g,v-1}(k, n_2, \ldots, \hat{n}_j, \ldots, n_v; b, c) \\
&\quad \cdot x_1^{k+2} x_2^{n_2} \cdots \widehat{x_j^{n_j}} \cdots x_v^{n_v} \\
&\quad - \sum_{n_2=0}^{\infty} \cdots \widehat{\sum_{n_j=0}^{\infty}} \cdots \sum_{n_v=0}^{\infty} n_j \widetilde{M}_{g,v-1}(n_j - 2, n_2, \ldots, \hat{n}_j, \ldots, n_v; b, c) \\
&\quad \cdot x_1^{n_j} x_2^{n_2} \cdots \widehat{x_j^{n_j}} \cdots x_v^{n_v} \Bigg] \\
&= c^2 \Bigg[\sum_{k=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \widehat{\sum_{n_j=0}^{\infty}} \cdots \sum_{n_v=0}^{\infty} \Bigg(\sum_{n_1 + n_j = k+2} n_j \Bigg) \widetilde{M}_{g,v-1}(k, n_2, \ldots, \hat{n}_j, \ldots, n_v; b, c) \\
&\quad \cdot x_1^{k+2} x_2^{n_2} \cdots \widehat{x_j^{n_j}} \cdots x_v^{n_v} \\
&\quad - \sum_{\ell=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \widehat{\sum_{n_j=0}^{\infty}} \cdots \sum_{n_v=0}^{\infty} (\ell + 2) \widetilde{M}_{g,v-1}(\ell, n_2, \ldots, \hat{n}_j, \ldots, n_v; b, c)
\end{aligned}
$$

$$
\begin{aligned}
&\quad \cdot x_1^{\ell+2} x_2^{n_2} \cdots \widehat{x_j^{n_j}} \cdots x_v^{n_v} \Bigg] \\
&= c^2 x_1 \Bigg[\frac{1}{2} \sum_{k=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \widehat{\sum_{n_j=0}^{\infty}} \cdots \sum_{n_v=0}^{\infty} (k+2)(k+3) \widetilde{M}_{g,v-1}(k, n_2, \ldots, \hat{n}_j, \ldots, n_v; b, c) \\
&\quad \cdot x_1^{k+1} x_2^{n_2} \cdots \widehat{x_j^{n_j}} \cdots x_v^{n_v} \\
&\quad - \sum_{\ell=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_v=0}^{\infty} (\ell+2) \widetilde{M}_{g,v-1}(\ell, n_2, \ldots, \hat{n}_j, \ldots, n_v; b, c) \\
&\quad \cdot x_1^{\ell+1} x_2^{n_2} \cdots \widehat{x_j^{n_j}} \cdots x_v^{n_v} \Bigg] \\
&= c^2 x_1 \Bigg\{ \frac{1}{2} \frac{\partial^2}{\partial x_1^2} \Big[x_1^3 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, \hat{x}_j, \ldots, x_v) \Big] \\
&\quad - \frac{\partial}{\partial x_1} \Big[x_1^2 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, \hat{x}_j, \ldots, x_v) \Big] \Bigg\}
\end{aligned}
$$

Here, we have used that:

$$
\begin{aligned}
&\frac{\partial^2}{\partial x_1^2} \Big[x_1^3 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, \hat{x}_j, \ldots, x_v) \Big] \\
&= \frac{\partial^2}{\partial x_1^2} \Bigg[\sum_{k=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_v=0}^{\infty} \widetilde{M}_{g,v-1}(k, n_2, \ldots, \hat{n}_j, \ldots, n_v; b, c)\, x_1^{k+3} x_2^{n_2} \cdots x_v^{n_v} \Bigg] \\
&= \sum_{k=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_v=0}^{\infty} (k+3)(k+2) \widetilde{M}_{g,v-1}(k, n_2, \ldots, \hat{n}_j, \ldots, n_v; b, c)\, x_1^{k+1} x_2^{n_2} \cdots x_v^{n_v}
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial x_1} \Big[x_1^2 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, \hat{x}_j, \ldots, x_v) \Big] \\
&= \frac{\partial}{\partial x_1} \Bigg[\sum_{\ell=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_v=0}^{\infty} \widetilde{M}_{g,v-1}(\ell, n_2, \ldots, \hat{n}_j, \ldots, n_v; b, c)\, x_1^{\ell+2} x_2^{n_2} \cdots x_v^{n_v} \Bigg] \\
&= \sum_{\ell=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_v=0}^{\infty} (\ell+2) \widetilde{M}_{g,v-1}(\ell, n_2, \ldots, \hat{n}_j, \ldots, n_v; b, c)\, x_1^{\ell+1} x_2^{n_2} \cdots x_v^{n_v}
\end{aligned}
$$

For notational convenience, we will write

$$
G := G_{g,v-1}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, \hat{x}_j, \ldots, x_v). \tag{7.16}
$$

Then, this becomes

$$
\begin{aligned}
&c^2 x_1 \left(\frac{1}{2} \frac{\partial^2}{\partial x_1^2} [x_1^3 \cdot G] - \frac{\partial}{\partial x_1} [x_1^2 \cdot G] \right) \\
&= c^2 x_1 \left[\frac{1}{2} \left(6x_1 \cdot G + 6x_1^2 \cdot \frac{\partial}{\partial x_1} G + x_1^3 \cdot \frac{\partial^2}{\partial x_1^2} G \right) - \left(2x_1 \cdot G + x_1^2 \cdot \frac{\partial}{\partial x_1} G \right) \right] \\
&= c^2 \left[x_1^2 \cdot G + 2x_1^3 \cdot \frac{\partial}{\partial x_1} G + \frac{1}{2} x_1^4 \cdot \frac{\partial^2}{\partial x_1^2} G \right]
\end{aligned}
$$

Now, the jth term in (7.7) is

$$
\begin{aligned}
&c^2 \left\{ \frac{x_1 x_j}{(x_1 - x_j)^2} \left[x_1^2 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_1, x_2, \dots, \hat{x}_j, \dots, x_v) - x_j^2 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_j, x_2, \dots, \hat{x}_j, \dots, x_v) \right] \right. \\
&\left. - \frac{x_1 x_j}{x_1 - x_j} \cdot \frac{\partial}{\partial x_j} \left[x_j^2 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_j, x_2, \dots, \hat{x}_j, \dots, x_v) \right] \right\} \\
&= c^2 \frac{1}{(x_1 - x_j)^2} \left[x_1 x_j \big[x_1^2 \, G_{g,v-1}^{\widetilde{M}(b,c)}(x_1, x_2, \dots, \hat{x}_j, \dots, x_v) - x_j^2 \, G_{g,v-1}^{\widetilde{M}(b,c)}(x_j, x_2, \dots, \hat{x}_j, \dots, x_v) \big] \right. \\
&\left. - x_1 x_j (x_1 - x_j) \, \frac{\partial}{\partial x_j} \big[x_j^2 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_j, x_2, \dots, \hat{x}_j, \dots, x_v) \big] \right]
\end{aligned}
$$

Letting $x_1 = x_j + h$, and taking the limit as $h \to 0$, we obtain

$$
\begin{aligned}
&\lim_{h \to 0} c^2 \frac{1}{h^2} \Big[(x_j + h) x_j \big[(x_j + h)^2 \, G_{g,v-1}^{\widetilde{M}(b,c)}(x_j + h, x_2, \dots, \hat{x}_j, \dots, x_v) \\
&\quad - x_j^2 \, G_{g,v-1}^{\widetilde{M}(b,c)}(x_j, x_2, \dots, \hat{x}_j, \dots, x_v) \big] \\
&\quad - (x_j + h) x_j h \, \frac{\partial}{\partial x_j} \big[x_j^2 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_j, x_2, \dots, \hat{x}_j, \dots, x_v) \big] \Big] \\
&= \lim_{h \to 0} c^2 \frac{1}{2h} \Big[x_j \big[(x_j + h)^2 \, G_{g,v-1}^{\widetilde{M}(b,c)}(x_j + h, x_2, \dots, \hat{x}_j, \dots, x_v) \\
&\quad - x_j^2 \, G_{g,v-1}^{\widetilde{M}(b,c)}(x_j, x_2, \dots, \hat{x}_j, \dots, x_v) \big] \\
&\quad + (x_j + h) x_j \big[2(x_j + h) \, G_{g,v-1}^{\widetilde{M}(b,c)}(x_j + h, x_2, \dots, \hat{x}_j, \dots, x_v) \\
&\quad + (x_j + h)^2 \frac{\partial}{\partial h} G_{g,v-1}^{\widetilde{M}(b,c)}(x_j + h, x_2, \dots, \hat{x}_j, \dots, x_v) \big] \\
&\quad - (x_j + 2h) x_j \, \frac{\partial}{\partial x_j} \big[x_j^2 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_j, x_2, \dots, \hat{x}_j, \dots, x_v) \big] \Big] \\
&= \lim_{h \to 0} c^2 \frac{1}{2h} \Big[x_j \big[(x_j + h)^2 \, G_{g,v-1}^{\widetilde{M}(b,c)}(x_j + h, x_2, \dots, \hat{x}_j, \dots, x_v)
\end{aligned}
$$

$$
\begin{aligned}
&- x_j^2 \, G_{g,v-1}^{\widetilde{M}(b,c)}(x_j, x_2, \ldots, \hat{x_j}, \ldots, x_v)\big] \\
&+ (x_j + h)x_j \cdot \frac{\partial}{\partial(x_j+h)}\big[(x_j+h)^2 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_j+h, x_2, \ldots, \hat{x_j}, \ldots, x_v)\big] \\
&- (x_j + 2h)x_j \cdot \frac{\partial}{\partial x_j}\big[x_j^2 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_j, x_2, \ldots, \hat{x_j}, \ldots, x_v)\big]\Bigg]
\end{aligned}
$$

Here, we applied L'Hopital's Rule, since this has indeterminate form $\frac{0}{0}$, and we also used the observation that

$$
\begin{aligned}
\frac{\partial}{\partial x_1} G_{g,v-1}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, \hat{x_j}, \ldots, x_v) &= \frac{\partial h}{\partial x_1}\frac{\partial}{\partial h} G_{g,v-1}^{\widetilde{M}(b,c)}(x_j+h, x_2, \ldots, \hat{x_j}, \ldots, x_v) \\
&= \frac{\partial}{\partial h} G_{g,v-1}^{\widetilde{M}(b,c)}(x_j+h, x_2, \ldots, \hat{x_j}, \ldots, x_v)
\end{aligned}
$$

Again, the term in brackets evaluates to zero, so we may apply L'Hopital's rule a second time to see that this equals:

$$
\begin{aligned}
&\lim_{h\to 0} c^2 \frac{1}{2}\Bigg[2x_j(x_j+h)\, G_{g,v-1}^{\widetilde{M}(b,c)}(x_j+h, x_2, \ldots, \hat{x_j}, \ldots, x_v) \\
&+ x_j(x_j+h)^2 \cdot \frac{\partial}{\partial(x_j+h)} G_{g,v-1}^{\widetilde{M}(b,c)}(x_j+h, x_2, \ldots, \hat{x_j}, \ldots, x_v) \\
&+ x_j \cdot \frac{\partial}{\partial(x_j+h)}\big[(x_j+h)^2 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_j+h, x_2, \ldots, \hat{x_j}, \ldots, x_v)\big] \\
&+ (x_j+h)x_j \cdot \frac{\partial^2}{\partial(x_j+h)^2}\big[(x_j+h)^2 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_j+h, x_2, \ldots, \hat{x_j}, \ldots, x_v)\big] \\
&- 2x_j \cdot \frac{\partial}{\partial x_j}\big[x_j^2 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_j, x_2, \ldots, \hat{x_j}, \ldots, x_v)\big]\Bigg] \\
= {} & c^2 \frac{1}{2}\Bigg[2x_j^2\, G_{g,v-1}^{\widetilde{M}(b,c)}(x_j, x_2, \ldots, \hat{x_j}, \ldots, x_v) \\
&+ x_j^3 \cdot \frac{\partial}{\partial x_j} G_{g,v-1}^{\widetilde{M}(b,c)}(x_j, x_2, \ldots, \hat{x_j}, \ldots, x_v) \\
&+ x_j \cdot \frac{\partial}{\partial x_j}\big[x_j^2 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_j, x_2, \ldots, \hat{x_j}, \ldots, x_v)\big] \\
&+ x_j^2 \cdot \frac{\partial^2}{\partial x_j^2}\big[x_j^2 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_j, x_2, \ldots, \hat{x_j}, \ldots, x_v)\big] \\
&- 2x_j \cdot \frac{\partial}{\partial x_j}\big[x_j^2 \cdot G_{g,v-1}^{\widetilde{M}(b,c)}(x_j, x_2, \ldots, \hat{x_j}, \ldots, x_v)\big]\Bigg]
\end{aligned}
$$

In the shortened notation $G := G_{g,v-1}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, \hat{x_j}, \ldots, x_v)$ that we used earlier (which implies $G = G_{g,v-1}^{\widetilde{M}(b,c)}(x_j, x_2, \ldots, \hat{x_j}, \ldots, x_v)$ since we are assuming $x_1 = x_j$), we see that this becomes

$$\begin{aligned}
&c^2 \frac{1}{2}\left[2x_j^2\, G + x_j^3 \cdot \frac{\partial}{\partial x_j} G + x_j^2 \cdot \frac{\partial^2}{\partial x_j^2}\left[x_j^2 \cdot G\right] - 2x_j^2 G - x_j^3 \cdot \frac{\partial}{\partial x_j} G\right] \\
&= c^2 \frac{1}{2}\left[x_j^2 \cdot \frac{\partial}{\partial x_j}\left[2x_j \cdot G + x_j^2 \cdot \frac{\partial}{\partial x_j} G\right]\right] \\
&= c^2 \frac{1}{2} x_j^2 \cdot \left[2 \cdot G + 2x_j \cdot \frac{\partial}{\partial x_j} G + 2x_j \cdot \frac{\partial}{\partial x_j} G + x_j^2 \cdot \frac{\partial^2}{\partial x_j^2} G\right] \\
&= c^2 \left[x_j^2 \cdot G + 2x_j^3 \cdot \frac{\partial}{\partial x_j} G + \frac{1}{2} x_j^4 \cdot \frac{\partial^2}{\partial x_j^2} G\right]
\end{aligned}$$

which is the same as we obtained above, if we replace x_j by x_1.

This completes the proof of the theorem.

□

7.1. Examples of the Generating Function for Generalized bc-Motzkin Numbers, for Some Cases of Small (g, v)

Now we will use Theorem 7.0.1 to obtain explicit formulas for the bc-Motzkin generating function for some cases of small (g, v).

When $g + v = 2$ and $g + v = 3$, we discuss the results for the case of general (b, c), as well as the particular case of generalized Catalan numbers when $(b, c) = (0, 1)$ and the case when $(b, c) = (2, 1)$, which will be studied later in Chapters 8 and 9 when discussing closed-form expressions and identities for generalized Catalan numbers.

For $g + v = 4$, we only discuss the case when $(b, c) = (0, 1)$, i.e. the case of generalized Catalan numbers, due to the lengthy computations involved when we work with general b and c.

Much of this work was done using the computer algebra system Mathematica to aid in computations.

Case $(g, v) = (0, 2)$**.**

We will first look at the bc-Motzkin generating function when $(g, v) = (0, 2)$.

PROPOSITION 7.1.1. *We have*

$$G_{0,2}^{\widetilde{M}(b,c)}(x_1, x_2) = \frac{x_1 x_2}{4\sqrt{1 - 2bx_1 + (b^2 - 4c^2)x_1^2}\sqrt{1 - 2bx_2 + (b^2 - 4c^2)x_2^2}} \cdot \left[\left(\frac{\sqrt{1 - 2bx_1 + (b^2 - 4c^2)x_1^2} - \sqrt{1 - 2bx_2 + (b^2 - 4c^2)x_2^2}}{x_1 - x_2}\right)^2 - (b^2 - 4c^2)\right] \quad (7.17)$$

if $x_1 \neq x_2$. *Further, if we take the limit as* $x_2 \to x_1 =: x$, *then we obtain*

$$G_{0,2}^{\widetilde{M}(b,c)}(x, x) = \lim_{x_2 \to x_1 =: x} G_{0,2}^{\widetilde{M}(b,c)}(x_1, x_2) = c^2 x^2 [1 - 2bx + (b^2 - 4c^2)x^2]^{-2}. \quad (7.18)$$

PROOF. From Theorem 7.0.1, we see that

$$G_{0,2}^{\widetilde{M}(b,c)}(x_1, x_2) = [1 - 2bx_1 + (b^2 - 4c^2)x_1^2]^{-1/2}\left[\mathcal{II}_{0,2}^{\widetilde{M}(b,c)}(x_1, x_2)\right],$$

where

$$\mathcal{II}_{0,2}^{\widetilde{M}(b,c)}(x_1, x_2) = c^2\left\{\frac{x_1 x_2}{(x_1 - x_2)^2}\left[x_1^2 \cdot G_{0,1}^{\widetilde{M}(b,c)}(x_1) - x_2^2 \cdot G_{0,1}^{\widetilde{M}(b,c)}(x_2)\right] - \frac{x_1 x_2}{x_1 - x_2} \cdot \frac{\partial}{\partial x_2}\left[x_2^2 \cdot G_{0,1}^{\widetilde{M}(b,c)}(x_2)\right]\right\}.$$

And, we may compute

$$\frac{d}{dx}\left[x^2 \cdot G_{0,1}^{\widetilde{M}(b,c)}(x)\right] = \frac{-1}{2c^2}\left[b + (-b + (b^2 - 4c^2)x)[1 - 2bx + (b^2 - 4c^2)x^2]^{-1/2}\right].$$

Thus, plugging this in to the above equation, we obtain

$$\begin{aligned}
&\mathcal{II}_{0,2}^{\widetilde{M}(b,c)}(x_1, x_2) \\
&= c^2\left\{\frac{x_1 x_2}{(x_1 - x_2)^2}\left[x_1^2 \cdot \frac{1 - bx_1 - [1 - 2bx_1 + (b^2 - 4c^2)x_1^2]^{1/2}}{2c^2 x_1^2}\right.\right. \\
&\quad \left. - x_2^2 \cdot \frac{1 - bx_2 - [1 - 2bx_2 + (b^2 - 4c^2)x_2^2]^{1/2}}{2c^2 x_2^2}\right] \\
&\quad \left. - \frac{x_1 x_2}{x_1 - x_2} \cdot \frac{-1}{2c^2}\left[b + (-b + (b^2 - 4c^2)x_2)[1 - 2bx_2 + (b^2 - 4c^2)x_2^2]^{-1/2}\right]\right\}
\end{aligned}$$

$$= \frac{x_1x_2}{2(x_1-x_2)^2}\Big[\big[1 - bx_1 - [1-2bx_1+(b^2-4c^2)x_1^2]^{1/2}\big]$$

$$- \big[1 - bx_2 - [1-2bx_2+(b^2-4c^2)x_2^2]^{1/2}\big]$$

$$+ (x_1-x_2)\big[b + (-b+(b^2-4c^2)x_2)[1-2bx_2+(b^2-4c^2)x_2^2]^{-1/2}\big]\Big]$$

$$= \frac{x_1x_2}{2(x_1-x_2)^2}\Big[- b(x_1-x_2) - [1-2bx_1+(b^2-4c^2)x_1^2]^{1/2}$$

$$+ [1-2bx_2+(b^2-4c^2)x_2^2]^{1/2}$$

$$+ b(x_1-x_2) + (x_1-x_2)(-b+(b^2-4c^2)x_2)[1-2bx_2+(b^2-4c^2)x_2^2]^{-1/2}\Big]$$

$$= \frac{x_1x_2}{2(x_1-x_2)^2}\Big[- [1-2bx_1+(b^2-4c^2)x_1^2]^{1/2} + \big[(1-2bx_2+(b^2-4c^2)x_2^2)$$

$$+ (x_1-x_2)(-b+(b^2-4c^2)x_2)\big][1-2bx_2+(b^2-4c^2)x_2^2]^{-1/2}\Big]$$

$$= \frac{x_1x_2}{2(x_1-x_2)^2}\Big[- (1-2bx_1+(b^2-4c^2)x_1^2)^{1/2}$$

$$+ \big[(1-2bx_2+(b^2-4c^2)x_2^2) + (-bx_1+(b^2-4c^2)x_1x_2) - (-bx_2+(b^2-4c^2)x_2^2)\big]$$

$$\cdot [1-2bx_2+(b^2-4c^2)x_2^2]^{-1/2}\Big]$$

$$= \frac{x_1x_2}{2(x_1-x_2)^2}\Big[- [1-2bx_1+(b^2-4c^2)x_1^2]^{1/2}$$

$$+ (1-b(x_1+x_2)+(b^2-4c^2)x_1x_2)[1-2bx_2+(b^2-4c^2)x_2^2]^{-1/2}\Big]$$

Thus, we see that

$$G_{0,2}^{\widetilde{M}(b,c)}(x_1,x_2)$$

$$= \frac{x_1x_2}{2(x_1-x_2)^2}\Big[-1 + (1-b(x_1+x_2)+(b^2-4c^2)x_1x_2)$$

$$\cdot [1-2bx_1+(b^2-4c^2)x_1^2]^{-1/2}[1-2bx_2+(b^2-4c^2)x_2^2]^{-1/2}\Big]$$

$$= \frac{x_1x_2}{2(x_1-x_2)^2}$$

$$\cdot \left[\frac{-\sqrt{1-2bx_1+(b^2-4c^2)x_1^2}\sqrt{1-2bx_2+(b^2-4c^2)x_2^2} + 1 - b(x_1+x_2) + (b^2-4c^2)x_1x_2}{\sqrt{1-2bx_1+(b^2-4c^2)x_1^2}\sqrt{1-2bx_2+(b^2-4c^2)x_2^2}}\right]$$

Now, observe

$$
\begin{aligned}
&\left(\sqrt{1-2bx_1+(b^2-4c^2)x_1^2}-\sqrt{1-2bx_2+(b^2-4c^2)x_2^2}\right)^2\\
&=[1-2bx_1+(b^2-4c^2)x_1^2]+[1-2bx_2+(b^2-4c^2)x_2^2]\\
&\quad-2\sqrt{1-2bx_1+(b^2-4c^2)x_1^2}\sqrt{1-2bx_2+(b^2-4c^2)x_2^2}\\
&=2-2b(x_1+x_2)+(b^2-4c^2)(x_1^2+x_2^2)\\
&\quad-2\sqrt{1-2bx_1+(b^2-4c^2)x_1^2}\sqrt{1-2bx_2+(b^2-4c^2)x_2^2}
\end{aligned}
$$

We may use this to write the above result in a simpler form. We thus have

$$
\begin{aligned}
&G_{0,2}^{\widetilde{M}(b,c)}(x_1,x_2)\\
&=\frac{x_1x_2}{2(x_1-x_2)^2}\cdot\frac{1}{\sqrt{1-2bx_1+(b^2-4c^2)x_1^2}\sqrt{1-2bx_2+(b^2-4c^2)x_2^2}}\\
&\quad\cdot\left[\frac{1}{2}\left(\sqrt{1-2bx_1+(b^2-4c^2)x_1^2}-\sqrt{1-2bx_2+(b^2-4c^2)x_2^2}\right)^2\right.\\
&\quad\quad\left.-\frac{1}{2}(b^2-4c^2)(x_1^2+x_2^2)+(b^2-4c^2)x_1x_2\right]\\
&=\frac{x_1x_2}{4(x_1-x_2)^2}\cdot\frac{1}{\sqrt{1-2bx_1+(b^2-4c^2)x_1^2}\sqrt{1-2bx_2+(b^2-4c^2)x_2^2}}\\
&\quad\left[\left(\sqrt{1-2bx_1+(b^2-4c^2)x_1^2}-\sqrt{1-2bx_2+(b^2-4c^2)x_2^2}\right)^2\right.\\
&\quad\quad\left.-(b^2-4c^2)(x_1^2-2x_1x_2+x_2^2)\right]\\
&=\frac{x_1x_2}{4\sqrt{1-2bx_1+(b^2-4c^2)x_1^2}\sqrt{1-2bx_2+(b^2-4c^2)x_2^2}}\\
&\quad\cdot\left[\left(\frac{\sqrt{1-2bx_1+(b^2-4c^2)x_1^2}-\sqrt{1-2bx_2+(b^2-4c^2)x_2^2}}{x_1-x_2}\right)^2-\frac{(b^2-4c^2)(x_1-x_2)^2}{(x_1-x_2)^2}\right]\\
&=\frac{x_1x_2}{4\sqrt{1-2bx_1+(b^2-4c^2)x_1^2}\sqrt{1-2bx_2+(b^2-4c^2)x_2^2}}\\
&\quad\cdot\left[\left(\frac{\sqrt{1-2bx_1+(b^2-4c^2)x_1^2}-\sqrt{1-2bx_2+(b^2-4c^2)x_2^2}}{x_1-x_2}\right)^2-(b^2-4c^2)\right]
\end{aligned}
$$

as was claimed.

For the second part of the above claim, observe that, by L'Hopital's Rule, we have:

$$
\begin{aligned}
&\lim_{h\to 0}\frac{\sqrt{1-2bx+(b^2-4c^2)x^2}-\sqrt{1-2b(x+h)+(b^2-4c^2)(x+h)^2}}{x-(x+h)}\\
&=\lim_{h\to 0}\frac{-\frac{1}{2}(1-2b(x+h)+(b^2-4c^2)(x+h)^2)^{-1/2}(-2b+2(b^2-4c^2)(x+h))}{-1}\\
&=\lim_{h\to 0}(1-2b(x+h)+(b^2-4c^2)(x+h)^2)^{-1/2}(-b+(b^2-4c^2)(x+h))\\
&=(1-2bx+(b^2-4c^2)x^2)^{-1/2}(-b+(b^2-4c^2)x)\\
&=\frac{-b+(b^2-4c^2)x}{\sqrt{1-2bx+(b^2-4c^2)x^2}}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&G_{0,2}^{\widetilde{M}(b,c)}(x,x)\\
&=\lim_{x_2\to x_1=:x}G_{0,2}^{\widetilde{M}(b,c)}(x_1,x_2)\\
&=\frac{x^2}{4[1-2bx+(b^2-4c^2)x^2]}\left[\left(\frac{-b+(b^2-4c^2)x}{\sqrt{1-2bx+(b^2-4c^2)x^2}}\right)^2-(b^2-4c^2)\right]\\
&=\frac{x^2}{4[1-2bx+(b^2-4c^2)x^2]}\left[\frac{(-b+(b^2-4c^2)x)^2-(1-2bx+(b^2-4c^2)x^2)(b^2-4c^2)}{1-2bx+(b^2-4c^2)x^2}\right]\\
&=\frac{x^2}{4[1-2bx+(b^2-4c^2)x^2]^2}\\
&\quad\cdot\left[b^2-2b(b^2-4c^2)x+(b^2-4c^2)^2x^2-((b^2-4c^2)-2b(b^2-4c^2)x+(b^2-4c^2)^2x^2)\right]\\
&=\frac{x^2}{4[1-2bx+(b^2-4c^2)x^2]^2}\left[b^2-(b^2-4c^2)\right]\\
&=\frac{c^2x^2}{[1-2bx+(b^2-4c^2)x^2]^2}
\end{aligned}
$$

as was claimed.

□

REMARK 7.1.1. Observe that, if $b=0$ and $c=1$, then we obtain the generating function for the $(0,2)$ Catalan numbers:

$$G_{0,2}^{\widetilde{M}(0,1)}(x_1,x_2)=\frac{x_1x_2}{4\sqrt{1-4x_1^2}\sqrt{1-4x_2^2}}\left[\left(\frac{\sqrt{1-4x_1^2}-\sqrt{1-4x_2^2}}{x_1-x_2}\right)^2+4\right] \tag{7.19}$$

Also, if we let $b = 2$ and $c = 1$, then we obtain:

$$G_{0,2}^{\widetilde{M}(2,1)}(x_1, x_2) = \frac{x_1 x_2}{4\sqrt{1-4x_1}\sqrt{1-4x_2}} \left(\frac{\sqrt{1-4x_1} - \sqrt{1-4x_2}}{x_1 - x_2} \right)^2 \qquad (7.20)$$

We will use these results later, in Chapter 9, to obtain an identity for the $(0, 2)$ Catalan numbers.

Remark 7.1.2. We will see that, of the following examples, only $G_{0,2}^{\widetilde{M}(b,c)}(x_1, x_2)$ is undefined at $x_1 = x_2$; for all other examples we will look at, the $x_1 - x_j$ terms will cancel with a factor in the numerator when the entire formula is put over a common denominator. Later in this chapter, we will conjecture that this is true more generally.

Case $(g, v) = (1, 1)$**.**

We will next apply Theorem 7.0.1 to the case when $(g, v) = (1, 1)$.

Proposition 7.1.2. *We have*

$$G_{1,1}^{\widetilde{M}(b,c)}(x) = c^4 x^4 [1 - 2bx + (b^2 - 4c^2)x^2]^{-5/2}. \qquad (7.21)$$

Proof. From Theorem 7.0.1, we see that

$$G_{1,1}^{\widetilde{M}(b,c)}(x) = [1 - 2bx + (b^2 - 4c^2)x^2]^{-1/2} \left[c^2 x^2 \, G_{0,2}^{\widetilde{M}(b,c)}(x, x) \right].$$

Hence, plugging in the result of Proposition 7.1.1 when $x_1 = x_2 = x$, we obtain

$$G_{1,1}^{\widetilde{M}(b,c)}(x) = c^4 x^4 [1 - 2bx + (b^2 - 4c^2)x^2]^{-5/2}$$

as was claimed.

□

Remark 7.1.3. In particular, when $b = 0$ and $c = 1$, we see that the generating function for the $(1, 1)$ Catalan numbers is:

$$G_{1,1}^{\widetilde{M}(0,1)}(x) = x^4 (1 - 4x^2)^{-5/2}. \qquad (7.22)$$

And, when $b = 2$ and $c = 1$, we have

$$G_{1,1}^{\widetilde{M}(2,1)}(x) = x^4(1-4x)^{-5/2}. \tag{7.23}$$

We will use these results later, in Chapters 8 and 9, to obtain a closed-form expression and an identity for the $(1,1)$ Catalan numbers.

Case $(g, v) = (0, 3)$**.**

We will now apply Theorem 7.0.1 to the case when $(g, v) = (0, 3)$.

PROPOSITION 7.1.3. *We have*

$$\begin{aligned} &G_{0,3}^{\widetilde{M}(b,c)}(x,y,z) \\ &\quad = \frac{2c^4xyz[x+y+z-2b(xy+xz+yz)+[-(b^2-4c^2)+4b^2]xyz]}{[1-2bx+(b^2-4c^2)x^2]^{3/2}[1-2by+(b^2-4c^2)y^2]^{3/2}[1-2bz+(b^2-4c^2)z^2]^{3/2}} \end{aligned} \tag{7.24}$$

PROOF. From Theorem 7.0.1, we see that we have:

$$\begin{aligned} &G_{0,3}^{\widetilde{M}(b,c)}(x,y,z) \\ &\quad = [1-2bx+(b^2-4c^2)x^2]^{-1/2}\left[\mathcal{II}_{0,3}^{\widetilde{M}(b,c)}(x,y,z) + 2c^2x^2G_{0,2}^{\widetilde{M}(b,c)}(x,y)G_{0,2}^{\widetilde{M}(b,c)}(x,z)\right] \end{aligned}$$

where

$$\begin{aligned} &\mathcal{II}_{0,3}^{\widetilde{M}(b,c)}(x,y,z) \\ &= c^2\Bigg\{\frac{xy}{(x-y)^2}\Bigg[x^2\cdot G_{0,2}^{\widetilde{M}(b,c)}(x,z) \\ &\qquad - y^2\cdot G_{0,2}^{\widetilde{M}(b,c)}(y,z)\Bigg] - \frac{xy}{x-y}\cdot\frac{\partial}{\partial y}\left[y^2\cdot G_{0,2}^{\widetilde{M}(b,c)}(y,z)\right] \\ &\quad + \frac{xz}{(x-z)^2}\Bigg[x^2\cdot G_{0,2}^{\widetilde{M}(b,c)}(x,y) \\ &\qquad - z^2\cdot G_{0,2}^{\widetilde{M}(b,c)}(z,y)\Bigg] - \frac{xz}{x-z}\cdot\frac{\partial}{\partial z}\left[z^2\cdot G_{0,2}^{\widetilde{M}(b,c)}(z,y)\right]\Bigg\} \end{aligned}$$

Using Mathematica, we may compute the above simplified form of $G_{0,3}^{\widetilde{M}(b,c)}(x,y,z)$.

$\square$

Remark 7.1.4. In the particular case when $b = 0$ and $c = 1$, we obtain the generating function for the $(0,3)$ Catalan numbers,

$$G_{0,3}^{\widetilde{M}(0,1)}(x,y,z) = 2xyz\frac{x+y+z+4xyz}{(1-4x^2)^{3/2}(1+4y^2)^{3/2}(1+4z^2)^{3/2}} \tag{7.25}$$

And, when $b = 2$ and $c = 1$, we also obtain

$$G_{0,3}^{\widetilde{M}(2,1)}(x,y,z) = 2xyz\frac{x+y+z-4(xy+xz+yz)+16xyz}{(1-4x)^{3/2}(1-4y)^{3/2}(1-4z)^{3/2}} \tag{7.26}$$

Case $(g,v) = (1,2)$**.**

We will now apply Theorem 7.0.1 to the case when $(g,v) = (1,2)$.

Proposition 7.1.4. *We have*

$$G_{1,2}^{\widetilde{M}(b,c)}(x,y) = \frac{c^6xy}{[1-2bx+(b^2-4c^2)x^2]^{7/2}[1-2by+(b^2-4c^2)y^2]^{7/2}}P_{1,2}^{\widetilde{M}(b,c)}(x,y), \tag{7.27}$$

where

$$\begin{aligned}
P_{1,2}^{\widetilde{M}(b,c)}(x,y) &= 5x^4 - 5bx^5 + 4x^3y - 33bx^4y + 29b^2x^5y + 4c^2x^5y + 3x^2y^2 - 25bx^3y^2 \\
&+ 91b^2x^4y^2 - 52c^2x^4y^2 - 69b^3x^5y^2 + 36bc^2x^5y^2 + 4xy^3 - 25bx^2y^3 \\
&+ 75b^2x^3y^3 - 52c^2x^3y^3 - 141b^3x^4y^3 + 260bc^2x^4y^3 + 87b^4x^5y^3 \\
&- 184b^2c^2x^5y^3 - 16c^4x^5y^3 + 5y^4 - 33bxy^4 + 91b^2x^2y^4 - 52c^2x^2y^4 \\
&- 141b^3x^3y^4 + 260bc^2x^3y^4 + 141b^4x^4y^4 - 520b^2c^2x^4y^4 + 208c^4x^4y^4 \\
&- 63b^5x^5y^4 + 296b^3c^2x^5y^4 - 176bc^4x^5y^4 - 5by^5 + 29b^2xy^5 + 4c^2xy^5 \\
&- 69b^3x^2y^5 + 36bc^2x^2y^5 + 87b^4x^3y^5 - 184b^2c^2x^3y^5 - 16c^4x^3y^5 \\
&- 63b^5x^4y^5 + 296b^3c^2x^4y^5 - 176bc^4x^4y^5 + 21b^6x^5y^5 - 148b^4c^2x^5y^5 \\
&+ 176b^2c^4x^5y^5 + 320c^6x^5y^5
\end{aligned}$$

Proof. Again, from Theorem 7.0.1, we see that we have

$$G_{1,2}^{\widetilde{M}(b,c)}(x,y) = [1-2bx+(b^2-4c^2)x^2]^{-1/2}\Big[\mathcal{I}\mathcal{I}_{1,2}^{\widetilde{M}(b,c)}(x,y)$$

$$+ c^2x^2\, G_{0,3}^{\widetilde{M}(b,c)}(x,x,y) + 2c^2x^2 G_{0,2}^{\widetilde{M}(b,c)}(x,y) G_{1,1}^{\widetilde{M}(b,c)}(x)\Big]$$

where

$$\begin{aligned}
&\mathcal{II}_{1,2}^{\widetilde{M}(b,c)}(x,y)\\
&\qquad = c^2\left\{\frac{xy}{(x-y)^2}\left[x^2\cdot G_{1,1}^{\widetilde{M}(b,c)}(x) - y^2\cdot G_{1,1}^{\widetilde{M}(b,c)}(y)\right] - \frac{xy}{x-y}\cdot\frac{\partial}{\partial y}\left[y^2\cdot G_{1,1}^{\widetilde{M}(b,c)}(y)\right]\right\}
\end{aligned}$$

Using Mathematica, we may compute the above result for $G_{1,2}^{\widetilde{M}(b,c)}(x,y)$.

□

Remark 7.1.5. When $b=0$ and $c=1$, we obtain the generating function for the $(1,2)$ Catalan numbers, which is

$$G_{1,2}^{\widetilde{M}(0,1)}(x,y) = \frac{xy}{(1-4x^2)^{7/2}(1-4y^2)^{7/2}} P_{1,2}^{\widetilde{M}(0,1)}(x,y), \tag{7.28}$$

where

$$\begin{aligned}
P_{1,2}^{\widetilde{M}(0,1)}(x,y) &= 5x^4 + 4x^3y + 4x^5y + 3x^2y^2 - 52x^4y^2 + 4xy^3 - 52x^3y^3 - 16x^5y^3\\
&\quad + 5y^4 - 52x^2y^4 + 208x^4y^4 + 4xy^5 - 16x^3y^5 + 320x^5y^5
\end{aligned}$$

And, for $b=2$ and $c=1$, we obtain

$$G_{1,2}^{\widetilde{M}(2,1)}(x,y) = \frac{xy}{(1-4x)^{7/2}(1-4y)^{7/2}} P_{1,2}^{\widetilde{M}(2,1)}(x,y) \tag{7.29}$$

where

$$\begin{aligned}
P_{1,2}^{\widetilde{M}(2,1)}(x,y) &= 640x^5y^3 - 480x^5y^2 + 120x^5y - 10x^5 + 384x^4y^4 - 608x^4y^3\\
&\quad + 312x^4y^2 - 66x^4y + 5x^4 + 640x^3y^5 - 608x^3y^4 + 248x^3y^3\\
&\quad - 50x^3y^2 + 4x^3y - 480x^2y^5 + 312x^2y^4 - 50x^2y^3 + 3x^2y^2\\
&\quad + 120xy^5 - 66xy^4 + 4xy^3 - 10y^5 + 5y^4
\end{aligned}$$

Case $(g, v) = (2, 1)$**.**

We will now apply Theorem 7.0.1 to the case when $(g, v) = (2, 1)$.

PROPOSITION 7.1.5. *We have*

$$G_{2,1}^{\widetilde{M}(b,c)}(x) = 21c^8x^8[1 - 2bx + (b^2 - 4c^2)x^2]^{-9/2} + 105c^{10}x^{10}[1 - 2bx + (b^2 - 4c^2)x^2]^{-11/2}. \quad (7.30)$$

PROOF. From Theorem 7.0.1, we see that

$$G_{2,1}^{\widetilde{M}(b,c)}(x) = c^2x^2[1 - 2bx + (b^2 - 4c^2)x^2]^{-1/2}\left[G_{1,2}^{\widetilde{M}(b,c)}(x, x) + \left[G_{1,1}^{\widetilde{M}(b,c)}(x)\right]^2\right]$$

Plugging in the result of Proposition 7.1.4 when $x = y$ and using Mathematica to simplify the result, we obtain

$$G_{2,1}^{\widetilde{M}(b,c)}(x) = 21c^8x^8[1 - 2bx + (b^2 - 4c^2)x^2]^{-9/2} + 105c^{10}x^{10}[1 - 2bx + (b^2 - 4c^2)x^2]^{-11/2}$$

as was claimed.

□

REMARK 7.1.6. In particular, when $b = 0$ and $c = 1$, we see that the generating function for the $(2, 1)$ Catalan numbers is

$$G_{2,1}^{\widetilde{M}(0,1)}(x) = 21x^8(1 - 4x^2)^{-9/2} + 105x^{10}(1 - 4x^2)^{-11/2}. \quad (7.31)$$

And, when $b = 2$ and $c = 1$, we have

$$G_{2,1}^{\widetilde{M}(2,1)}(x) = 21x^8(1 - 4x)^{-9/2} + 105x^{10}(1 - 4x)^{-11/2}. \quad (7.32)$$

Case $(g, v) = (0, 4)$**.**

We will now look at the case when $(g, v) = (0, 4)$. Theorem 7.0.1 gives:

$$G_{0,4}^{\widetilde{M}(b,c)}(x, y, z, w) = [1 - 2bx + (b^2 - 4c^2)x^2]^{-1/2}\Bigg[\mathcal{II}_{0,4}^{\widetilde{M}(b,c)}(x, y, z, w)$$
$$+ 2c^2x^2\Bigg(G_{0,2}^{\widetilde{M}(b,c)}(x, y)G_{0,3}^{\widetilde{M}(b,c)}(x, z, w)$$

$$
\begin{aligned}
&+ G_{0,2}^{\widetilde{M}(b,c)}(x,z) G_{0,3}^{\widetilde{M}(b,c)}(x,y,w) \\
&+ G_{0,2}^{\widetilde{M}(b,c)}(x,w) G_{0,3}^{\widetilde{M}(b,c)}(x,y,z) \Big) \Big]
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathcal{I}\mathcal{I}_{0,4}^{\widetilde{M}(b,c)}(x,y,z,w) \\
&= c^2 \Bigg\{ \frac{xy}{(x-y)^2} \left[x^2 \cdot G_{0,3}^{\widetilde{M}(b,c)}(x,z,w) - y^2 \cdot G_{0,3}^{\widetilde{M}(b,c)}(y,z,w) \right] \\
&\quad - \frac{xy}{x-y} \cdot \frac{\partial}{\partial y} \left[y^2 \cdot G_{0,3}^{\widetilde{M}(b,c)}(y,z,w) \right] \\
&\quad + \frac{xz}{(x-z)^2} \Bigg[x^2 \cdot G_{0,3}^{\widetilde{M}(b,c)}(x,y,w) \\
&\quad - z^2 \cdot G_{0,3}^{\widetilde{M}(b,c)}(z,y,w) \Bigg] - \frac{xz}{x-z} \cdot \frac{\partial}{\partial z} \left[z^2 \cdot G_{0,3}^{\widetilde{M}(b,c)}(z,y,w) \right] \\
&\quad + \frac{xw}{(x-w)^2} \Bigg[x^2 \cdot G_{0,3}^{\widetilde{M}(b,c)}(x,y,z) \\
&\quad - w^2 \cdot G_{0,3}^{\widetilde{M}(b,c)}(w,y,z) \Bigg] - \frac{xw}{x-w} \cdot \frac{\partial}{\partial w} \left[w^2 \cdot G_{0,3}^{\widetilde{M}(b,c)}(w,y,z) \right] \Bigg\}
\end{aligned}
$$

Letting $b = 0$ and $c = 1$ and using Mathematica for the computations, we see that the generating function for the $(0,4)$ Catalan numbers is

$$
\begin{aligned}
&G_{0,4}^{\widetilde{M}(0,1)}(x,y,z,w) \\
&\qquad = \frac{-2wxyz}{(1-4w^2)^{5/2}(1-4x^2)^{5/2}(1-4y^2)^{5/2}(1-4z^2)^{5/2}} P_{0,4}^{\widetilde{M}(0,1)}(x,y,z,w) \qquad (7.33)
\end{aligned}
$$

where

$$
\begin{aligned}
P_{0,4}^{\widetilde{M}(0,1)}(x,y,z,w) &= -3w^2 - 4wx + 4w^3x - 3x^2 + 24w^2x^2 + 4wx^3 + 32w^3x^3 - 4wy \\
&\quad + 4w^3y - 4xy + 4w^2xy + 4wx^2y + 32w^3x^2y + 4x^3y + 32w^2x^3y \\
&\quad - 3y^2 + 24w^2y^2 + 4wxy^2 + 32w^3xy^2 + 24x^2y^2 - 144w^2x^2y^2 \\
&\quad + 32wx^3y^2 - 320w^3x^3y^2 + 4wy^3 + 32w^3y^3 + 4xy^3 + 32w^2xy^3 \\
&\quad + 32wx^2y^3 - 320w^3x^2y^3 + 32x^3y^3 - 320w^2x^3y^3 - 4wz + 4w^3z
\end{aligned}
$$

$$
\begin{aligned}
&-4xz+4w^2xz+4wx^2z+32w^3x^2z+4x^3z+32w^2x^3z-4yz\\
&+4w^2yz-24wxyz+48w^3xyz+4x^2yz+32w^2x^2yz+48wx^3yz\\
&+4wy^2z+32w^3y^2z+4xy^2z+32w^2xy^2z+32wx^2y^2z\\
&-320w^3x^2y^2z+32x^3y^2z-320w^2x^3y^2z+4y^3z+32w^2y^3z\\
&+48wxy^3z+32x^2y^3z-320w^2x^2y^3z-768w^3x^3y^3z-3z^2\\
&+24w^2z^2+4wxz^2+32w^3xz^2+24x^2z^2-144w^2x^2z^2\\
&+32wx^3z^2-320w^3x^3z^2+4wyz^2+32w^3yz^2+4xyz^2\\
&+32w^2xyz^2+32wx^2yz^2-320w^3x^2yz^2+32x^3yz^2\\
&-320w^2x^3yz^2+24y^2z^2-144w^2y^2z^2+32wxy^2z^2\\
&-320w^3xy^2z^2-144x^2y^2z^2+768w^2x^2y^2z^2-320wx^3y^2z^2\\
&+2048w^3x^3y^2z^2+32wy^3z^2-320w^3y^3z^2+32xy^3z^2\\
&-320w^2xy^3z^2-320wx^2y^3z^2+2048w^3x^2y^3z^2-320x^3y^3z^2\\
&+2048w^2x^3y^3z^2+4wz^3+32w^3z^3+4xz^3+32w^2xz^3\\
&+32wx^2z^3-320w^3x^2z^3+32x^3z^3-320w^2x^3z^3+4yz^3\\
&+32w^2yz^3+48wxyz^3+32x^2yz^3-320w^2x^2yz^3-768w^3x^3yz^3\\
&+32wy^2z^3-320w^3y^2z^3+32xy^2z^3-320w^2xy^2z^3\\
&-320wx^2y^2z^3+2048w^3x^2y^2z^3-320x^3y^2z^3+2048w^2x^3y^2z^3\\
&+32y^3z^3-320w^2y^3z^3-768w^3xy^3z^3-320x^2y^3z^3\\
&+2048w^2x^2y^3z^3-768wx^3y^3z^3+6144w^3x^3y^3z^3
\end{aligned}
$$

Case $(g, v) = (1, 3)$**.**

We will now look at the case when $(g, v) = (1, 3)$. Theorem 7.0.1 gives:

$$\begin{aligned}
&G_{1,3}^{\widetilde{M}(b,c)}(x, y, z) \\
&= [1 - 2bx + (b^2 - 4c^2)x^2]^{-1/2} \Bigg[\mathcal{II}_{1,3}^{\widetilde{M}(b,c)}(x, y, z) + c^2x^2\, G_{0,4}^{\widetilde{M}(b,c)}(x, x, y, z) \\
&\quad + 2c^2x^2 \Bigg(G_{0,3}^{\widetilde{M}(b,c)}(x, y, z) G_{1,1}^{\widetilde{M}(b,c)}(x) + G_{0,2}^{\widetilde{M}(b,c)}(x, y) G_{1,2}^{\widetilde{M}(b,c)}(x, z) \\
&\qquad + G_{0,2}^{\widetilde{M}(b,c)}(x, z) G_{1,2}^{\widetilde{M}(b,c)}(x, y) \Bigg) \Bigg]
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{II}_{1,3}^{\widetilde{M}(b,c)}(x, y, z) = c^2 \Bigg\{ & \frac{xy}{(x-y)^2} \left[x^2 \cdot G_{1,2}^{\widetilde{M}(b,c)}(x, z) - y^2 \cdot G_{1,2}^{\widetilde{M}(b,c)}(y, z) \right] \\
& - \frac{xy}{x-y} \cdot \frac{\partial}{\partial y} \left[y^2 \cdot G_{1,2}^{\widetilde{M}(b,c)}(y, z) \right] \\
& + \frac{xz}{(x-z)^2} \left[x^2 \cdot G_{1,2}^{\widetilde{M}(b,c)}(x, y) - z^2 \cdot G_{1,2}^{\widetilde{M}(b,c)}(z, y) \right] \\
& - \frac{xz}{x-z} \cdot \frac{\partial}{\partial z} \left[z^2 \cdot G_{1,2}^{\widetilde{M}(b,c)}(z, y) \right] \Bigg\}
\end{aligned}$$

Letting $b = 0$ and $c = 1$ and using Mathematica for the computations, we see that the generating function for the $(1, 3)$ Catalan numbers is

$$G_{1,3}^{\widetilde{M}(0,1)}(x, y, z) = \frac{2xyz}{(1-4x^2)^{9/2}(1-4y^2)^{9/2}(1-4z^2)^{9/2}} P_{1,3}^{\widetilde{M}(0,1)}(x, y, z) \tag{7.34}$$

where

$$\begin{aligned}
P_{1,3}^{\widetilde{M}(0,1)}(x, y, z) = {} & 15x^5 + 10x^7 + 15x^4y + 10x^6y + 12x^3y^2 - 216x^5y^2 - 168x^7y^2 \\
& + 12x^2y^3 - 216x^4y^3 - 168x^6y^3 + 15xy^4 - 216x^3y^4 + 1248x^5y^4 \\
& + 864x^7y^4 + 15y^5 - 216x^2y^5 + 1248x^4y^5 + 864x^6y^5 + 10xy^6 \\
& - 168x^3y^6 + 864x^5y^6 - 5888x^7y^6 + 10y^7 - 168x^2y^7 + 864x^4y^7 \\
& - 5888x^6y^7 + 15x^4z + 10x^6z + 12x^3yz + 24x^5yz - 8x^7yz \\
& + 9x^2y^2z - 192x^4y^2z - 216x^6y^2z + 12xy^3z - 192x^3y^3z
\end{aligned}$$

$$
\begin{aligned}
&-240x^5y^3z-96x^7y^3z+15y^4z-192x^2y^4z+1056x^4y^4z\\
&+1248x^6y^4z+24xy^5z-240x^3y^5z+4608x^5y^5z-2688x^7y^5z\\
&+10y^6z-216x^2y^6z+1248x^4y^6z-6656x^6y^6z-8xy^7z\\
&-96x^3y^7z-2688x^5y^7z-5120x^7y^7z+12x^3z^2-216x^5z^2\\
&-168x^7z^2+9x^2yz^2-192x^4yz^2-216x^6yz^2+9xy^2z^2\\
&-360x^3y^2z^2+3168x^5y^2z^2+2592x^7y^2z^2+12y^3z^2\\
&-360x^2y^3z^2+3024x^4y^3z^2+2976x^6y^3z^2-192xy^4z^2\\
&+3024x^3y^4z^2-14976x^5y^4z^2-16512x^7y^4z^2-216y^5z^2\\
&+3168x^2y^5z^2-14976x^4y^5z^2-17280x^6y^5z^2-216xy^6z^2\\
&+2976x^3y^6z^2-17280x^5y^6z^2+89088x^7y^6z^2-168y^7z^2\\
&+2592x^2y^7z^2-16512x^4y^7z^2+89088x^6y^7z^2+12x^2z^3\\
&-216x^4z^3-168x^6z^3+12xyz^3-192x^3yz^3-240x^5yz^3\\
&-96x^7yz^3+12y^2z^3-360x^2y^2z^3+3024x^4y^2z^3\\
&+2976x^6y^2z^3-192xy^3z^3+1872x^3y^3z^3+5760x^5y^3z^3\\
&-384x^7y^3z^3-216y^4z^3+3024x^2y^4z^3-13824x^4y^4z^3\\
&-19584x^6y^4z^3-240xy^5z^3+5760x^3y^5z^3-69120x^5y^5z^3\\
&+38400x^7y^5z^3-168y^6z^3+2976x^2y^6z^3-19584x^4y^6z^3\\
&+95232x^6y^6z^3-96xy^7z^3-384x^3y^7z^3+38400x^5y^7z^3\\
&+73728x^7y^7z^3+15xz^4-216x^3z^4+1248x^5z^4+864x^7z^4\\
&+15yz^4-192x^2yz^4+1056x^4yz^4+1248x^6yz^4-192xy^2z^4\\
&+3024x^3y^2z^4-14976x^5y^2z^4-16512x^7y^2z^4-216y^3z^4\\
&+3024x^2y^3z^4-13824x^4y^3z^4-19584x^6y^3z^4+1056xy^4z^4\\
&-13824x^3y^4z^4+48384x^5y^4z^4+121344x^7y^4z^4+1248y^5z^4
\end{aligned}
$$

$$
\begin{aligned}
&- 14976x^2y^5z^4 + 48384x^4y^5z^4 + 127488x^6y^5z^4 + 1248xy^6z^4 \\
&- 19584x^3y^6z^4 + 127488x^5y^6z^4 - 491520x^7y^6z^4 + 864y^7z^4 \\
&- 16512x^2y^7z^4 + 121344x^4y^7z^4 - 491520x^6y^7z^4 + 15z^5 \\
&- 216x^2z^5 + 1248x^4z^5 + 864x^6z^5 + 24xyz^5 - 240x^3yz^5 \\
&+ 4608x^5yz^5 - 2688x^7yz^5 - 216y^2z^5 + 3168x^2y^2z^5 \\
&- 14976x^4y^2z^5 - 17280x^6y^2z^5 - 240xy^3z^5 + 5760x^3y^3z^5 \\
&- 69120x^5y^3z^5 + 38400x^7y^3z^5 + 1248y^4z^5 - 14976x^2y^4z^5 \\
&+ 48384x^4y^4z^5 + 127488x^6y^4z^5 + 4608xy^5z^5 - 69120x^3y^5z^5 \\
&+ 423936x^5y^5z^5 - 239616x^7y^5z^5 + 864y^6z^5 - 17280x^2y^6z^5 \\
&+ 127488x^4y^6z^5 - 503808x^6y^6z^5 - 2688xy^7z^5 + 38400x^3y^7z^5 \\
&- 239616x^5y^7z^5 - 344064x^7y^7z^5 + 10xz^6 - 168x^3z^6 + 864x^5z^6 \\
&- 5888x^7z^6 + 10yz^6 - 216x^2yz^6 + 1248x^4yz^6 - 6656x^6yz^6 \\
&- 216xy^2z^6 + 2976x^3y^2z^6 - 17280x^5y^2z^6 + 89088x^7y^2z^6 \\
&- 168y^3z^6 + 2976x^2y^3z^6 - 19584x^4y^3z^6 + 95232x^6y^3z^6 \\
&+ 1248xy^4z^6 - 19584x^3y^4z^6 + 127488x^5y^4z^6 - 491520x^7y^4z^6 \\
&+ 864y^5z^6 - 17280x^2y^5z^6 + 127488x^4y^5z^6 - 503808x^6y^5z^6 \\
&- 6656xy^6z^6 + 95232x^3y^6z^6 - 503808x^5y^6z^6 + 1204224x^7y^6z^6 \\
&- 5888y^7z^6 + 89088x^2y^7z^6 - 491520x^4y^7z^6 + 1204224x^6y^7z^6 \\
&+ 10z^7 - 168x^2z^7 + 864x^4z^7 - 5888x^6z^7 - 8xyz^7 - 96x^3yz^7 \\
&- 2688x^5yz^7 - 5120x^7yz^7 - 168y^2z^7 + 2592x^2y^2z^7 - 16512x^4y^2z^7 \\
&+ 89088x^6y^2z^7 - 96xy^3z^7 - 384x^3y^3z^7 + 38400x^5y^3z^7 \\
&+ 73728x^7y^3z^7 + 864y^4z^7 - 16512x^2y^4z^7 + 121344x^4y^4z^7 \\
&- 491520x^6y^4z^7 - 2688xy^5z^7 + 38400x^3y^5z^7 - 239616x^5y^5z^7
\end{aligned}
$$

$$- 344064x^7y^5z^7 - 5888y^6z^7 + 89088x^2y^6z^7 - 491520x^4y^6z^7$$

$$+ 1204224x^6y^6z^7 - 5120xy^7z^7 + 73728x^3y^7z^7 - 344064x^5y^7z^7$$

$$+ 1671168x^7y^7z^7$$

Case $(g, v) = (2, 2)$**.**

We will now look at the case when $(g, v) = (2, 2)$. Theorem 7.0.1 gives:

$$G_{2,2}^{\widetilde{M}(b,c)}(x,y) = [1 - 2bx + (b^2 - 4c^2)x^2]^{-1/2}\Big[\mathcal{II}_{2,2}^{\widetilde{M}(b,c)}(x,y) + c^2x^2\, G_{1,3}^{\widetilde{M}(b,c)}(x,x,y)$$
$$+ 2c^2x^2[G_{0,2}^{\widetilde{M}(b,c)}(x,y)G_{2,1}^{\widetilde{M}(b,c)}(x) + G_{1,1}^{\widetilde{M}(b,c)}(x)G_{1,2}^{\widetilde{M}(b,c)}(x,y)]\Big]$$

where

$$\mathcal{II}_{2,2}^{\widetilde{M}(b,c)}(x,y)$$
$$= c^2\left\{\frac{xy}{(x-y)^2}\left[x^2 \cdot G_{2,1}^{\widetilde{M}(b,c)}(x) - y^2 \cdot G_{2,1}^{\widetilde{M}(b,c)}(y)\right] - \frac{xy}{x-y} \cdot \frac{\partial}{\partial y}\left[y^2 \cdot G_{2,1}^{\widetilde{M}(b,c)}(y)\right]\right\}$$

Letting $b = 0$ and $c = 1$ and using Mathematica for the computations, we see that the generating function for the $(2, 2)$ Catalan numbers is

$$G_{2,2}^{\widetilde{M}(0,1)}(x,y) = \frac{-xy}{(1-4x^2)^{13/2}(1-4y^2)^{13/2}}P_{2,2}^{\widetilde{M}(0,1)}(x,y) \tag{7.35}$$

where

$$P_{2,2}^{\widetilde{M}(0,1)}(x,y) = -189x^8 - 399x^{10} - 168x^7y - 462x^9y - 84x^{11}y - 147x^6y^2$$
$$+ 4011x^8y^2 + 9408x^{10}y^2 - 156x^5y^3 + 3664x^7y^3 + 10516x^9y^3$$
$$+ 1696x^{11}y^3 - 165x^4y^4 + 3317x^6y^4 - 33736x^8y^4 - 92368x^{10}y^4$$
$$- 156x^3y^5 + 3342x^5y^5 - 30980x^7y^5 - 101120x^9y^5 - 12992x^{11}y^5$$
$$- 147x^2y^6 + 3317x^4y^6 - 27624x^6y^6 + 129648x^8y^6 + 487936x^{10}y^6$$
$$- 168xy^7 + 3664x^3y^7 - 30980x^5y^7 + 133248x^7y^7 + 491072x^9y^7$$

$$+ 77312x^{11}y^7 - 189y^8 + 4011x^2y^8 - 33736x^4y^8 + 129648x^6y^8$$
$$- 202752x^8y^8 - 1415936x^{10}y^8 - 462xy^9 + 10516x^3y^9 - 101120x^5y^9$$
$$+ 491072x^7y^9 - 2519552x^9y^9 + 560128x^{11}y^9 - 399y^{10} + 9408x^2y^{10}$$
$$- 92368x^4y^{10} + 487936x^6y^{10} - 1415936x^8y^{10} + 2951168x^{10}y^{10}$$
$$- 84xy^{11} + 1696x^3y^{11} - 12992x^5y^{11} + 77312x^7y^{11}$$
$$+ 560128x^9y^{11} + 1736704x^{11}y^{11}$$

Case $(g, v) = (3, 1)$.

We will now look at the case when $(g, v) = (3, 1)$. Theorem 7.0.1 gives:

$$G_{3,1}^{\widetilde{M}(b,c)}(x) = [1 - 2bx + (b^2 - 4c^2)x^2]^{-1/2}\left[c^2x^2\, G_{2,2}^{\widetilde{M}(b,c)}(x, x) + 2c^2x^2G_{1,1}^{\widetilde{M}(b,c)}(x)G_{2,1}^{\widetilde{M}(b,c)}(x, x)\right]$$

Letting $b = 0$ and $c = 1$ and using Mathematica for the computations, we see that the generating function for the $(3, 1)$ Catalan numbers is

$$\begin{aligned}
G_{3,1}^{\widetilde{M}(0,1)}(x) &= \frac{11x^{12}(135 + 558x^2 + 158x^4)}{(1 - 4x^2)^{17/2}} \\
&= \frac{11x^{12}[135(1 - 8x^2 + 16x^4) + 1638x^2 - 2318x^4]}{(1 - 4x^2)^{17/2}} \\
&= \frac{11x^{12}[135(1 - 4x^2)^2 + 1638x^2(1 - 4x^2) + 4234x^4]}{(1 - 4x^2)^{17/2}} \\
&= 1485x^{12}(1 - 4x^2)^{-13/2} + 18018x^{14}(1 - 4x^2)^{-15/2} + 46574x^{16}(1 - 4x^2)^{-17/2}
\end{aligned}$$

Hence,

$$G_{3,1}^{\widetilde{M}(0,1)}(x) = 1485x^{12}(1 - 4x^2)^{-13/2} + 18018x^{14}(1 - 4x^2)^{-15/2} + 46574x^{16}(1 - 4x^2)^{-17/2} \quad (7.36)$$

7.2. Conjectured General Form of the Generating Function for Generalized bc-Motzkin Numbers

Now, based on these examples of $G_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v)$ for small g and v, we would like to form a conjecture regarding the general form of this generating function for the generalized bc-Motzkin numbers.

For notational convenience, we will write

$$R^{(b,c)}(x_i) = [1 - 2bx_i + (b^2 - 4c^2)x_i^2]. \tag{7.37}$$

We make the following conjecture.

Conjecture 7.2.1. *For $(g, v) \neq (0,1), (0,2)$, the generating function $G_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v)$ for the generalized bc-Motzkin numbers is of the form*

$$G_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v) = \frac{x_1 x_2 \cdots x_v}{[R^{(b,c)}(x_1)R^{(b,c)}(x_2) \cdots R^{(b,c)}(x_v)]^{\alpha(g,v)/2}} P_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v), \tag{7.38}$$

where:

(i) the exponent $\alpha(g, v) = 6g - 3 + 2v$,
(ii) $P_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v)$ is a polynomial, and
(iii) for $v \geq 2$, the polynomial $P_{g,v}^{\widetilde{M}(b,c)}(x_1, x_1, x_2, \ldots, x_{v-1})$ is divisible by $[R^{(b,c)}(x_1)]^{\beta(g,v)}$, where $\beta(g, v) = 3g - 3 + v$.

Remark 7.2.1. For the above examples, if we let $\alpha(g, v)$ denote the exponent of the square root in the denominator, and let $\varphi(g - 1, v + 1)$ denote the power of the factor of $R^{(b,c)}(x_1)$ in $P_{g-1,v+1}^{\widetilde{M}(b,c)}(x_1, x_1, x_2, \ldots, x_v)$, then we see that:

- if $(g, v) = (1, 1)$, then $\alpha(1, 1) = 5 = 6 - 3 + 2$
- if $(g, v) = (0, 3)$, then $\alpha(0, 3) = 3 = 0 - 3 + 6$
- if $(g, v) = (1, 2)$, then $\alpha(1, 2) = 7 = 6 - 3 + 4$
- if $(g, v) = (2, 1)$, then $\alpha(2, 1) = 11 = 12 - 3 + 2$
- if $(g, v) = (0, 4)$, then $\alpha(0, 4) = 5 = 0 - 3 + 8$
- if $(g, v) = (1, 3)$, then $\alpha(1, 3) = 9 = 6 - 3 + 6$
- if $(g, v) = (2, 2)$, then $\alpha(2, 2) = 13 = 12 - 3 + 4$

- if $(g, v) = (3, 1)$, then $\alpha(3, 1) = 17 = 18 - 3 + 2$

And, for $v \neq 0$, using Mathematica for the computations, we see that

- if $(g-1, v+1) = (0, 3)$, then $\varphi(g-1, v+1) = 0 = 3 - 5 + 2$
- if $(g-1, v+1) = (1, 2)$, then $\varphi(g-1, v+1) = 2 = 6 - 5 + 1$
- if $(g-1, v+1) = (0, 4)$, then $\varphi(g-1, v+1) = 1 = 3 - 5 + 3$
- if $(g-1, v+1) = (1, 3)$, then $\varphi(g-1, v+1) = 3 = 6 - 5 + 2$
- if $(g-1, v+1) = (2, 2)$, then $\varphi(g-1, v+1) = 5 = 9 - 5 + 1$

(Note that, for the cases when $g + v = 4$, we have only computed the $(b, c) = (0, 1)$ case.)

So, we predict that

$$\alpha(g, v) = 6g - 3 + 2v \quad \text{and} \quad \varphi(g-1, v+1) = 3g - 5 + v.$$

I.e., we predict that $P_{g-1,v+1}^{\widetilde{M}(b,c)}(x_1, x_1, x_2, \ldots, x_v)$ is divisible by

$$[R^{(b,c)}(x_1)]^{3g+v-5} = [R^{(b,c)}(x_1)]^{[\alpha(g,v)-7]/2},$$

where $v \neq 0$.

Note that this implies $P_{g,v}^{\widetilde{M}(b,c)}(x_1, x_1, x_2, \ldots, x_{v-1})$ is divisible by

$$[R^{(b,c)}(x_1)]^{3(g+1)+(v-1)-5} = [R^{(b,c)}(x_1)]^{3g-3+v} = [R^{(b,c)}(x_1)]^{\beta(g,v)},$$

where $v \neq 1$.

Thus, this conjecture is true for $g + v \leq 3$ (and also for the special case of $g + v = 4$ when $(b, c) = (0, 1)$).

We would now like to discuss how one might go about proving this result.

From Theorem 7.0.1, we see that we may proceed by induction and look at the four terms separately. We assume that the conjecture holds for all (g_0, v_0) with $g_0 + v_0 < g + v$ and $g_0 < g$.

We may then show that (i) and (ii) in the above conjecture hold for the third and fourth terms, as follows.

For the third term, since we are assuming that $P^{\widetilde{M}(b,c)}_{g-1,v+1}(x_1,x_1,x_2,\dots,x_v)$ is divisible by $[R^{(b,c)}(x_1)]^{\beta(g-1,v+1)}$, we may write

$$P^{\widetilde{M}(b,c)}_{g-1,v+1}(x_1,x_1,x_2,\dots,x_v) = [R^{(b,c)}(x_1)]^{\beta(g-1,v+1)} Q^{\widetilde{M}(b,c)}_{g-1,v+1}(x_1,x_1,x_2,\dots,x_v),$$

for some polynomial $Q^{\widetilde{M}(b,c)}_{g-1,v+1}(x_1,x_1,x_2,\dots,x_v)$.

Thus:

$$\begin{aligned}
&\mathrm{III}^{\widetilde{M}(b,c)}_{g,v}(x_1,x_2,\dots,x_v)\\
&:= [R^{(b,c)}(x_1)]^{-1/2}\left[c^2x_1^2\, G^{\widetilde{M}(b,c)}_{g-1,v+1}(x_1,x_1,x_2,\dots,x_v)\right]\\
&= c^2x_1^2[R^{(b,c)}(x_1)]^{-1/2}\left[\frac{x_1(x_1x_2\cdots x_v)}{[R^{(b,c)}(x_1)R^{(b,c)}(x_1)\cdots R^{(b,c)}(x_v)]^{\alpha(g-1,v+1)/2}} P^{\widetilde{M}(b,c)}_{g-1,v+1}(x_1,x_1,x_2,\dots,x_v)\right]\\
&= c^2x_1^3\frac{x_1x_2\cdots x_v}{[R^{(b,c)}(x_1)]^{\alpha(g,v)-4+1/2-\beta(g-1,v+1)}[R^{(b,c)}(x_2)\cdots R^{(b,c)}(x_v)]^{\alpha(g,v)/2-2}}\\
&\quad\cdot Q^{\widetilde{M}(b,c)}_{g-1,v+1}(x_1,x_1,x_2,\dots,x_v)\\
&= c^2x_1^3\frac{x_1x_2\cdots x_v}{[R^{(b,c)}(x_1)]^{\alpha(g,v)/2}[R^{(b,c)}(x_2)\cdots R^{(b,c)}(x_v)]^{\alpha(g,v)/2-2}} Q^{\widetilde{M}(b,c)}_{g-1,v+1}(x_1,x_1,x_2,\dots,x_v)\\
&= \frac{x_1x_2\cdots x_v}{[R^{(b,c)}(x_1)\cdots R^{(b,c)}(x_v)]^{\alpha(g,v)/2}}\left[c^2x_1^3[R^{(b,c)}(x_2)\cdots R^{(b,c)}(x_v)]^2 Q^{\widetilde{M}(b,c)}_{g-1,v+1}(x_1,x_1,x_2,\dots,x_v)\right]
\end{aligned}$$

where we have observed that

$$\begin{aligned}
\alpha(g-1,v+1)/2 &= (6g-6-3+2v+2)/2\\
&= (6g-7+2v)/2\\
&= (\alpha(g,v)-4)/2\\
&= \alpha(g,v)/2-2
\end{aligned}$$

and

$$\begin{aligned}
\alpha(g,v)-4+1/2-\beta(g-1,v+1) &= (6g-3+2v)-4+1/2-[3(g-1)-3+(v+1)]\\
&= (6g-3+2v)/2 = \alpha(g,v)/2.
\end{aligned}$$

This shows that (i) and (ii) in the above conjecture hold for the third term.

For the fourth term, we have:

$$\begin{aligned}
&\mathrm{IV}_{g,v}^{\widetilde{M}(b,c)}(x_1,x_2,\ldots,x_v)\\
&:= [R^{(b,c)}(x_1)]^{-1/2}\Bigg[c^2x_1^2 \sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{stable}} G_{g_1,|I|+1}^{\widetilde{M}(b,c)}(x_1,x_I)G_{g_2,|J|+1}^{\widetilde{M}(b,c)}(x_1,x_J)\Bigg]\\
&= c^2x_1^2[R^{(b,c)}(x_1)]^{-1/2}\\
&\quad\cdot\Bigg[\sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{stable}} \frac{x_1\prod_{i\in I}x_i}{[R^{(b,c)}(x_1)]^{\alpha(g_1,|I|+1)/2}\prod_{i\in I}[R^{(b,c)}(x_i)]^{\alpha(g_1,|I|+1)/2}}P_{g_1,|I|+1}^{\widetilde{M}(b,c)}(x_1,x_I)\\
&\quad\cdot\frac{x_1\prod_{j\in J}x_j}{[R^{(b,c)}(x_1)]^{\alpha(g_2,|J|+1)/2}\prod_{j\in J}[R^{(b,c)}(x_j)]^{\alpha(g_2,|J|+1)/2}}P_{g_2,|J|+1}^{\widetilde{M}(b,c)}(x_1,x_J)\Bigg]\\
&= c^2x_1^3(x_1\cdots x_v)[R^{(b,c)}(x_1)]^{-\alpha(g,v)/2}\\
&\quad\cdot\Bigg[\sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{stable}} \frac{1}{\prod_{i\in I}[R^{(b,c)}(x_i)]^{\alpha(g_1,|I|+1)/2}\prod_{j\in J}[R^{(b,c)}(x_j)]^{\alpha(g_2,|J|+1)/2}}\\
&\quad\cdot P_{g_1,|I|+1}^{\widetilde{M}(b,c)}(x_1,x_I)P_{g_2,|J|+1}^{\widetilde{M}(b,c)}(x_1,x_J)\Bigg]\\
&= c^2x_1^3(x_1\cdots x_v)[R^{(b,c)}(x_1)]^{-\alpha(g,v)/2}\\
&\quad\cdot\Bigg[\sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{stable}} \frac{\prod_{i\in I}[R^{(b,c)}(x_i)]^{\alpha(g_2,|J|+1)/2+1/2}\prod_{j\in J}[R^{(b,c)}(x_j)]^{\alpha(g_1,|I|+1)/2+1/2}}{\prod_{i\in I}[R^{(b,c)}(x_i)]^{\alpha(g,v)/2}\prod_{j\in J}[R^{(b,c)}(x_j)]^{\alpha(g,v)/2}}\\
&\quad\cdot P_{g_1,|I|+1}^{\widetilde{M}(b,c)}(x_1,x_I)P_{g_2,|J|+1}^{\widetilde{M}(b,c)}(x_1,x_J)\Bigg]\\
&= \frac{x_1\cdots x_v}{[R^{(b,c)}(x_1)\cdots R^{(b,c)}(x_v)]^{-\alpha(g,v)/2}}\\
&\quad\cdot\Bigg[c^2x_1^3 \sum_{g_1+g_2=g,\ I\sqcup J=\{2,\ldots,v\},\ \text{stable}} \prod_{i\in I}[R^{(b,c)}(x_i)]^{3g_2+|J|}\prod_{j\in J}[R^{(b,c)}(x_j)]^{3g_1+|I|}\\
&\quad\cdot P_{g_1,|I|+1}^{\widetilde{M}(b,c)}(x_1,x_I)P_{g_2,|J|+1}^{\widetilde{M}(b,c)}(x_1,x_J)\Bigg]
\end{aligned}$$

where we have observed that

$$\begin{aligned}
\alpha(g_1,|I|+1)/2+\alpha(g_2,|J|+1)/2+1/2 &= [(6g_1+2|I|-1)+(6g_2+2|J|-1)+1]/2\\
&= [6(g_1+g_2)-3+2(|I|+|J|+1)]/2\\
&= (6g-3+2v)/2
\end{aligned}$$

$$= \alpha(g, v)/2$$

Further,

$$\begin{aligned}
\alpha(g_1, |I| + 1)/2 + 1/2 &= [(6g_1 + 2|I| - 1) + 1]/2 \\
&= (6g_1 + 2|I|)/2 \\
&= 3g_1 + |I|
\end{aligned}$$

and, similarly,

$$\begin{aligned}
\alpha(g_2, |J| + 1)/2 + 1/2 &= [(6g_2 + 2|J| - 1) + 1]/2 \\
&= 3g_2 + |J|
\end{aligned}$$

This shows that (i) and (ii) in the above conjecture hold for the fourth term.

It still remains to be shown whether item (iii) of the above conjecture holds for the third and fourth terms. Further, it also still needs to be determined whether all three items hold for the first and second terms appearing in Theorem 7.0.1. Based on my attempts to prove this part of the conjecture, it appears likely that these first and second terms will need to be considered together, and, upon putting both over a common denominator, the $(x_1 - x_j)$ terms in the denominator will be shown to cancel with factors in the numerator.

Lastly, we do not yet have a conjectured form for the polynomial $P_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v)$. Knowing what this polynomial looks like in the general case is likely to help greatly in work towards finding a general formula for closed-form expressions of generalized Catalan numbers, as will be discussed in the next chapter.

CHAPTER 8

Closed-Form Expressions of Generalized Catalan Numbers for Some Cases of Small (g, v)

We would now like to use the generating functions which were obtained in Chapter 7 to find closed-form expressions for generalized Catalan numbers, in the cases when $(g, v) = (1, 1), (2, 1)$, and $(3, 1)$, as well as when $(g, v) = (0, 3)$. We will also discuss how we might use the results of Conjecture 7.2.1, if it is indeed true, to obtain closed-form expressions in the case of general (g, v).

First, we will look at the case when $(g, v) = (0, 1)$. Recall that we have the following power series expansion.

Lemma 8.0.1. *We have*

$$(1+x)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} x^k = 1 - \sum_{k=0}^{\infty} \frac{2}{k+1}\binom{2k}{k}\left(-\frac{x}{4}\right)^{k+1}. \tag{8.1}$$

Therefore, for the (relatively well-known) $(g, v) = (0, 1)$ case of the aerated Catalan numbers, we may compute:

$$\begin{aligned} G^C_{0,1}(x) &= \frac{1}{2x^2}\left[1 - (1-4x^2)^{1/2}\right] \\ &= \frac{1}{2x^2}\left[1 - \left(1 - \sum_{k=0}^{\infty} \frac{2}{k+1}\binom{2k}{k}\left(-\frac{(-4x^2)}{4}\right)^{k+1}\right)\right] \\ &= \frac{1}{2x^2}\left[\sum_{k=0}^{\infty} \frac{2}{k+1}\binom{2k}{k}(x^2)^{k+1}\right] \\ &= \sum_{k=0}^{\infty} \frac{1}{k+1}\binom{2k}{k}(x^2)^k \end{aligned}$$

Thus, as expected, we obtain the $(0,1)$ Catalan numbers,

$$\begin{cases} C_{0,1}(2k) = \dfrac{1}{k+1}\dbinom{2k}{k}, \\ C_{0,1}(2k+1) = 0. \end{cases} \tag{8.2}$$

More generally, we have the following power series expansion, which will be applied to obtain closed form expressions for some generalized Catalan numbers later in this chapter.

PROPOSITION 8.0.1. *For $n \geq 0$, we have:*

$$(1+x)^{-(2n+1)/2} = \frac{n!}{(2n)!}\sum_{k=0}^{\infty}\frac{(k+n)!}{k!}\binom{2(k+n)}{k+n}\left(-\frac{x}{4}\right)^k. \tag{8.3}$$

PROOF. We may compute:

$$\begin{aligned}
(1+x)^{-(2n+1)/2} &= \sum_{k=0}^{\infty}\binom{-(2n+1)/2}{k}x^k \\
&= 1 + \sum_{k=1}^{\infty}\frac{-\frac{2n+1}{2}(-\frac{2n+1}{2}-1)(-\frac{2n+1}{2}-2)\cdots(-\frac{2n+1}{2}-k+1)}{k!}x^k \\
&= 1 + \sum_{k=1}^{\infty}(-1)^k\frac{(2n+1)(2n+2)(2n+3)\cdots(2n+2k-1)}{k!\cdot 2^k}x^k \\
&= 1 + \frac{1}{1\cdot 3\cdots(2n-1)}\sum_{k=1}^{\infty}\frac{(-1)^k}{k!\cdot 2^k}\frac{(2k+2n)!}{2^{k+n}(k+n)!}x^k \\
&= 1 + \frac{1}{[1\cdot 3\cdots(2n-1)]\cdot 2^n}\sum_{k=1}^{\infty}\frac{(2k+2n)!}{k!(k+n)!}\left(-\frac{x}{4}\right)^k \\
&= 1 + \frac{1}{[1\cdot 3\cdots(2n-1)]\cdot 2^n} \\
&\qquad \cdot\sum_{k=1}^{\infty}(k+1)(k+2)\cdots(k+n)\binom{2(k+n)}{k+n}\left(-\frac{x}{4}\right)^k \\
&= \frac{1}{[1\cdot 3\cdots(2n-1)]\cdot 2^n} \\
&\qquad \cdot\sum_{k=0}^{\infty}(k+1)(k+2)\cdots(k+n)\binom{2(k+n)}{k+n}\left(-\frac{x}{4}\right)^k \\
&= \frac{2^n\cdot n!}{(2n)!\cdot 2^n}\sum_{k=0}^{\infty}\frac{(k+n)!}{k!}\binom{2(k+n)}{k+n}\left(-\frac{x}{4}\right)^k
\end{aligned}$$

$$= \frac{n!}{(2n)!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \binom{2(k+n)}{k+n} \left(-\frac{x}{4}\right)^k$$

where we have used that

$$\frac{(1)(2)\cdots(n)\binom{2n}{n}}{[1\cdot 3\cdots(2n-1)]\cdot 2^n} = \frac{n!\cdot\frac{(2n)!}{n!n!}}{\frac{(2n)!}{2^n\cdot n!}\cdot 2^n} = 1.$$

□

REMARK 8.0.1. Thus, we see that:

$$\begin{aligned} x^n(1-4x)^{-(2n+1)/2} &= \frac{n!}{(2n)!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \binom{2(k+n)}{k+n} x^{k+n} \\ &= \frac{n!}{(2n)!} \sum_{\ell=n}^{\infty} \frac{\ell!}{(\ell-n)!} \binom{2\ell}{\ell} x^{\ell} \end{aligned} \tag{8.4}$$

Case $(g, v) = (1, 1)$**.**

When $(g, v) = (1, 1)$, we have the following result.

PROPOSITION 8.0.2. *The* $(1, 1)$ *Catalan numbers are given as follows.*

(i) $C_{1,1}(2k+1) = 0$ *for all* k*.*

(ii) $C_{1,1}(2k) = 0$ *if* $k \leq 1$*.*

(iii) $C_{1,1}(2k) = \dfrac{k(k-1)}{12}\dbinom{2k}{k}$ *if* $k \geq 2$*.*

PROOF. From equation (8.4) with $n = 2$, we see that:

$$\begin{aligned} x^2(1-4x)^{-5/2} &= \frac{2!}{4!} \sum_{\ell=2}^{\infty} \frac{\ell!}{(\ell-2)!} \binom{2\ell}{\ell} x^{\ell} \\ &= \frac{1}{12} \sum_{\ell=2}^{\infty} \ell(\ell-1) \binom{2\ell}{\ell} x^{\ell} \end{aligned}$$

So, using equation (7.22) for the $(1, 1)$ Catalan number generating function, we see that

$$\begin{aligned} G_{1,1}^C(x) &= x^4(1-4x^2)^{-5/2} \\ &= \sum_{\ell=2}^{\infty} \frac{\ell(\ell-1)}{12} \binom{2\ell}{\ell} (x^2)^{\ell} \end{aligned}$$

This proves the above claim. □

Remark 8.0.2. Observe that, for example, from Proposition 8.0.2 we may compute:

$$C_{1,1}(4) = \frac{2}{12}\binom{4}{2} = 1,$$

$$C_{1,1}(6) = \frac{3(2)}{12}\binom{6}{3} = 10,$$

$$C_{1,1}(8) = \frac{4(3)}{12}\binom{8}{4} = 70,$$

and

$$C_{1,1}(10) = \frac{5(4)}{12}\binom{10}{5} = 420.$$

As expected, these numbers are the same as obtained in Appendix A for $C_{1,1}$ by applying the Catalan recursion formula.

Case $(g, v) = (2, 1)$**.**

When $(g, v) = (2, 1)$, we have the following result.

Proposition 8.0.3. *The* $(2, 1)$ *Catalan numbers are given as follows.*

(i) $C_{2,1}(2k + 1) = 0$ *for all* k*.*

(ii) $C_{2,1}(2k) = 0$ *if* $k \leq 3$*.*

(iii) $C_{2,1}(2k) = \dfrac{5k-2}{4 \cdot 5 \cdot 8 \cdot 9} k(k-1)(k-2)(k-3)\dbinom{2k}{k}$ *if* $k \geq 4$*.*

Proof. Applying equation (8.4) when $n = 4$, we see that

$$\begin{aligned} x^4(1-4x)^{-9/2} &= \frac{4!}{8!}\sum_{\ell=4}^{\infty} \frac{\ell!}{(\ell-4)!}\binom{2\ell}{\ell}x^\ell \\ &= \frac{1}{5 \cdot 6 \cdot 7 \cdot 8}\sum_{\ell=4}^{\infty} \ell(\ell-1)(\ell-2)(\ell-3)\binom{2\ell}{\ell}x^\ell \end{aligned}$$

And, when $n = 5$, we have

$$\begin{aligned} x^5(1-4x)^{-11/2} &= \frac{5!}{10!}\sum_{\ell=5}^{\infty} \frac{\ell!}{(\ell-5)!}\binom{2\ell}{\ell}x^\ell \\ &= \frac{1}{6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}\sum_{\ell=5}^{\infty} \ell(\ell-1)(\ell-2)(\ell-3)(\ell-4)\binom{2\ell}{\ell}x^\ell \end{aligned}$$

Therefore, using equation (7.31) for the $(2,1)$ Catalan number generating function, and simplifying the result, gives

$$\begin{aligned}
G^C_{2,1}(x) &= 21x^8(1-4x^2)^{-9/2} + 105x^{10}(1-4x^2)^{-11/2} \\
&= \frac{3\cdot 7}{5\cdot 6\cdot 7\cdot 8}\sum_{\ell=4}^{\infty} \ell(\ell-1)(\ell-2)(\ell-3)\binom{2\ell}{\ell}(x^2)^\ell \\
&\quad + \frac{3\cdot 5\cdot 7}{6\cdot 7\cdot 8\cdot 9\cdot 10}\sum_{\ell=5}^{\infty} \ell(\ell-1)(\ell-2)(\ell-3)(\ell-4)\binom{2\ell}{\ell}(x^2)^\ell \\
&= \frac{1}{5\cdot 2\cdot 8}\sum_{\ell=4}^{\infty} \ell(\ell-1)(\ell-2)(\ell-3)\binom{2\ell}{\ell}(x^2)^\ell \\
&\quad + \frac{1}{2\cdot 8\cdot 9\cdot 2}\sum_{\ell=4}^{\infty} \ell(\ell-1)(\ell-2)(\ell-3)(\ell-4)\binom{2\ell}{\ell}(x^2)^\ell \\
&= \sum_{\ell=4}^{\infty} \ell(\ell-1)(\ell-2)(\ell-3)\left[\frac{1}{5\cdot 2\cdot 8} + \frac{1}{2\cdot 8\cdot 9\cdot 2}(\ell-4)\right]\binom{2\ell}{\ell}(x^2)^\ell \\
&= \frac{1}{4\cdot 5\cdot 8\cdot 9}\sum_{\ell=4}^{\infty} \ell(\ell-1)(\ell-2)(\ell-3)\Big[18+5(\ell-4)\Big]\binom{2\ell}{\ell}(x^2)^\ell \\
&= \sum_{\ell=4}^{\infty} \frac{5\ell-2}{4\cdot 5\cdot 8\cdot 9}\ell(\ell-1)(\ell-2)(\ell-3)\binom{2\ell}{\ell}(x^2)^\ell
\end{aligned}$$

This proves the above claim. □

REMARK 8.0.3. From Proposition 8.0.3, we may compute:

$$C_{2,1}(8) = \frac{4\cdot 3\cdot 2\cdot 1}{5\cdot 2\cdot 8}\binom{8}{4} = 21$$

and

$$C_{2,1}(10) = \frac{5\cdot 5-2}{4\cdot 5\cdot 8\cdot 9}5(4)(3)(2)\binom{10}{5} = 483$$

As expected, these numbers are the same as obtained in Appendix A for $C_{2,1}$ by applying the Catalan recursion formula.

Case $(g, v) = (3, 1)$**.**

When $(g, v) = (3, 1)$, we have the following result.

Proposition 8.0.4. *The* $(3, 1)$ *Catalan numbers are given as follows.*

(i) $C_{3,1}(2k + 1) = 0$ *for all* k*.*

(ii) $C_{3,1}(2k) = 0$ *if* $k \leq 5$*.*

(iii)
$$C_{3,1}(2k) = \left[\frac{1}{2^6 \cdot 7} + \frac{1}{2^6 \cdot 3 \cdot 5}(k-6) + \frac{29 \cdot 73}{2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 13}(k-6)(k-7)\right] \cdot k(k-1)(k-2)(k-3)(k-4)(k-5)\binom{2k}{k} \quad \text{if } k \geq 6$$

Proof. Applying equation (8.4), we see that

$$x^{12}(1-4x^2)^{-13/2} = \frac{6!}{12!}\sum_{\ell=6}^{\infty}\frac{\ell!}{(\ell-6)!}\binom{2\ell}{\ell}x^{2\ell},$$

$$x^{14}(1-4x^2)^{-15/2} = \frac{7!}{14!}\sum_{\ell=7}^{\infty}\frac{\ell!}{(\ell-7)!}\binom{2\ell}{\ell}x^{2\ell},$$

and

$$x^{16}(1-4x^2)^{-17/2} = \frac{8!}{16!}\sum_{\ell=8}^{\infty}\frac{\ell!}{(\ell-8)!}\binom{2\ell}{\ell}x^{2\ell}.$$

And, using equation (7.36) for the $(3, 1)$ Catalan number generating function, we have

$$\begin{aligned}
G_{3,1}^{\widetilde{M}(0,1)}(x) &= 1485x^{12}(1-4x^2)^{-13/2} + 18018x^{14}(1-4x^2)^{-15/2} + 46574x^{16}(1-4x^2)^{-17/2} \\
&= (3^3 \cdot 5 \cdot 11)\frac{6!}{12!}\sum_{\ell=6}^{\infty}\frac{\ell!}{(\ell-6)!}\binom{2\ell}{\ell}x^{2\ell} \\
&\quad + (2 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13)\frac{7!}{14!}\sum_{\ell=7}^{\infty}\frac{\ell!}{(\ell-7)!}\binom{2\ell}{\ell}x^{2\ell} \\
&\quad + (2 \cdot 11 \cdot 29 \cdot 73)\frac{8!}{16!}\sum_{\ell=8}^{\infty}\frac{\ell!}{(\ell-8)!}\binom{2\ell}{\ell}x^{2\ell} \\
&= \sum_{\ell=6}^{\infty}\left[\frac{3^3 \cdot 5 \cdot 11}{7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}\ell(\ell-1)(\ell-2)(\ell-3)(\ell-4)(\ell-5)\right. \\
&\quad + \frac{2 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13}{8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14}\ell(\ell-1)(\ell-2)(\ell-3)(\ell-4)(\ell-5)(\ell-6) \\
&\quad + \frac{2 \cdot 11 \cdot 29 \cdot 73}{9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15 \cdot 16}
\end{aligned}$$

$$
\begin{aligned}
&\cdot \ell(\ell-1)(\ell-2)(\ell-3)(\ell-4)(\ell-5)(\ell-6)(\ell-7)\Big]\binom{2\ell}{\ell}x^{2\ell} \\
&= \sum_{\ell=6}^{\infty} \ell(\ell-1)(\ell-2)(\ell-3)(\ell-4)(\ell-5)\left[\frac{1}{2\cdot 4\cdot 7\cdot 8} + \frac{1}{8\cdot 10\cdot 12}(\ell-6)\right. \\
&\quad \left. + \frac{29\cdot 73}{9\cdot 10\cdot 12\cdot 13\cdot 14\cdot 15\cdot 8}(\ell-6)(\ell-7)\right]\binom{2\ell}{\ell}x^{2\ell} \\
&= \sum_{\ell=6}^{\infty}\left[\frac{1}{2^6\cdot 7} + \frac{1}{2^6\cdot 3\cdot 5}(\ell-6) + \frac{29\cdot 73}{2^7\cdot 3^4\cdot 5^2\cdot 7\cdot 13}(\ell-6)(\ell-7)\right] \\
&\quad \cdot \ell(\ell-1)(\ell-2)(\ell-3)(\ell-4)(\ell-5)\binom{2\ell}{\ell}x^{2\ell}
\end{aligned}
$$

This proves the above claim. □

REMARK 8.0.4. From Proposition 8.0.4, we may compute:

$$C_{3,1}(\mu) = 0 \quad \text{for} \quad \mu < 12.$$

And,

$$
\begin{aligned}
C_{3,1}(12) &= \left[\frac{1}{2^6\cdot 7} + \frac{1}{2^6\cdot 3\cdot 5}(6-6) + \frac{29\cdot 73}{2^7\cdot 3^4\cdot 5^2\cdot 7\cdot 13}(6-6)(6-7)\right] \\
&\quad \cdot 6(6-1)(6-2)(6-3)(6-4)(6-5)\binom{12}{6} \\
&= \frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{2^6\cdot 7}\cdot\binom{12}{6} \\
&= 1485
\end{aligned}
$$

As expected, these numbers are the same as obtained in Appendix A for $C_{3,1}$ by applying the Catalan recursion formula.

REMARK 8.0.5. The above method used to determine closed-form expressions for the generalized Catalan numbers when $(g, v) = (0, 1), (1, 1), (2, 1)$, and $(3, 1)$ should work to obtain closed-form expressions for higher-order Catalan numbers of the form $(g, 1)$. However, this method is currently computationally intensive, since we must first determine the generating function for the $(g, 1)$ Catalan numbers, which, via the recursive formula, requires us to also determine generating functions for all lower-order Catalan numbers.

Further, even after computing the closed-form expressions in the above cases, we do not yet have a conjecture for a general closed-form expression of the $(g, 1)$ Catalan numbers. However, if Conjecture 7.2.1 for the general form of the bc-Motzkin numbers generating functions is proved, along with obtaining a general form for the polynomial $P_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v)$, then it seems likely that a general formula giving closed-form expressions for the $(g, 1)$ Catalan numbers will follow.

Additionally, given the rather complicated nature of the formulas for the generating functions of generalized Catalan numbers not of the form $(g, 1)$, it is more challenging to determine a closed-form expression for these generalized Catalan numbers. The easiest one appears to be the $(0, 3)$-Catalan numbers, and we have found a closed-form expression for these numbers as shown below. However, determining whether the (necessarily symmetric) polynomials $P_{g,v}^{\widetilde{M}(b,c)}(x_1, x_2, \ldots, x_v)$ follow a particular pattern may help in progress towards determining a general formula that gives closed-form expressions for more cases of Catalan numbers.

Case $(g, v) = (0, 3)$**.**

When $(g, v) = (0, 3)$, we have the following result.

Proposition 8.0.5. *The $(0, 3)$ Catalan numbers are given as follows.*

(i) $C_{0,3}(a, b, c) = 0$ *if* a*,*b*, or* c *equals zero.*

(ii) $C_{0,3}(a, b, c) = 0$ *if* $a + b + c$ *is odd.*

(iii) $C_{0,3}(a, b, c) = (n \cdot m \cdot k)\binom{2n}{n}\binom{2m}{m}\binom{2k}{k}$ *if* $a = 2n$*,* $b = 2m$*, and* $c = 2k$ *are all even.*

(iv) $C_{0,3}(a, b, c) = \frac{1}{4}(n \cdot m \cdot k)\binom{2n}{n}\binom{2m}{m}\binom{2k}{k}$ *if* $a = 2n$ *is even, and* $b = 2m - 1$ *and* $c = 2k - 1$ *are odd.*

(v) $C_{0,3}(a, b, c) = \frac{1}{4}(n \cdot m \cdot k)\binom{2n}{n}\binom{2m}{m}\binom{2k}{k}$ *if* $b = 2m$ *is even, and* $a = 2n - 1$ *and* $c = 2k - 1$ *are odd.*

(vi) $C_{0,3}(a, b, c) = \frac{1}{4}(n \cdot m \cdot k)\binom{2n}{n}\binom{2m}{m}\binom{2k}{k}$ *if* $c = 2k$ *is even, and* $a = 2n - 1$ *and* $b = 2m - 1$ *are odd.*

PROOF. Applying equation (8.4), we see that

$$x(1-4x^2)^{-3/2} = \frac{1}{2x}\sum_{\ell=1}^{\infty} \ell \binom{2\ell}{\ell} x^{2\ell}.$$

And, using equation (7.25) for the $(0,3)$ Catalan number generating function, we have

$$\begin{aligned}
&G_{0,3}^{\widetilde{M}(0,1)}(x,y,z)\\
&= \frac{2xyz(x+y+z+4xyz)}{(1-4x^2)^{3/2}(1-4y^2)^{3/2}(1-4z^2)^{3/2}}\\
&= 2(x+y+z+4xyz)\left(\frac{1}{2x}\sum_{n=1}^{\infty} n\binom{2n}{n}x^{2n}\right)\left(\frac{1}{2y}\sum_{m=1}^{\infty} m\binom{2m}{m}y^{2m}\right)\left(\frac{1}{2z}\sum_{k=1}^{\infty} k\binom{2k}{k}z^{2k}\right)\\
&= \frac{x+y+z+4xyz}{4xyz}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}(n\cdot m\cdot k)\binom{2n}{n}\binom{2m}{m}\binom{2k}{k}x^{2n}y^{2m}z^{2k}\\
&= \left(\frac{1}{4yz}+\frac{1}{4xz}+\frac{1}{4xy}+1\right)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}(n\cdot m\cdot k)\binom{2n}{n}\binom{2m}{m}\binom{2k}{k}x^{2n}y^{2m}z^{2k}\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}(n\cdot m\cdot k)\binom{2n}{n}\binom{2m}{m}\binom{2k}{k}x^{2n}y^{2m}z^{2k}\\
&\quad+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{4}(n\cdot m\cdot k)\binom{2n}{n}\binom{2m}{m}\binom{2k}{k}x^{2n}y^{2m-1}z^{2k-1}\\
&\quad+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{4}(n\cdot m\cdot k)\binom{2n}{n}\binom{2m}{m}\binom{2k}{k}x^{2n-1}y^{2m}z^{2k-1}\\
&\quad+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{4}(n\cdot m\cdot k)\binom{2n}{n}\binom{2m}{m}\binom{2k}{k}x^{2n-1}y^{2m-1}z^{2k}
\end{aligned}$$

This proves the above claim. □

REMARK 8.0.6. From Proposition 8.0.5, we may compute the following generalized Catalan numbers.

For the numbers satisfying $a+b+c=4$,

$$C_{0,3}(1,1,2) = \frac{1}{4}(1\cdot 1\cdot 1)\binom{2}{1}\binom{2}{1}\binom{2}{1} = 2$$

For the numbers satisfying $a+b+c=6$,

$$C_{0,3}(1,1,4) = \frac{1}{4}(1\cdot 1\cdot 2)\binom{2}{1}\binom{2}{1}\binom{4}{2} = 2\cdot\binom{4}{2} = 12$$

$$C_{0,3}(1,2,3) = \frac{1}{4}(1\cdot 1\cdot 2)\binom{2}{1}\binom{2}{1}\binom{4}{2} = 12$$

$$C_{0,3}(2,2,2) = (1\cdot 1\cdot 1)\binom{2}{1}\binom{2}{1}\binom{2}{1} = 8$$

For the numbers satisfying $a+b+c=8$,

$$C_{0,3}(1,1,6) = \frac{1}{4}(1\cdot 1\cdot 3)\binom{2}{1}\binom{2}{1}\binom{6}{3} = 3\cdot\binom{6}{3} = 60$$

$$C_{0,3}(1,2,5) = \frac{1}{4}(1\cdot 1\cdot 3)\binom{2}{1}\binom{2}{1}\binom{6}{3} = 60$$

$$C_{0,3}(1,3,4) = \frac{1}{4}(1\cdot 2\cdot 2)\binom{2}{1}\binom{4}{2}\binom{4}{2} = 2\cdot 6\cdot 6 = 72$$

$$C_{0,3}(2,2,4) = (1\cdot 1\cdot 2)\binom{2}{1}\binom{2}{1}\binom{4}{2} = 8\cdot 6 = 48$$

$$C_{0,3}(2,3,3) = \frac{1}{4}(1\cdot 2\cdot 2)\binom{2}{1}\binom{4}{2}\binom{4}{2} = 72$$

As expected, these numbers are the same as obtained in Appendix A for $C_{0,3}$ by applying the Catalan recursion formula.

CHAPTER 9

Identities for Some Generalized Catalan Numbers

We will use the generating functions for the $(1,1)$ and $(0,2)$ bc-Motzkin numbers, which were obtained in Chapter 7, to prove some new identities for the $(1,1)$ and $(0,2)$ Catalan numbers, respectively.

First, however, we will use the generating function obtained for the $(0,1)$ bc-Motzkin numbers in Proposition 7.0.1 to prove the following relatively well-known identity on Catalan numbers.

Proposition 9.0.1. *The $(0,1)$ Catalan numbers and the $(0,1)$-bc-Motzkin numbers satisfy the following relation, for all n:*

$$C_{0,1}(2n+2) = \widetilde{M}_{0,1}(n,2,1). \tag{9.1}$$

Since, by definition,

$$\widetilde{M}_{0,1}(n,2,1) := \sum_{\mu=0}^{n} \binom{n}{\mu} C_{0,1}(\mu) 2^{n-\mu}, \tag{9.2}$$

this therefore implies the following relation, which involves only $(0,1)$ Catalan numbers:

$$C_{0,1}(2n+2) = \sum_{\mu=0}^{n} \binom{n}{\mu} C_{0,1}(\mu) 2^{n-\mu}. \tag{9.3}$$

Remark 9.0.1. In the literature, e.g. see [**27**], this identity is often written as

$$C_{n+1} = \sum_{k} \binom{n}{2k} C_k 2^{n-2k}.$$

Proof. We know from Remark 7.0.1 the Catalan generating function is

$$\begin{aligned} G_{0,1}^{C}(x) &= \sum_{n=0}^{\infty} C_n x^n \\ &= \sum_{n=0}^{\infty} C_{0,1}(2n)\, x^n \\ &= \frac{1-\sqrt{1-4x}}{2x}. \end{aligned}$$

Thus, we may compute

$$\begin{aligned}\sum_{n=0}^{\infty} C_{n+1} x^n &= \sum_{n=0}^{\infty} C_{0,1}(2n+2)\, x^n \\ &= \frac{1}{x}\Big[- C_{0,1}(0) + \sum_{n=0}^{\infty} C_{0,1}(2n)\, x^n \Big].\end{aligned}$$

So,

$$\begin{aligned}\sum_{n=0}^{\infty} C_{0,1}(2n+2)\, x^n &= \frac{1}{x}\Big[- C_{0,1}(0) + \frac{1-\sqrt{1-4x}}{2x}\Big] \\ &= \frac{1-2x-\sqrt{1-4x}}{2x^2}.\end{aligned}$$

And, we know from Proposition 7.0.1 that

$$G_{0,1}^{\widetilde{M}(2,1)}(x) = \frac{1-2x-\sqrt{1-4x}}{2x^2}.$$

Therefore, we see that

$$\widetilde{M}_{0,1}(n;2,1) = C_{0,1}(2n+2).$$

Hence the claimed formula does indeed hold. □

We now wish to employ a similar procedure to obtain identities on some generalized Catalan numbers for small (g, v), using the generating function for the generalized bc-Motzkin numbers.

We will now prove the following result for $(1,1)$ Catalan numbers.

THEOREM 9.0.1. *The $(1,1)$ Catalan numbers and the $(1,1)$-bc-Motzkin numbers satisfy the following relation, for all n:*

$$C_{1,1}(2n-4) = \widetilde{M}_{1,1}(n;2,1). \tag{9.4}$$

Since, by definition,

$$\widetilde{M}_{1,1}(n;2,1) := \sum_{\mu=0}^{n} \binom{n}{\mu} C_{1,1}(\mu) 2^{n-\mu}, \tag{9.5}$$

this therefore implies the following relation, which involves only $(1,1)$ Catalan numbers:

$$C_{1,1}(2n-4) = \sum_{\mu=0}^{n} \binom{n}{\mu} C_{1,1}(\mu) 2^{n-\mu}. \tag{9.6}$$

We will prove this result by showing that the generating function for $C_{1,1}(2n-4)$ is equal to the generating function for $\widetilde{M}_{1,1}(n;2,1)$.

Proof. Recall from Proposition 7.1.2 that

$$G_{1,1}^{\widetilde{M}(b,c)}(x) = c^4x^4[1-2bx+(b^2-4c^2)x^2]^{-5/2}.$$

Now, letting $b=2$ and $c=1$ gives

$$G_{1,1}^{\widetilde{M}(2,1)}(x) = x^4(1-4x)^{-5/2}.$$

And, letting $b=0$ and $c=1$ gives

$$G_{1,1}^{\widetilde{M}(0,1)}(x) = x^4(1-4x^2)^{-5/2}.$$

I.e,

$$\sum_{n=0}^{\infty} \widetilde{M}_{1,1}(n;2,1)x^n = x^4(1-4x)^{-5/2}$$

and

$$\sum_{n=0}^{\infty} C_{1,1}(n)x^n = \sum_{n=0}^{\infty} \widetilde{M}_{1,1}(n;0,1)x^n = x^4(1-4x^2)^{-5/2}.$$

Thus,

$$\sum_{n=0}^{\infty} C_{1,1}(2n)x^n = x^2(1-4x)^{-5/2}$$

So,

$$\begin{aligned}
\sum_{n=0}^{\infty} C_{1,1}(2n-4)x^n &= \sum_{n=2}^{\infty} C_{1,1}(2n-4)x^n \\
&= \sum_{k+2=2}^{\infty} C_{1,1}(2(k+2)-4)x^{k+2} \\
&= x^2\sum_{k=0}^{\infty} C_{1,1}(2k)x^k \\
&= x^2[x^2(1-4x)^{-5/2}] \\
&= x^4(1-4x)^{-5/2}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \widetilde{M}_{1,1}(n;2,1)x^n$$

This proves that we do indeed have

$$C_{1,1}(2n-4) = \widetilde{M}_{1,1}(n;2,1) = \sum_{\mu=0}^{n} \binom{n}{\mu} C_{1,1}(\mu)2^{n-\mu}$$

as was claimed.

□

We would next like to prove the following result for $(0,2)$ Catalan numbers.

THEOREM 9.0.2. *We have*

$$\sum_{a+b=2n-2} C_{0,2}(a,b) = \sum_{n_1+n_2=n} \widetilde{M}_{0,2}(n_1,n_2;2,1). \tag{9.7}$$

Since, by definition

$$\widetilde{M}_{0,2}(n_1,n_2;2,1) = \sum_{\mu_1=0}^{n_1} \sum_{\mu_2=0}^{n_2} \binom{n_1}{\mu_1}\binom{n_1}{\mu_1} C_{0,2}(\mu_1,\mu_2)\, 2^{(n_1+n_2)-(\mu_1+\mu_2)}, \tag{9.8}$$

this therefore implies the following relation, which only involves $(0,2)$ Catalan numbers.

$$\sum_{a+b=2n-2} C_{0,2}(a,b) = \sum_{n_1+n_2=n} \Bigg[\sum_{\mu_1=0}^{n_1} \sum_{\mu_2=0}^{n_2} \binom{n_1}{\mu_1}\binom{n_1}{\mu_1} C_{0,2}(\mu_1,\mu_2)\, 2^{(n_1+n_2)-(\mu_1+\mu_2)} \Bigg]. \tag{9.9}$$

PROOF. First, observe that the generating function for $\sum_{a+b=2n-2} C_{0,2}(a,b)$ is

$$\begin{aligned}
\sum_{n=0}^{\infty} \sum_{a+b=2n-2} C_{0,2}(a,b)\, x^n &= \sum_{k+1=1}^{\infty} \sum_{a+b=2(k+1)-2} C_{0,2}(a,b)\, x^{k+1} \\
&= x \sum_{k=0}^{\infty} \sum_{a+b=2k} C_{0,2}(a,b)\, x^k \\
&= x \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} C_{0,2}(a,b)\, x^a x^b
\end{aligned}$$

And, recall from Proposition 7.1.1 that

$$G_{0,2}^{\widetilde{M}(b,c)}(x,x) = c^2x^2[1-2bx+(b^2-4c^2)x^2]^{-2}$$

So,

$$\sum_{a=0}^{\infty}\sum_{b=0}^{\infty} C_{0,2}(a,b)\ (x^2)^a (x^2)^b = G_{0,2}^{\widetilde{M}(0,1)}(x,x)$$

$$= x^2(1 - 0 + (0-4)x^2)^{-2}$$

$$= x^2(1-4x^2)^{-2}$$

and hence

$$\sum_{a=0}^{\infty}\sum_{b=0}^{\infty} C_{0,2}(a,b)\ x^a x^b = x(1-4x)^{-2}.$$

Further,

$$\sum_{n=0}^{\infty}\sum_{n_1+n_2=n} \widetilde{M}_{0,2}(n_1,n_2;2,1)\, x^n = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \widetilde{M}_{0,2}(n_1,n_2;2,1)\, x^{n_1} x^{n_2}$$

$$= G_{0,2}^{\widetilde{M}(2,1)}(x,x)$$

$$= x^2(1-4x+(2^2-4)x^2)^{-2}$$

$$= x^2(1-4x)^{-2}$$

Thus, we see that we do indeed have

$$\sum_{a+b=2n-2} C_{0,2}(a,b) = \sum_{n_1+n_2=n} \widetilde{M}_{0,2}(n_1,n_2;2,1)$$

as was claimed.

$\square$

Remark 9.0.2. Upon applying this approach to $(2,1)$ and to $(0,3)$ Catalan numbers, I was unable to find any relationship between the Catalan number generating function and the bc-Motzkin number generating function when $b=2$ and $c=1$. It remains to be determined what, if any, identities similar to the ones given above are satisfied by Catalan numbers of higher order.

APPENDIX A

Exact Values for Generalized Catalan Numbers for Some Small (g, v)

Recall from Proposition 2.0.1 that the generalized Catalan numbers are defined by the recursion

$$\begin{aligned} C_{g,v}(\vec{\mu}) = &\sum_{j=2}^{v} \mu_j C_{g,v-1}(\mu_1 + \mu_j - 2, \mu_2, \ldots, \hat{\mu_j}, \ldots, \mu_v) \\ &+ \sum_{\alpha+\beta=\mu_1-2} \Bigg[C_{g-1,v+1}(\alpha, \beta, \mu_2, \ldots, \mu_v) \\ &+ \sum_{g_1+g_2=g, I \sqcup J=\{2,\ldots,v\}} C_{g_1,|I|+1}(\alpha, \mu_I) C_{g_2,|J|+1}(\beta, \mu_J) \Bigg] \end{aligned}$$

Remark A.0.1. We have the following observations.

- By the definition of the generalized Catalan numbers,

$$C_{g,v}(\vec{\mu}) = C_{g,v}(\vec{\mu}_P),$$

  where $\vec{\mu}_P$ denotes any permutation of the elements of $\vec{\mu}$.
- Further, if $\mu_1 + \mu_2 + \cdots + \mu_v$ is odd, then $C_{g,v}(\vec{\mu}) = 0$.
- And, if $\mu_i = 0$ for any i and for $(g, v) \neq (0, 1)$, then $C_{g,v}(\vec{\mu}) = 0$.

Using Mathematica to aid in the computations, we may compute exact values of these numbers for some small g, v, and n.

Case $(g, v) = (0, 1)$**.**

The $(0, 1)$ Catalan numbers (also known as the aerated Catalan numbers) are defined by

$$C_{0,1}(2m) = \frac{1}{m+1}\binom{2m}{m}. \qquad \text{(A.1)}$$

Thus, for $2m \leq 10$, we may compute:

$C_{0,1}(0) = 1$

$C_{0,1}(2) = 1$

$C_{0,1}(4) = 2$

$C_{0,1}(6) = 5$

$C_{0,1}(8) = 14$

$C_{0,1}(10) = 42$

Case $(g, v) = (0, 2)$**.**

The $(0, 2)$ Catalan numbers are defined recursively by

$$C_{0,2}(\mu_1, \mu_2) = \mu_2 C_{0,1}(\mu_1 + \mu_2 - 2) + 2 \sum_{\alpha+\beta=\mu_1-2} C_{0,1}(\alpha) C_{0,2}(\beta, \mu_2). \tag{A.2}$$

Thus, for $\mu_1 + \mu_2 \leq 10$, we may compute:

$C_{0,2}(1, 1) = 1$

$C_{0,2}(1, 3) = 3$

$C_{0,2}(2, 2) = 2$

$C_{0,2}(1, 5) = 10$

$C_{0,2}(2, 4) = 8$

$C_{0,2}(3, 3) = 12$

$C_{0,2}(1, 7) = 35$

$C_{0,2}(2, 6) = 30$

$C_{0,2}(3, 5) = 45$

$C_{0,2}(4, 4) = 36$

$C_{0,2}(1, 9) = 126$

$C_{0,2}(2, 8) = 112$

$C_{0,2}(3, 7) = 168$

$C_{0,2}(4, 6) = 144$

$C_{0,2}(5, 5) = 180$

Case $(g,v)=(1,1)$**.**

The $(1,1)$ Catalan numbers are defined recursively by

$$C_{1,1}(\mu)=\sum_{\alpha+\beta=\mu_1-2}\big[C_{0,2}(\alpha,\beta)+2C_{0,1}(\alpha)C_{1,1}(\beta)\big]. \tag{A.3}$$

Thus, for $\mu\leq 10$, we may compute:

$C_{1,1}(0)=0$

$C_{1,1}(2)=0$

$C_{1,1}(4)=1$

$C_{1,1}(6)=10$

$C_{1,1}(8)=70$

$C_{1,1}(10)=420$

Case $(g,v)=(0,3)$**.**

The $(0,3)$ Catalan numbers are defined recursively by

$$\begin{aligned}C_{0,3}(\mu_1,\mu_2,\mu_3)&=\mu_2C_{0,2}(\mu_1+\mu_2-2,\mu_3)+\mu_3C_{0,2}(\mu_1+\mu_3-2,\mu_2)\\&\quad+\sum_{a+b=\mu_1-2}\big[2C_{0,1}(a)C_{0,3}(b,\mu_2,\mu_3)+2C_{0,2}(a,\mu_2)C_{0,2}(b,\mu_3)\big]\end{aligned} \tag{A.4}$$

Thus, for $\mu_1+\mu_2+\mu_3\leq 10$, we may compute:

$C_{0,3}(1,1,2)=2$

$C_{0,3}(1,1,4)=12$

$C_{0,3}(1,2,3)=12$

$C_{0,3}(2,2,2)=8$

$C_{0,3}(1,1,6)=60$

$C_{0,3}(1,2,5)=60$

$C_{0,3}(1,3,4)=72$

$C_{0,3}(2,2,4)=48$

$C_{0,3}(2,3,3)=72$

$C_{0,3}(1,1,8)=280$

$C_{0,3}(1, 2, 7) = 280$

$C_{0,3}(1, 3, 6) = 360$

$C_{0,3}(1, 4, 5) = 360$

$C_{0,3}(2, 2, 6) = 240$

$C_{0,3}(2, 3, 5) = 360$

$C_{0,3}(2, 4, 4) = 288$

$C_{0,3}(3, 3, 4) = 432$

Case $(g, v) = (1, 2)$**.**

The $(1, 2)$ Catalan numbers are defined recursively by

$$\begin{aligned} C_{1,2}(\mu_1, \mu_2) &= \mu_2 C_{1,1}(\mu_1 + \mu_2 - 2) \\ &\quad + \sum_{a+b=\mu_1-2} \big[C_{0,3}(a, b, \mu_2) + 2C_{0,1}(a)C_{1,2}(b, \mu_2) + 2C_{1,1}(a)C_{0,2}(b, \mu_2)\big] \end{aligned} \tag{A.5}$$

Thus, for $\mu_1 + \mu_2 \leq 10$, we may compute:

$C_{1,2}(1, 1) = 0$

$C_{1,2}(1, 3) = 0$

$C_{1,2}(2, 2) = 0$

$C_{1,2}(1, 5) = 5$

$C_{1,2}(2, 4) = 4$

$C_{1,2}(3, 3) = 3$

$C_{1,2}(1, 7) = 70$

$C_{1,2}(2, 6) = 60$

$C_{1,2}(3, 5) = 60$

$C_{1,2}(4, 4) = 60$

$C_{1,2}(1, 9) = 630$

$C_{1,2}(2, 8) = 560$

$C_{1,2}(3, 7) = 630$

$C_{1,2}(4, 6) = 600$

$C_{1,2}(5,5) = 600$

Case $(g, v) = (2, 1)$**.**

The $(2, 1)$ Catalan numbers are defined recursively by

$$C_{2,1}(\mu) = \sum_{a+b=\mu_1-2} \big[C_{1,2}(a,b) + 2C_{0,1}(a)C_{2,1}(b) + C_{1,1}(a)C_{1,1}(b)\big]. \tag{A.6}$$

Thus, for $\mu \le 10$, we may compute:

$C_{2,1}(\mu) = 0$ if $\mu \le 6$

$C_{2,1}(8) = 21$

$C_{2,1}(10) = 483$

Case $(g, v) = (0, 4)$**.**

The $(0, 4)$ Catalan numbers are defined recursively by

$$\begin{aligned} C_{0,4}(\mu_1, \mu_2, \mu_3, \mu_4) = {} & \mu_2 C_{0,3}(\mu_1 + \mu_2 - 2, \mu_3, \mu_4) + \mu_3 C_{0,3}(\mu_1 + \mu_3 - 2, \mu_2, \mu_4) \\ & + \mu_4 C_{0,3}(\mu_1 + \mu_4 - 2, \mu_2, \mu_3) \\ & + \sum_{a+b=\mu_1-2} \big[2C_{0,1}(\alpha)C_{0,4}(\beta, \mu_2, \mu_3, \mu_4) \\ & + 2C_{0,2}(\alpha, \mu_2)C_{0,3}(\beta, \mu_3, \mu_4) + 2C_{0,2}(\alpha, \mu_3)C_{0,3}(\beta, \mu_2, \mu_4) \\ & + 2C_{0,2}(\alpha, \mu_4)C_{0,3}(\beta, \mu_2, \mu_3)\big] \end{aligned} \tag{A.7}$$

Thus, for $\mu_1 + \mu_2 + \mu_3 + \mu_4 \le 10$, we may compute:

$C_{0,4}(1,1,1,1) = 0$

$C_{0,4}(1,1,1,3) = 6$

$C_{0,4}(1,1,2,2) = 8$

$C_{0,4}(1,1,1,5) = 60$

$C_{0,4}(1,1,2,4) = 72$

$C_{0,4}(1,1,3,3) = 72$

$C_{0,4}(1,2,2,3) = 72$

$C_{0,4}(2,2,2,2) = 48$

$C_{0,4}(1,1,1,7) = 420$

$C_{0,4}(1,1,2,6) = 480$

$C_{0,4}(1,1,3,5) = 540$

$C_{0,4}(1,1,4,4) = 576$

$C_{0,4}(1,2,2,5) = 480$

$C_{0,4}(1,2,3,4) = 576$

$C_{0,4}(1,3,3,3) = 648$

$C_{0,4}(2,2,2,4) = 384$

$C_{0,4}(2,2,3,3) = 576$

Case $(g, v) = (1, 3)$**.**

The $(1,3)$ Catalan numbers are defined recursively by

$$\begin{aligned} C_{1,3}(\mu_1,\mu_2,\mu_3) = {} & \mu_2 C_{1,2}(\mu_1+\mu_2-2,\mu_3) + \mu_3 C_{1,2}(\mu_1+\mu_3-2,\mu_2) \\ & + \sum_{a+b=\mu_1-2} \Big[C_{0,4}(a,b,\mu_2,\mu_3) + 2C_{0,1}(a)C_{1,3}(b,\mu_2,\mu_3) \\ & + 2C_{0,2}(a,\mu_2)C_{1,2}(b,\mu_3) + 2C_{0,2}(a,\mu_3)C_{1,2}(b,\mu_2) \\ & + 2C_{0,3}(a,\mu_2,\mu_3)C_{1,1}(b)\Big] \end{aligned} \tag{A.8}$$

Thus, for $\mu_1+\mu_2+\mu_3 \le 10$, we may compute:

$C_{1,3}(1,1,2) = 0$

$C_{1,3}(1,1,4) = 0$

$C_{1,3}(1,2,3) = 0$

$C_{1,3}(2,2,2) = 0$

$C_{1,3}(1,1,6) = 30$

$C_{1,3}(1,2,5) = 30$

$C_{1,3}(1,3,4) = 24$

$C_{1,3}(2,2,4) = 24$

$C_{1,3}(2,3,3) = 18$

$C_{1,3}(1,1,8) = 560$

$C_{1,3}(1,2,7) = 560$

$C_{1,3}(1,3,6) = 540$

$C_{1,3}(1,4,5) = 540$

$C_{1,3}(2,2,6) = 480$

$C_{1,3}(2,3,5) = 480$

$C_{1,3}(2,4,4) = 480$

$C_{1,3}(3,3,4) = 468$

Case $(g, v) = (2, 2)$**.**

The $(2, 2)$ Catalan numbers are defined recursively by

$$\begin{aligned} C_{2,2}(\mu_1, \mu_2) = \mu_2 C_{2,1}(\mu_1 + \mu_2 - 2) & \\ + \sum_{a+b=\mu_1-2} \Big[& C_{1,3}(a, b, \mu_2) + 2C_{0,1}(a)C_{2,2}(b, \mu_2) \\ & + 2C_{0,2}(a, \mu_2)C_{2,1}(b) + 2C_{1,1}(a)C_{1,2}(b, \mu_2) \Big] \end{aligned} \tag{A.9}$$

Thus, for $\mu_1 + \mu_2 \leq 12$, we may compute:

$C_{2,2}(\mu_1, \mu_2) = 0$ if $\mu \leq 8$

$C_{2,2}(1,9) = 189$

$C_{2,2}(2,8) = 168$

$C_{2,2}(3,7) = 147$

$C_{2,2}(4,6) = 156$

$C_{2,2}(5,5) = 165$

$C_{2,2}(1,11) = 5313$

$C_{2,2}(2,10) = 4830$

$C_{2,2}(3,9) = 4725$

$C_{2,2}(4,8) = 4760$

$C_{2,2}(5,7) = 4795$

$C_{2,2}(6,6) = 4770$

Case $(g, v) = (3, 1)$**.**

The $(3, 1)$ Catalan numbers are defined recursively by

$$C_{3,1}(\mu) = \sum_{a+b=\mu-2} \big[C_{2,2}(a, b) + 2C_{0,1}(a)C_{3,1}(b) + 2C_{1,1}(a)C_{2,1}(b)\big]. \qquad \text{(A.10)}$$

Thus, for $\mu \leq 14$, we may compute:

$C_{3,1}(\mu) = 0$ if $\mu \leq 10$

$C_{3,1}(12) = 1485$

$C_{3,1}(14) = 56628$

APPENDIX B

Exact Values for Generalized bc-Motzkin Numbers for Some Small (g, v)

Recall from Definition 3.0.2 that the generalized bc-Motzkin numbers are defined by

$$\widetilde{M}_{g,v}(\vec{n}; b, c) = \sum_{\mu_1=0}^{n_1} \cdots \sum_{\mu_v=0}^{n_v} \binom{n_1}{\mu_1} \cdots \binom{n_v}{\mu_v} C_{g,v}(\vec{\mu})\, b^{|\vec{n}|-|\vec{\mu}|} c^{|\vec{\mu}|}$$

Remark B.0.1. By the definition of the generalized bc-Motzkin numbers,

$$\widetilde{M}_{g,v}(\vec{n}) = \widetilde{M}_{g,v}(\vec{n}_P),$$

where $\vec{n}_P$ denotes any permutation of the elements of $\vec{n}$.

Using Mathematica to aid in the computations, we may compute exact values of these numbers for some small g, v, and n.

Case $(g, v) = (0, 1)$**.**

For $n \leq 10$, we may compute:

$$\begin{aligned} \widetilde{M}_{0,1}(0; b, c) &= C_{0,1}(0) \\ &= 1 \end{aligned}$$

$$\begin{aligned} \widetilde{M}_{0,1}(1; b, c) &= bC_{0,1}(0) \\ &= b \end{aligned}$$

$$\begin{aligned} \widetilde{M}_{0,1}(2; b, c) &= b^2 C_{0,1}(0) + c^2 C_{0,1}(2) \\ &= b^2 + c^2 \end{aligned}$$

$$
\begin{aligned}
\widetilde{M}_{0,1}(3;b,c) &= b^3C_{0,1}(0) + 3bc^2C_{0,1}(2) \\
&= b^3 + 3bc^2
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,1}(4;b,c) &= b^4C_{0,1}(0) + 6b^2c^2C_{0,1}(2) + c^4C_{0,1}(4) \\
&= b^4 + 6b^2c^2 + 2c^4
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,1}(5;b,c) &= b^5C_{0,1}(0) + 10b^3c^2C_{0,1}(2) + 5bc^4C_{0,1}(4) \\
&= b^5 + 10b^3c^2 + 10bc^4
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,1}(6;b,c) &= b^6C_{0,1}(0) + 15b^4c^2C_{0,1}(2) + 15b^2c^4C_{0,1}(4) + c^6C_{0,1}(6) \\
&= b^6 + 15b^4c^2 + 30b^2c^4 + 5c^6
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,1}(7;b,c) &= b^7C_{0,1}(0) + 21b^5c^2C_{0,1}(2) + 35b^3c^4C_{0,1}(4) + 7bc^6C_{0,1}(6) \\
&= b^7 + 21b^5c^2 + 70b^3c^4 + 35bc^6
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,1}(8;b,c) &= b^8C_{0,1}(0) + 28b^6c^2C_{0,1}(2) + 70b^4c^4C_{0,1}(4) + 28b^2c^6C_{0,1}(6) \\
&\quad + c^8C_{0,1}(8) \\
&= b^8 + 28b^6c^2 + 140b^4c^4 + 140b^2c^6 + 14c^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,1}(9;b,c) &= b^9C_{0,1}(0) + 36b^7c^2C_{0,1}(2) + 126b^5c^4C_{0,1}(4) + 84b^3c^6C_{0,1}(6) \\
&\quad + 9bc^8C_{0,1}(8) \\
&= b^9 + 36b^7c^2 + 252b^5c^4 + 420b^3c^6 + 126bc^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,1}(10;b,c) &= b^{10}C_{0,1}(0) + 45b^8c^2C_{0,1}(2) + 210b^6c^4C_{0,1}(4) + 210b^4c^6C_{0,1}(6) \\
&\quad + 45b^2c^8C_{0,1}(8) + c^{10}C_{0,1}(10) \\
&= b^{10} + 45b^8c^2 + 420b^6c^4 + 1050b^4c^6 + 630b^2c^8 + 42c^{10}
\end{aligned}
$$

Case $(g, v) = (0, 2)$**.**

For $n_1 + n_2 \leq 10$, we may compute:

$$\begin{aligned}\widetilde{M}_{0,2}(1,1;b,c) &= c^2 C_{0,2}(1,1)\\ &= c^2\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{0,2}(1,2;b,c) &= 2bc^2 C_{0,2}(1,1)\\ &= 2bc^2\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{0,2}(1,3;b,c) &= 3b^2c^2 C_{0,2}(1,1) + c^4 C_{0,2}(1,3)\\ &= 3b^2c^2 + 3c^4\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{0,2}(2,2;b,c) &= 4b^2c^2 C_{0,2}(1,1) + c^4 C_{0,2}(2,2)\\ &= 4b^2c^2 + 2c^4\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{0,2}(1,4;b,c) &= 4b^3c^2 C_{0,2}(1,1) + 4bc^4 C_{0,2}(1,3)\\ &= 4b^3c^2 + 12bc^4\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{0,2}(2,3;b,c) &= 6b^3c^2 C_{0,2}(1,1) + 2bc^4 C_{0,2}(1,3) + 3bc^4 C_{0,2}(2,2)\\ &= 6b^3c^2 + 12bc^4\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{0,2}(1,5;b,c) &= 5b^4c^2 C_{0,2}(1,1) + 10b^2c^4 C_{0,2}(1,3) + c^6 C_{0,2}(1,5)\\ &= 5b^4c^2 + 30b^2c^4 + 10c^6\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{0,2}(2,4;b,c) &= 8b^4c^2 C_{0,2}(1,1) + 8b^2c^4 C_{0,2}(1,3) + 6b^2c^4 C_{0,2}(2,2) + c^6 C_{0,2}(2,4)\\ &= 8b^4c^2 + 36b^2c^4 + 8c^6\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{0,2}(3,3;b,c) &= 9b^4c^2 C_{0,2}(1,1) + 6b^2c^4 C_{0,2}(1,3) + 9b^2c^4 C_{0,2}(2,2) + c^6 C_{0,2}(3,3)\\ &= 9b^4c^2 + 36b^2c^4 + 12c^6\end{aligned}$$

$$
\begin{aligned}
\widetilde{M}_{0,2}(1,6;b,c) &= 6b^5c^2C_{0,2}(1,1) + 20b^3c^4C_{0,2}(1,3) + 6bc^6C_{0,2}(1,5) \\
&= 6b^5c^2 + 60b^3c^4 + 60bc^6
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,2}(2,5;b,c) &= 10b^5c^2C_{0,2}(1,1) + 20b^3c^4C_{0,2}(1,3) + 2bc^6C_{0,2}(1,5) \\
&\quad + 10b^3c^4C_{0,2}(2,2) + 5bc^6C_{0,2}(2,4) \\
&= 10b^5c^2 + 80b^3c^4 + 60bc^6
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,2}(3,4;b,c) &= 12b^5c^2C_{0,2}(1,1) + 16b^3c^4C_{0,2}(1,3) + 18b^3c^4C_{0,2}(2,2) \\
&\quad + 3bc^6C_{0,2}(2,4) + 4bc^6C_{0,2}(3,3) \\
&= 12b^5c^2 + 84b^3c^4 + 72bc^6
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,2}(1,7;b,c) &= 7b^6c^2C_{0,2}(1,1) + 35b^4c^4C_{0,2}(1,3) + 21b^2c^6C_{0,2}(1,5) \\
&\quad + c^8C_{0,2}(1,7) \\
&= 7b^6c^2 + 105b^4c^4 + 210b^2c^6 + 35c^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,2}(2,6;b,c) &= 12b^6c^2C_{0,2}(1,1) + 40b^4c^4C_{0,2}(1,3) + 12b^2c^6C_{0,2}(1,5) \\
&\quad + 15b^4c^4C_{0,2}(2,2) + 15b^2c^6C_{0,2}(2,4) + c^8C_{0,2}(2,6) \\
&= 12b^6c^2 + 150b^4c^4 + 240b^2c^6 + 30c^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,2}(3,5;b,c) &= 15b^6c^2C_{0,2}(1,1) + 35b^4c^4C_{0,2}(1,3) + 3b^2c^6C_{0,2}(1,5) \\
&\quad + 30b^4c^4C_{0,2}(2,2) + 15b^2c^6C_{0,2}(2,4) + 10b^2c^6C_{0,2}(3,3) \\
&\quad + c^8C_{0,2}(3,5) \\
&= 15b^6c^2 + 165b^4c^4 + 270b^2c^6 + 45c^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,2}(4,4;b,c) &= 16b^6c^2C_{0,2}(1,1) + 32b^4c^4C_{0,2}(1,3) + 36b^4c^4C_{0,2}(2,2) \\
&\quad + 12b^2c^6C_{0,2}(2,4) + 16b^2c^6C_{0,2}(3,3) + c^8C_{0,2}(4,4) \\
&= 16b^6c^2 + 168b^4c^4 + 288b^2c^6 + 36c^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,2}(1,8;b,c) &= 8b^7c^2C_{0,2}(1,1) + 56b^5c^4C_{0,2}(1,3) + 56b^3c^6C_{0,2}(1,5) \\
&\quad + 8bc^8C_{0,2}(1,7) \\
&= 8b^7c^2 + 168b^5c^4 + 560b^3c^6 + 280bc^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,2}(2,7;b,c) &= 14b^7c^2C_{0,2}(1,1) + 70b^5c^4C_{0,2}(1,3) + 42b^3c^6C_{0,2}(1,5) \\
&\quad + 2bc^8C_{0,2}(1,7) + 21b^5c^4C_{0,2}(2,2) + 35b^3c^6C_{0,2}(2,4) \\
&\quad + 7bc^8C_{0,2}(2,6) \\
&= 14b^7c^2 + 252b^5c^4 + 700b^3c^6 + 280bc^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,2}(3,6;b,c) &= 18b^7c^2C_{0,2}(1,1) + 66b^5c^4C_{0,2}(1,3) + 18b^3c^6C_{0,2}(1,5) \\
&\quad + 45b^5c^4C_{0,2}(2,2) + 45b^3c^6C_{0,2}(2,4) + 3bc^8C_{0,2}(2,6) \\
&\quad + 20b^3c^6C_{0,2}(3,3) + 6bc^8C_{0,2}(3,5) \\
&= 18b^7c^2 + 288b^5c^4 + 780b^3c^6 + 360bc^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,2}(4,5;b,c) &= 20b^7c^2C_{0,2}(1,1) + 60b^5c^4C_{0,2}(1,3) + 4b^3c^6C_{0,2}(1,5) \\
&\quad + 60b^5c^4C_{0,2}(2,2) + 40b^3c^6C_{0,2}(2,4) + 40b^3c^6C_{0,2}(3,3) \\
&\quad + 4bc^8C_{0,2}(3,5) + 5bc^8C_{0,2}(4,4) \\
&= 20b^7c^2 + 300b^5c^4 + 840b^3c^6 + 360bc^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,2}(1,9;b,c) &= 9b^8c^2C_{0,2}(1,1) + 84b^6c^4C_{0,2}(1,3) + 126b^4c^6C_{0,2}(1,5) \\
&\quad + 36b^2c^8C_{0,2}(1,7) + c^{10}C_{0,2}(1,9) \\
&= 9b^8c^2 + 252b^6c^4 + 1260b^4c^6 + 1260b^2c^8 + 126c^{10}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,2}(2,8;b,c) &= 16b^8c^2C_{0,2}(1,1) + 112b^6c^4C_{0,2}(1,3) + 112b^4c^6C_{0,2}(1,5) \\
&\quad + 16b^2c^8C_{0,2}(1,7) + 28b^6c^4C_{0,2}(2,2) + 70b^4c^6C_{0,2}(2,4) \\
&\quad + 28b^2c^8C_{0,2}(2,6) + c^{10}C_{0,2}(2,8) \\
&= 16b^8c^2 + 392b^6c^4 + 1680b^4c^6 + 1400b^2c^8 + 112c^{10}
\end{aligned}
$$

$$\begin{aligned}\widetilde{M}_{0,2}(3,7;b,c) &= 21b^8c^2C_{0,2}(1,1) + 112b^6c^4C_{0,2}(1,3) + 63b^4c^6C_{0,2}(1,5)\\ &\quad + 3b^2c^8C_{0,2}(1,7) + 63b^6c^4C_{0,2}(2,2) + 105b^4c^6C_{0,2}(2,4)\\ &\quad + 21b^2c^8C_{0,2}(2,6) + 35b^4c^6C_{0,2}(3,3) + 21b^2c^8C_{0,2}(3,5)\\ &\quad + c^{10}C_{0,2}(3,7)\\ &= 21b^8c^2 + 462b^6c^4 + 1890b^4c^6 + 1680b^2c^8 + 168c^{10}\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{0,2}(4,6;b,c) &= 24b^8c^2C_{0,2}(1,1) + 104b^6c^4C_{0,2}(1,3) + 24b^4c^6C_{0,2}(1,5)\\ &\quad + 90b^6c^4C_{0,2}(2,2) + 105b^4c^6C_{0,2}(2,4) + 6b^2c^8C_{0,2}(2,6)\\ &\quad + 80b^4c^6C_{0,2}(3,3) + 24b^2c^8C_{0,2}(3,5) + 15b^2c^8C_{0,2}(4,4)\\ &\quad + c^{10}C_{0,2}(4,6)\\ &= 24b^8c^2 + 492b^6c^4 + 2040b^4c^6 + 1800b^2c^8 + 144c^{10}\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{0,2}(5,5;b,c) &= 25b^8c^2C_{0,2}(1,1) + 100b^6c^4C_{0,2}(1,3) + 10b^4c^6C_{0,2}(1,5)\\ &\quad + 100b^6c^4C_{0,2}(2,2) + 100b^4c^6C_{0,2}(2,4) + 100b^4c^6C_{0,2}(3,3)\\ &\quad + 20b^2c^8C_{0,2}(3,5) + 25b^2c^8C_{0,2}(4,4) + c^{10}C_{0,2}(5,5)\\ &= 25b^8c^2 + 500b^6c^4 + 2100b^4c^6 + 1800b^2c^8 + 180c^{10}\end{aligned}$$

Case $(g,v) = (1,1)$**.**

For $n \leq 10$, we may compute:

$$\widetilde{M}_{1,1}(1;b,c) = 0$$

$$\widetilde{M}_{1,1}(2;b,c) = 0$$

$$\begin{aligned}\widetilde{M}_{1,1}(3;b,c) &= 3bc^2C_{1,1}(2)\\ &= 0\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{1,1}(4;b,c) &= 6b^2c^2C_{1,1}(2) + c^4C_{1,1}(4)\\ &= c^4\end{aligned}$$

$$\widetilde{M}_{1,1}(5;b,c) = 10b^3c^2C_{1,1}(2) + 5bc^4C_{1,1}(4)$$
$$= 5bc^4$$

$$\widetilde{M}_{1,1}(6;b,c) = 15b^4c^2C_{1,1}(2) + 15b^2c^4C_{1,1}(4) + c^6C_{1,1}(6)$$
$$= 15b^2c^4 + 10c^6$$

$$\widetilde{M}_{1,1}(7;b,c) = 21b^5c^2C_{1,1}(2) + 35b^3c^4C_{1,1}(4) + 7bc^6C_{1,1}(6)$$
$$= 35b^3c^4 + 70bc^6$$

$$\widetilde{M}_{1,1}(8;b,c) = 28b^6c^2C_{1,1}(2) + 70b^4c^4C_{1,1}(4) + 28b^2c^6C_{1,1}(6) + c^8C_{1,1}(8)$$
$$= 70b^4c^4 + 280b^2c^6 + 70c^8$$

$$\widetilde{M}_{1,1}(9;b,c) = 36b^7c^2C_{1,1}(2) + 126b^5c^4C_{1,1}(4) + 84b^3c^6C_{1,1}(6) + 9bc^8C_{1,1}(8)$$
$$= 126b^5c^4 + 840b^3c^6 + 630bc^8$$

$$\begin{aligned}\widetilde{M}_{1,1}(10;b,c) &= 45b^8c^2C_{1,1}(2) + 210b^6c^4C_{1,1}(4) + 210b^4c^6C_{1,1}(6)\\ &\quad + 45b^2c^8C_{1,1}(8) + c^{10}C_{1,1}(10)\\ &= 210b^6c^4 + 2100b^4c^6 + 3150b^2c^8 + 420c^{10}\end{aligned}$$

Case $(g,v) = (0,3)$**.**

For $n_1 + n_2 + n_3 \leq 10$, we may compute:

$$\widetilde{M}_{0,3}(1,1,1;b,c) = 0$$

$$\widetilde{M}_{0,3}(1,1,2;b,c) = c^4C_{0,3}(1,1,2)$$
$$= 2c^4$$

$$\widetilde{M}_{0,3}(1,1,3;b,c) = 3bc^4C_{0,3}(1,1,2)$$
$$= 6bc^4$$

$$\widetilde{M}_{0,3}(1,2,2;b,c) = 4bc^4C_{0,3}(1,1,2)$$
$$= 8bc^4$$

$$
\begin{aligned}
\widetilde{M}_{0,3}(1,1,4;b,c) &= 6b^2c^4C_{0,3}(1,1,2) + c^6C_{0,3}(1,1,4)\\
&= 12b^2c^4 + 12c^6\\
\widetilde{M}_{0,3}(1,2,3;b,c) &= 9b^2c^4C_{0,3}(1,1,2) + c^6C_{0,3}(1,2,3)\\
&= 18b^2c^4 + 12c^6\\
\widetilde{M}_{0,3}(2,2,2;b,c) &= 12b^2c^4C_{0,3}(1,1,2) + c^6C_{0,3}(2,2,2)\\
&= 24b^2c^4 + 8c^6
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,3}(1,1,5;b,c) &= 10b^3c^4C_{0,3}(1,1,2) + 5bc^6C_{0,3}(1,1,4)\\
&= 20b^3c^4 + 60bc^6\\
\widetilde{M}_{0,3}(1,2,4;b,c) &= 16b^3c^4C_{0,3}(1,1,2) + 2bc^6C_{0,3}(1,1,4) + 4bc^6C_{0,3}(1,2,3)\\
&= 32b^3c^4 + 72bc^6\\
\widetilde{M}_{0,3}(1,3,3;b,c) &= 18b^3c^4C_{0,3}(1,1,2) + 6bc^6C_{0,3}(1,2,3)\\
&= 36b^3c^4 + 72bc^6\\
\widetilde{M}_{0,3}(2,2,3;b,c) &= 24b^3c^4C_{0,3}(1,1,2) + 4bc^6C_{0,3}(1,2,3) + 3bc^6C_{0,3}(2,2,2)\\
&= 48b^3c^4 + 72bc^6
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,3}(1,1,6;b,c) &= 15b^4c^4C_{0,3}(1,1,2) + 15b^2c^6C_{0,3}(1,1,4) + c^8C_{0,3}(1,1,6)\\
&= 30b^4c^4 + 180b^2c^6 + 60c^8\\
\widetilde{M}_{0,3}(1,2,5;b,c) &= 25b^4c^4C_{0,3}(1,1,2) + 10b^2c^6C_{0,3}(1,1,4) + 10b^2c^6C_{0,3}(1,2,3)\\
&\quad + c^8C_{0,3}(1,2,5)\\
&= 50b^4c^4 + 240b^2c^6 + 60c^8\\
\widetilde{M}_{0,3}(1,3,4;b,c) &= 30b^4c^4C_{0,3}(1,1,2) + 3b^2c^6C_{0,3}(1,1,4) + 18b^2c^6C_{0,3}(1,2,3)\\
&\quad + c^8C_{0,3}(1,3,4)\\
&= 60b^4c^4 + 252b^2c^6 + 72c^8
\end{aligned}
$$

$$\begin{aligned}\widetilde{M}_{0,3}(2,2,4;b,c) &= 40b^4c^4C_{0,3}(1,1,2) + 4b^2c^6C_{0,3}(1,1,4) + 16b^2c^6C_{0,3}(1,2,3)\\ &\quad + 6b^2c^6C_{0,3}(2,2,2) + c^8C_{0,3}(2,2,4)\\ &= 80b^4c^4 + 288b^2c^6 + 48c^8\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{0,3}(2,3,3;b,c) &= 45b^4c^4C_{0,3}(1,1,2) + 18b^2c^6C_{0,3}(1,2,3) + 9b^2c^6C_{0,3}(2,2,2)\\ &\quad + c^8C_{0,3}(2,3,3)\\ &= 90b^4c^4 + 288b^2c^6 + 72c^8\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{0,3}(1,1,7;b,c) &= 21b^5c^4C_{0,3}(1,1,2) + 35b^3c^6C_{0,3}(1,1,4) + 7bc^8C_{0,3}(1,1,6)\\ &= 42b^5c^4 + 420b^3c^6 + 420bc^8\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{0,3}(1,2,6;b,c) &= 36b^5c^4C_{0,3}(1,1,2) + 30b^3c^6C_{0,3}(1,1,4) + 2bc^8C_{0,3}(1,1,6)\\ &\quad + 20b^3c^6C_{0,3}(1,2,3) + 6bc^8C_{0,3}(1,2,5)\\ &= 72b^5c^4 + 600b^3c^6 + 480bc^8\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{0,3}(1,3,5;b,c) &= 45b^5c^4C_{0,3}(1,1,2) + 15b^3c^6C_{0,3}(1,1,4) + 40b^3c^6C_{0,3}(1,2,3)\\ &\quad + 3bc^8C_{0,3}(1,2,5) + 5bc^8C_{0,3}(1,3,4)\\ &= 90b^5c^4 + 660b^3c^6 + 540bc^8\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{0,3}(1,4,4;b,c) &= 48b^5c^4C_{0,3}(1,1,2) + 8b^3c^6C_{0,3}(1,1,4) + 48b^3c^6C_{0,3}(1,2,3)\\ &\quad + 8bc^8C_{0,3}(1,3,4)\\ &= 96b^5c^4 + 672b^3c^6 + 576bc^8\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{0,3}(2,2,5;b,c) &= 60b^5c^4C_{0,3}(1,1,2) + 20b^3c^6C_{0,3}(1,1,4) + 40b^3c^6C_{0,3}(1,2,3)\\ &\quad + 4bc^8C_{0,3}(1,2,5) + 10b^3c^6C_{0,3}(2,2,2) + 5bc^8C_{0,3}(2,2,4)\\ &= 120b^5c^4 + 800b^3c^6 + 480bc^8\end{aligned}$$

$$
\begin{aligned}
\widetilde{M}_{0,3}(2,3,4;b,c) &= 72b^5c^4C_{0,3}(1,1,2) + 6b^3c^6C_{0,3}(1,1,4) + 52b^3c^6C_{0,3}(1,2,3) \\
&\quad + 2bc^8C_{0,3}(1,3,4) + 18b^3c^6C_{0,3}(2,2,2) + 3bc^8C_{0,3}(2,2,4) \\
&\quad + 4bc^8C_{0,3}(2,3,3) \\
&= 144b^5c^4 + 840b^3c^6 + 576bc^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,3}(3,3,3;b,c) &= 81b^5c^4C_{0,3}(1,1,2) + 54b^3c^6C_{0,3}(1,2,3) + 27b^3c^6C_{0,3}(2,2,2) \\
&\quad + 9bc^8C_{0,3}(2,3,3) \\
&= 162b^5c^4 + 864b^3c^6 + 648bc^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,3}(1,1,8;b,c) &= 28b^6c^4C_{0,3}(1,1,2) + 70b^4c^6C_{0,3}(1,1,4) + 28b^2c^8C_{0,3}(1,1,6) \\
&\quad + c^{10}C_{0,3}(1,1,8) \\
&= 56b^6c^4 + 840b^4c^6 + 1680b^2c^8 + 280c^{10}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,3}(1,2,7;b,c) &= 49b^6c^4C_{0,3}(1,1,2) + 70b^4c^6C_{0,3}(1,1,4) + 14b^2c^8C_{0,3}(1,1,6) \\
&\quad + 35b^4c^6C_{0,3}(1,2,3) + 21b^2c^8C_{0,3}(1,2,5) + c^{10}C_{0,3}(1,2,7) \\
&= 98b^6c^4 + 1260b^4c^6 + 2100b^2c^8 + 280c^{10}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,3}(1,3,6;b,c) &= 63b^6c^4C_{0,3}(1,1,2) + 45b^4c^6C_{0,3}(1,1,4) + 3b^2c^8C_{0,3}(1,1,6) \\
&\quad + 75b^4c^6C_{0,3}(1,2,3) + 18b^2c^8C_{0,3}(1,2,5) \\
&\quad + 15b^2c^8C_{0,3}(1,3,4) + c^{10}C_{0,3}(1,3,6) \\
&= 126b^6c^4 + 1440b^4c^6 + 2340b^2c^8 + 360c^{10}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,3}(1,4,5;b,c) &= 70b^6c^4C_{0,3}(1,1,2) + 25b^4c^6C_{0,3}(1,1,4) + 100b^4c^6C_{0,3}(1,2,3) \\
&\quad + 6b^2c^8C_{0,3}(1,2,5) + 30b^2c^8C_{0,3}(1,3,4) + c^{10}C_{0,3}(1,4,5) \\
&= 140b^6c^4 + 1500b^4c^6 + 2520b^2c^8 + 360c^{10}
\end{aligned}
$$

$$\begin{aligned}\widetilde{M}_{0,3}(2,2,6;b,c) &= 84b^6c^4C_{0,3}(1,1,2) + 60b^4c^6C_{0,3}(1,1,4) + 4b^2c^8C_{0,3}(1,1,6)\\ &\quad + 80b^4c^6C_{0,3}(1,2,3) + 24b^2c^8C_{0,3}(1,2,5)\\ &\quad + 15b^4c^6C_{0,3}(2,2,2) + 15b^2c^8C_{0,3}(2,2,4) + c^{10}C_{0,3}(2,2,6)\\ &= 168b^6c^4 + 1800b^4c^6 + 2400b^2c^8 + 240c^{10}\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{0,3}(2,3,5;b,c) &= 105b^6c^4C_{0,3}(1,1,2) + 30b^4c^6C_{0,3}(1,1,4) + 115b^4c^6C_{0,3}(1,2,3)\\ &\quad + 9b^2c^8C_{0,3}(1,2,5) + 10b^2c^8C_{0,3}(1,3,4)\\ &\quad + 30b^4c^6C_{0,3}(2,2,2) + 15b^2c^8C_{0,3}(2,2,4)\\ &\quad + 10b^2c^8C_{0,3}(2,3,3) + c^{10}C_{0,3}(2,3,5)\\ &= 210b^6c^4 + 1980b^4c^6 + 2700b^2c^8 + 360c^{10}\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{0,3}(2,4,4;b,c) &= 112b^6c^4C_{0,3}(1,1,2) + 16b^4c^6C_{0,3}(1,1,4) + 128b^4c^6C_{0,3}(1,2,3)\\ &\quad + 16b^2c^8C_{0,3}(1,3,4) + 36b^4c^6C_{0,3}(2,2,2)\\ &\quad + 12b^2c^8C_{0,3}(2,2,4) + 16b^2c^8C_{0,3}(2,3,3) + c^{10}C_{0,3}(2,4,4)\\ &= 224b^6c^4 + 2016b^4c^6 + 2880b^2c^8 + 288c^{10}\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{0,3}(3,3,4;b,c) &= 126b^6c^4C_{0,3}(1,1,2) + 9b^4c^6C_{0,3}(1,1,4) + 132b^4c^6C_{0,3}(1,2,3)\\ &\quad + 6b^2c^8C_{0,3}(1,3,4) + 54b^4c^6C_{0,3}(2,2,2)\\ &\quad + 9b^2c^8C_{0,3}(2,2,4) + 30b^2c^8C_{0,3}(2,3,3) + c^{10}C_{0,3}(3,3,4)\\ &= 252b^6c^4 + 2124b^4c^6 + 3024b^2c^8 + 432c^{10}\end{aligned}$$

Case $(g,v) = (1,2)$**.**

For $n_1 + n_2 \le 10$, we may compute:

$$\begin{aligned}\widetilde{M}_{1,2}(1,1;b,c) &= c^2C_{1,2}(1,1)\\ &= 0\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{1,2}(1,2;b,c) &= 2bc^2C_{1,2}(1,1)\\ &= 0\end{aligned}$$

$$\begin{aligned}
\widetilde{M}_{1,2}(1,3;b,c) &= 3b^2c^2C_{1,2}(1,1) + c^4C_{1,2}(1,3) \\
&= 0 \\
\widetilde{M}_{1,2}(2,2;b,c) &= 4b^2c^2C_{1,2}(1,1) + c^4C_{1,2}(2,2) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\widetilde{M}_{1,2}(1,4;b,c) &= 4b^3c^2C_{1,2}(1,1) + 4bc^4C_{1,2}(1,3) \\
&= 0 \\
\widetilde{M}_{1,2}(2,3;b,c) &= 6b^3c^2C_{1,2}(1,1) + 2bc^4C_{1,2}(1,3) + 3bc^4C_{1,2}(2,2) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\widetilde{M}_{1,2}(1,5;b,c) &= 5b^4c^2C_{1,2}(1,1) + 10b^2c^4C_{1,2}(1,3) + c^6C_{1,2}(1,5) \\
&= 5c^6 \\
\widetilde{M}_{1,2}(2,4;b,c) &= 8b^4c^2C_{1,2}(1,1) + 8b^2c^4C_{1,2}(1,3) + 6b^2c^4C_{1,2}(2,2) \\
&\qquad + c^6C_{1,2}(2,4) \\
&= 4c^6 \\
\widetilde{M}_{1,2}(3,3;b,c) &= 9b^4c^2C_{1,2}(1,1) + 6b^2c^4C_{1,2}(1,3) + 9b^2c^4C_{1,2}(2,2) \\
&\qquad + c^6C_{1,2}(3,3) \\
&= 3c^6
\end{aligned}$$

$$\begin{aligned}
\widetilde{M}_{1,2}(1,6;b,c) &= 6b^5c^2C_{1,2}(1,1) + 20b^3c^4C_{1,2}(1,3) + 6bc^6C_{1,2}(1,5) \\
&= 30bc^6 \\
\widetilde{M}_{1,2}(2,5;b,c) &= 10b^5c^2C_{1,2}(1,1) + 20b^3c^4C_{1,2}(1,3) + 2bc^6C_{1,2}(1,5) \\
&\qquad + 10b^3c^4C_{1,2}(2,2) + 5bc^6C_{1,2}(2,4) \\
&= 30bc^6
\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{1,2}(3,4;b,c) &= 12b^5c^2C_{1,2}(1,1) + 16b^3c^4C_{1,2}(1,3) + 18b^3c^4C_{1,2}(2,2)\\ &\quad + 3bc^6C_{1,2}(2,4) + 4bc^6C_{1,2}(3,3)\\ &= 24bc^6\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{1,2}(1,7;b,c) &= 7b^6c^2C_{1,2}(1,1) + 35b^4c^4C_{1,2}(1,3) + 21b^2c^6C_{1,2}(1,5)\\ &\quad + c^8C_{1,2}(1,7)\\ &= 105b^2c^6 + 70c^8\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{1,2}(2,6;b,c) &= 12b^6c^2C_{1,2}(1,1) + 40b^4c^4C_{1,2}(1,3) + 12b^2c^6C_{1,2}(1,5)\\ &\quad + 15b^4c^4C_{1,2}(2,2) + 15b^2c^6C_{1,2}(2,4) + c^8C_{1,2}(2,6)\\ &= 120b^2c^6 + 60c^8\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{1,2}(3,5;b,c) &= 15b^6c^2C_{1,2}(1,1) + 35b^4c^4C_{1,2}(1,3) + 3b^2c^6C_{1,2}(1,5)\\ &\quad + 30b^4c^4C_{1,2}(2,2) + 15b^2c^6C_{1,2}(2,4) + 10b^2c^6C_{1,2}(3,3)\\ &\quad + c^8C_{1,2}(3,5)\\ &= 105b^2c^6 + 60c^8\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{1,2}(4,4;b,c) &= 16b^6c^2C_{1,2}(1,1) + 32b^4c^4C_{1,2}(1,3) + 36b^4c^4C_{1,2}(2,2)\\ &\quad + 12b^2c^6C_{1,2}(2,4) + 16b^2c^6C_{1,2}(3,3) + c^8C_{1,2}(4,4)\\ &= 96b^2c^6 + 60c^8\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{1,2}(1,8;b,c) &= 8b^7c^2C_{1,2}(1,1) + 56b^5c^4C_{1,2}(1,3) + 56b^3c^6C_{1,2}(1,5)\\ &\quad + 8bc^8C_{1,2}(1,7)\\ &= 280b^3c^6 + 560bc^8\end{aligned}$$

$$
\begin{aligned}
\widetilde{M}_{1,2}(2,7;b,c) &= 14b^7c^2C_{1,2}(1,1) + 70b^5c^4C_{1,2}(1,3) + 42b^3c^6C_{1,2}(1,5) \\
&\qquad + 2bc^8C_{1,2}(1,7) + 21b^5c^4C_{1,2}(2,2) + 35b^3c^6C_{1,2}(2,4) \\
&\qquad + 7bc^8C_{1,2}(2,6) \\
&= 350b^3c^6 + 560bc^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{1,2}(3,6;b,c) &= 18b^7c^2C_{1,2}(1,1) + 66b^5c^4C_{1,2}(1,3) + 18b^3c^6C_{1,2}(1,5) \\
&\qquad + 45b^5c^4C_{1,2}(2,2) + 45b^3c^6C_{1,2}(2,4) + 3bc^8C_{1,2}(2,6) \\
&\qquad + 20b^3c^6C_{1,2}(3,3) + 6bc^8C_{1,2}(3,5) \\
&= 330b^3c^6 + 540bc^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{1,2}(4,5;b,c) &= 20b^7c^2C_{1,2}(1,1) + 60b^5c^4C_{1,2}(1,3) + 4b^3c^6C_{1,2}(1,5) \\
&\qquad + 60b^5c^4C_{1,2}(2,2) + 40b^3c^6C_{1,2}(2,4) + 40b^3c^6C_{1,2}(3,3) \\
&\qquad + 4bc^8C_{1,2}(3,5) + 5bc^8C_{1,2}(4,4) \\
&= 300b^3c^6 + 540bc^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{1,2}(1,9;b,c) &= 9b^8c^2C_{1,2}(1,1) + 84b^6c^4C_{1,2}(1,3) + 126b^4c^6C_{1,2}(1,5) \\
&\qquad + 36b^2c^8C_{1,2}(1,7) + c^{10}C_{1,2}(1,9) \\
&= 630b^4c^6 + 2520b^2c^8 + 630c^{10}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{1,2}(2,8;b,c) &= 16b^8c^2C_{1,2}(1,1) + 112b^6c^4C_{1,2}(1,3) + 112b^4c^6C_{1,2}(1,5) \\
&\qquad + 16b^2c^8C_{1,2}(1,7) + 28b^6c^4C_{1,2}(2,2) + 70b^4c^6C_{1,2}(2,4) \\
&\qquad + 28b^2c^8C_{1,2}(2,6) + c^{10}C_{1,2}(2,8) \\
&= 840b^4c^6 + 2800b^2c^8 + 560c^{10}
\end{aligned}
$$

$$\begin{aligned}\widetilde{M}_{1,2}(3,7;b,c) &= 21b^8c^2C_{1,2}(1,1) + 112b^6c^4C_{1,2}(1,3) + 63b^4c^6C_{1,2}(1,5) \\ &\quad + 3b^2c^8C_{1,2}(1,7) + 63b^6c^4C_{1,2}(2,2) + 105b^4c^6C_{1,2}(2,4) \\ &\quad + 21b^2c^8C_{1,2}(2,6) + 35b^4c^6C_{1,2}(3,3) + 21b^2c^8C_{1,2}(3,5) \\ &\quad + c^{10}C_{1,2}(3,7) \\ &= 840b^4c^6 + 2730b^2c^8 + 630c^{10}\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{1,2}(4,6;b,c) &= 24b^8c^2C_{1,2}(1,1) + 104b^6c^4C_{1,2}(1,3) + 24b^4c^6C_{1,2}(1,5) \\ &\quad + 90b^6c^4C_{1,2}(2,2) + 105b^4c^6C_{1,2}(2,4) + 6b^2c^8C_{1,2}(2,6) \\ &\quad + 80b^4c^6C_{1,2}(3,3) + 24b^2c^8C_{1,2}(3,5) + 15b^2c^8C_{1,2}(4,4) \\ &\quad + c^{10}C_{1,2}(4,6) \\ &= 780b^4c^6 + 2700b^2c^8 + 600c^{10}\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{1,2}(5,5;b,c) &= 25b^8c^2C_{1,2}(1,1) + 100b^6c^4C_{1,2}(1,3) + 10b^4c^6C_{1,2}(1,5) \\ &\quad + 100b^6c^4C_{1,2}(2,2) + 100b^4c^6C_{1,2}(2,4) + 100b^4c^6C_{1,2}(3,3) \\ &\quad + 20b^2c^8C_{1,2}(3,5) + 25b^2c^8C_{1,2}(4,4) + c^{10}C_{1,2}(5,5) \\ &= 750b^4c^6 + 2700b^2c^8 + 600c^{10}\end{aligned}$$

Case $(g,v) = (2,1)$**.**

For $n \leq 10$, we may compute:

$$\widetilde{M}_{2,1}(1;b,c) = 0$$

$$\begin{aligned}\widetilde{M}_{2,1}(2;b,c) &= c^2C_{2,1}(2) \\ &= 0\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{2,1}(3;b,c) &= 3bc^2C_{2,1}(2) \\ &= 0\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{2,1}(4;b,c) &= 6b^2c^2C_{2,1}(2) + c^4C_{2,1}(4) \\ &= 0\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{2,1}(5;b,c) &= 10b^3c^2C_{2,1}(2) + 5bc^4C_{2,1}(4)\\ &= 0\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{2,1}(6;b,c) &= 15b^4c^2C_{2,1}(2) + 15b^2c^4C_{2,1}(4) + c^6C_{2,1}(6)\\ &= 0\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{2,1}(7;b,c) &= 21b^5c^2C_{2,1}(2) + 35b^3c^4C_{2,1}(4) + 7bc^6C_{2,1}(6)\\ &= 0\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{2,1}(8;b,c) &= 28b^6c^2C_{2,1}(2) + 70b^4c^4C_{2,1}(4) + 28b^2c^6C_{2,1}(6) + c^8C_{2,1}(8)\\ &= 21c^8\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{2,1}(9;b,c) &= 36b^7c^2C_{2,1}(2) + 126b^5c^4C_{2,1}(4) + 84b^3c^6C_{2,1}(6) + 9bc^8C_{2,1}(8)\\ &= 189bc^8\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{2,1}(10;b,c) &= 45b^8c^2C_{2,1}(2) + 210b^6c^4C_{2,1}(4) + 210b^4c^6C_{2,1}(6)\\ &\quad + 45b^2c^8C_{2,1}(8) + c^{10}C_{2,1}(10)\\ &= 945b^2c^8 + 483c^{10}\end{aligned}$$

Case $(g,v) = (0,4)$**.**

For $n_1 + n_2 + n_3 + n_4 \leq 10$, we may compute:

$$\begin{aligned}\widetilde{M}_{0,4}(1,1,1,1;b,c) &= c^4C_{0,4}(1,1,1,1)\\ &= 0\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{0,4}(1,1,1,2;b,c) &= 2bc^4C_{0,4}(1,1,1,1)\\ &= 0\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{0,4}(1,1,1,3;b,c) &= 3b^2c^4C_{0,4}(1,1,1,1) + c^6C_{0,4}(1,1,1,3)\\ &= 6c^6\end{aligned}$$

$$
\begin{aligned}
\widetilde{M}_{0,4}(1,1,2,2;b,c) &= 4b^2c^4C_{0,4}(1,1,1,1) + c^6C_{0,4}(1,1,2,2) \\
&= 8c^6
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,4}(1,1,1,4;b,c) &= 4b^3c^4C_{0,4}(1,1,1,1) + 4bc^6C_{0,4}(1,1,1,3) \\
&= 24bc^6
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,4}(1,1,2,3;b,c) &= 6b^3c^4C_{0,4}(1,1,1,1) + 2bc^6C_{0,4}(1,1,1,3) \\
&\quad + 3bc^6C_{0,4}(1,1,2,2) \\
&= 36bc^6
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,4}(1,1,1,5;b,c) &= 5b^4c^4C_{0,4}(1,1,1,1) + 10b^2c^6C_{0,4}(1,1,1,3) \\
&\quad + c^8C_{0,4}(1,1,1,5) \\
&= 60b^2c^6 + 60c^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,4}(1,1,2,4;b,c) &= 8b^4c^4C_{0,4}(1,1,1,1) + 8b^2c^6C_{0,4}(1,1,1,3) \\
&\quad + 6b^2c^6C_{0,4}(1,1,2,2) + c^8C_{0,4}(1,1,2,4) \\
&= 96b^2c^6 + 72c^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,4}(1,1,3,3;b,c) &= 9b^4c^4C_{0,4}(1,1,1,1) + 6b^2c^6C_{0,4}(1,1,1,3) \\
&\quad + 9b^2c^6C_{0,4}(1,1,2,2) + c^8C_{0,4}(1,1,3,3) \\
&= 108b^2c^6 + 72c^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,4}(1,2,2,3;b,c) &= 12b^4c^4C_{0,4}(1,1,1,1) + 4b^2c^6C_{0,4}(1,1,1,3) \\
&\quad + 15b^2c^6C_{0,4}(1,1,2,2) + c^8C_{0,4}(1,2,2,3) \\
&= 144b^2c^6 + 72c^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,4}(2,2,2,2;b,c) &= 16b^4c^4C_{0,4}(1,1,1,1) + 24b^2c^6C_{0,4}(1,1,2,2) \\
&\quad + c^8C_{0,4}(2,2,2,2) \\
&= 192b^2c^6 + 48c^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,4}(1,1,1,6;b,c) &= 6b^5c^4C_{0,4}(1,1,1,1) + 20b^3c^6C_{0,4}(1,1,1,3) \\
&\quad + 6bc^8C_{0,4}(1,1,1,5) \\
&= 120b^3c^6 + 360bc^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,4}(1,1,2,5;b,c) &= 10b^5c^4C_{0,4}(1,1,1,1) + 20b^3c^6C_{0,4}(1,1,1,3) \\
&\quad + 2bc^8C_{0,4}(1,1,1,5) + 10b^3c^6C_{0,4}(1,1,2,2) \\
&\quad + 5bc^8C_{0,4}(1,1,2,4) \\
&= 200b^3c^6 + 480bc^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,4}(1,1,3,4;b,c) &= 12b^5c^4C_{0,4}(1,1,1,1) + 16b^3c^6C_{0,4}(1,1,1,3) \\
&\quad + 18b^3c^6C_{0,4}(1,1,2,2) + 3bc^8C_{0,4}(1,1,2,4) \\
&\quad + 4bc^8C_{0,4}(1,1,3,3) \\
&= 240b^3c^6 + 504bc^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,4}(1,2,2,4;b,c) &= 16b^5c^4C_{0,4}(1,1,1,1) + 16b^3c^6C_{0,4}(1,1,1,3) \\
&\quad + 28b^3c^6C_{0,4}(1,1,2,2) + 4bc^8C_{0,4}(1,1,2,4) \\
&\quad + 4bc^8C_{0,4}(1,2,2,3) \\
&= 320b^3c^6 + 576bc^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,4}(2,2,2,3;b,c) &= 24b^5c^4C_{0,4}(1,1,1,1) + 8b^3c^6C_{0,4}(1,1,1,3) \\
&\quad + 54b^3c^6C_{0,4}(1,1,2,2) + 6bc^8C_{0,4}(1,2,2,3) \\
&\quad + 3bc^8C_{0,4}(2,2,2,2) \\
&= 480b^3c^6 + 576bc^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,4}(1,1,1,7;b,c) &= 7b^6c^4C_{0,4}(1,1,1,1) + 35b^4c^6C_{0,4}(1,1,1,3) \\
&\quad + 21b^2c^8C_{0,4}(1,1,1,5) + c^{10}C_{0,4}(1,1,1,7) \\
&= 210b^4c^6 + 1260b^2c^8 + 420c^{10}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,4}(1,1,2,6;b,c) &= 12b^6c^4C_{0,4}(1,1,1,1) + 40b^4c^6C_{0,4}(1,1,1,3) \\
&\quad + 12b^2c^8C_{0,4}(1,1,1,5) + 15b^4c^6C_{0,4}(1,1,2,2) \\
&\quad + 15b^2c^8C_{0,4}(1,1,2,4) + c^{10}C_{0,4}(1,1,2,6) \\
&= 360b^4c^6 + 1800b^2c^8 + 480c^{10}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,4}(1,1,3,5;b,c) &= 15b^6c^4C_{0,4}(1,1,1,1) + 35b^4c^6C_{0,4}(1,1,1,3) \\
&\quad + 3b^2c^8C_{0,4}(1,1,1,5) + 30b^4c^6C_{0,4}(1,1,2,2) \\
&\quad + 15b^2c^8C_{0,4}(1,1,2,4) + 10b^2c^8C_{0,4}(1,1,3,3) \\
&\quad + c^{10}C_{0,4}(1,1,3,5) \\
&= 450b^4c^6 + 1980b^2c^8 + 540c^{10}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,4}(1,1,4,4;b,c) &= 16b^6c^4C_{0,4}(1,1,1,1) + 32b^4c^6C_{0,4}(1,1,1,3) \\
&\quad + 36b^4c^6C_{0,4}(1,1,2,2) + 12b^2c^8C_{0,4}(1,1,2,4) \\
&\quad + 16b^2c^8C_{0,4}(1,1,3,3) + c^{10}C_{0,4}(1,1,4,4) \\
&= 480b^4c^6 + 2016b^2c^8 + 576c^{10}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,4}(1,2,2,5;b,c) &= 20b^6c^4C_{0,4}(1,1,1,1) + 40b^4c^6C_{0,4}(1,1,1,3) \\
&\quad + 4b^2c^8C_{0,4}(1,1,1,5) + 45b^4c^6C_{0,4}(1,1,2,2) \\
&\quad + 20b^2c^8C_{0,4}(1,1,2,4) + 10b^2c^8C_{0,4}(1,2,2,3) \\
&\quad + c^{10}C_{0,4}(1,2,2,5) \\
&= 600b^4c^6 + 2400b^2c^8 + 480c^{10}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{0,4}(1,2,3,4;b,c) &= 24b^6c^4C_{0,4}(1,1,1,1) + 32b^4c^6C_{0,4}(1,1,1,3) \\
&\quad + 66b^4c^6C_{0,4}(1,1,2,2) + 9b^2c^8C_{0,4}(1,1,2,4) \\
&\quad + 8b^2c^8C_{0,4}(1,1,3,3) + 18b^2c^8C_{0,4}(1,2,2,3) \\
&\quad + c^{10}C_{0,4}(1,2,3,4) \\
&= 720b^4c^6 + 2520b^2c^8 + 576c^{10}
\end{aligned}
$$

$$\begin{aligned}\widetilde{M}_{0,4}(1,3,3,3;b,c) &= 27b^6c^4C_{0,4}(1,1,1,1) + 27b^4c^6C_{0,4}(1,1,1,3) \\ &\quad + 81b^4c^6C_{0,4}(1,1,2,2) + 9b^2c^8C_{0,4}(1,1,3,3) \\ &\quad + 27b^2c^8C_{0,4}(1,2,2,3) + c^{10}C_{0,4}(1,3,3,3) \\ &= 810b^4c^6 + 2592b^2c^8 + 648c^{10}\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{0,4}(2,2,2,4;b,c) &= 32b^6c^4C_{0,4}(1,1,1,1) + 32b^4c^6C_{0,4}(1,1,1,3) \\ &\quad + 96b^4c^6C_{0,4}(1,1,2,2) + 12b^2c^8C_{0,4}(1,1,2,4) \\ &\quad + 24b^2c^8C_{0,4}(1,2,2,3) + 6b^2c^8C_{0,4}(2,2,2,2) \\ &\quad + c^{10}C_{0,4}(2,2,2,4) \\ &= 960b^4c^6 + 2880b^2c^8 + 384c^{10}\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{0,4}(2,2,3,3;b,c) &= 36b^6c^4C_{0,4}(1,1,1,1) + 24b^4c^6C_{0,4}(1,1,1,3) \\ &\quad + 117b^4c^6C_{0,4}(1,1,2,2) + 4b^2c^8C_{0,4}(1,1,3,3) \\ &\quad + 30b^2c^8C_{0,4}(1,2,2,3) + 9b^2c^8C_{0,4}(2,2,2,2) \\ &\quad + c^{10}C_{0,4}(2,2,3,3) \\ &= 1080b^4c^6 + 2880b^2c^8 + 576c^{10}\end{aligned}$$

Case $(g,v) = (1,3)$**.**

For $n_1 + n_2 + n_3 \leq 10$, we may compute:

$$\widetilde{M}_{1,3}(1,1,1;b,c) = 0$$

$$\begin{aligned}\widetilde{M}_{1,3}(1,1,2;b,c) &= c^4C_{1,3}(1,1,2) \\ &= 0\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{1,3}(1,1,3;b,c) &= 3bc^4C_{1,3}(1,1,2) \\ &= 0\end{aligned}$$

$$
\begin{aligned}
\widetilde{M}_{1,3}(1,2,2;b,c) &= 4bc^4C_{1,3}(1,1,2)\\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{1,3}(1,1,4;b,c) &= 6b^2c^4C_{1,3}(1,1,2) + c^6C_{1,3}(1,1,4)\\
&= 0\\
\widetilde{M}_{1,3}(1,2,3;b,c) &= 9b^2c^4C_{1,3}(1,1,2) + c^6C_{1,3}(1,2,3)\\
&= 0\\
\widetilde{M}_{1,3}(2,2,2;b,c) &= 12b^2c^4C_{1,3}(1,1,2) + c^6C_{1,3}(2,2,2)\\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{1,3}(1,1,5;b,c) &= 10b^3c^4C_{1,3}(1,1,2) + 5bc^6C_{1,3}(1,1,4)\\
&= 0\\
\widetilde{M}_{1,3}(1,2,4;b,c) &= 16b^3c^4C_{1,3}(1,1,2) + 2bc^6C_{1,3}(1,1,4) + 4bc^6C_{1,3}(1,2,3)\\
&= 0\\
\widetilde{M}_{1,3}(1,3,3;b,c) &= 18b^3c^4C_{1,3}(1,1,2) + 6bc^6C_{1,3}(1,2,3)\\
&= 0\\
\widetilde{M}_{1,3}(2,2,3;b,c) &= 24b^3c^4C_{1,3}(1,1,2) + 4bc^6C_{1,3}(1,2,3) + 3bc^6C_{1,3}(2,2,2)\\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{1,3}(1,1,6;b,c) &= 15b^4c^4C_{1,3}(1,1,2) + 15b^2c^6C_{1,3}(1,1,4) + c^8C_{1,3}(1,1,6)\\
&= 30c^8\\
\widetilde{M}_{1,3}(1,2,5;b,c) &= 25b^4c^4C_{1,3}(1,1,2) + 10b^2c^6C_{1,3}(1,1,4) + 10b^2c^6C_{1,3}(1,2,3)\\
&\quad + c^8C_{1,3}(1,2,5)\\
&= 30c^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{1,3}(1,3,4;b,c) &= 30b^4c^4C_{1,3}(1,1,2) + 3b^2c^6C_{1,3}(1,1,4) + 18b^2c^6C_{1,3}(1,2,3) \\
&\quad + c^8C_{1,3}(1,3,4) \\
&= 24c^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{1,3}(2,2,4;b,c) &= 40b^4c^4C_{1,3}(1,1,2) + 4b^2c^6C_{1,3}(1,1,4) + 16b^2c^6C_{1,3}(1,2,3) \\
&\quad + 6b^2c^6C_{1,3}(2,2,2) + c^8C_{1,3}(2,2,4) \\
&= 24c^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{1,3}(2,3,3;b,c) &= 45b^4c^4C_{1,3}(1,1,2) + 18b^2c^6C_{1,3}(1,2,3) + 9b^2c^6C_{1,3}(2,2,2) \\
&\quad + c^8C_{1,3}(2,3,3) \\
&= 18c^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{1,3}(1,1,7;b,c) &= 21b^5c^4C_{1,3}(1,1,2) + 35b^3c^6C_{1,3}(1,1,4) + 7bc^8C_{1,3}(1,1,6) \\
&= 210bc^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{1,3}(1,2,6;b,c) &= 36b^5c^4C_{1,3}(1,1,2) + 30b^3c^6C_{1,3}(1,1,4) + 2bc^8C_{1,3}(1,1,6) \\
&\quad + 20b^3c^6C_{1,3}(1,2,3) + 6bc^8C_{1,3}(1,2,5) \\
&= 240bc^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{1,3}(1,3,5;b,c) &= 45b^5c^4C_{1,3}(1,1,2) + 15b^3c^6C_{1,3}(1,1,4) + 40b^3c^6C_{1,3}(1,2,3) \\
&\quad + 3bc^8C_{1,3}(1,2,5) + 5bc^8C_{1,3}(1,3,4) \\
&= 210bc^8
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{1,3}(1,4,4;b,c) &= 48b^5c^4C_{1,3}(1,1,2) + 8b^3c^6C_{1,3}(1,1,4) + 48b^3c^6C_{1,3}(1,2,3) \\
&\quad + 8bc^8C_{1,3}(1,3,4) \\
&= 192bc^8
\end{aligned}
$$

$$\begin{aligned}
\widetilde{M}_{1,3}(2,2,5;b,c) &= 60b^5c^4C_{1,3}(1,1,2) + 20b^3c^6C_{1,3}(1,1,4) + 40b^3c^6C_{1,3}(1,2,3) \\
&\quad + 4bc^8C_{1,3}(1,2,5) + 10b^3c^6C_{1,3}(2,2,2) + 5bc^8C_{1,3}(2,2,4) \\
&= 240bc^8
\end{aligned}$$

$$\begin{aligned}
\widetilde{M}_{1,3}(2,3,4;b,c) &= 72b^5c^4C_{1,3}(1,1,2) + 6b^3c^6C_{1,3}(1,1,4) + 52b^3c^6C_{1,3}(1,2,3) \\
&\quad + 2bc^8C_{1,3}(1,3,4) + 18b^3c^6C_{1,3}(2,2,2) + 3bc^8C_{1,3}(2,2,4) \\
&\quad + 4bc^8C_{1,3}(2,3,3) \\
&= 192bc^8
\end{aligned}$$

$$\begin{aligned}
\widetilde{M}_{1,3}(3,3,3;b,c) &= 81b^5c^4C_{1,3}(1,1,2) + 54b^3c^6C_{1,3}(1,2,3) + 27b^3c^6C_{1,3}(2,2,2) \\
&\quad + 9bc^8C_{1,3}(2,3,3) \\
&= 162bc^8
\end{aligned}$$

$$\begin{aligned}
\widetilde{M}_{1,3}(1,1,8;b,c) &= 28b^6c^4C_{1,3}(1,1,2) + 70b^4c^6C_{1,3}(1,1,4) + 28b^2c^8C_{1,3}(1,1,6) \\
&\quad + c^{10}C_{1,3}(1,1,8) \\
&= 840b^2c^8 + 560c^{10}
\end{aligned}$$

$$\begin{aligned}
\widetilde{M}_{1,3}(1,2,7;b,c) &= 49b^6c^4C_{1,3}(1,1,2) + 70b^4c^6C_{1,3}(1,1,4) + 14b^2c^8C_{1,3}(1,1,6) \\
&\quad + 35b^4c^6C_{1,3}(1,2,3) + 21b^2c^8C_{1,3}(1,2,5) + c^{10}C_{1,3}(1,2,7) \\
&= 1050b^2c^8 + 560c^{10}
\end{aligned}$$

$$\begin{aligned}
\widetilde{M}_{1,3}(1,3,6;b,c) &= 63b^6c^4C_{1,3}(1,1,2) + 45b^4c^6C_{1,3}(1,1,4) + 3b^2c^8C_{1,3}(1,1,6) \\
&\quad + 75b^4c^6C_{1,3}(1,2,3) + 18b^2c^8C_{1,3}(1,2,5) + 15b^2c^8C_{1,3}(1,3,4) \\
&\quad + c^{10}C_{1,3}(1,3,6) \\
&= 990b^2c^8 + 540c^{10}
\end{aligned}$$

$$\begin{aligned}
\widetilde{M}_{1,3}(1,4,5;b,c) &= 70b^6c^4C_{1,3}(1,1,2) + 25b^4c^6C_{1,3}(1,1,4) + 100b^4c^6C_{1,3}(1,2,3)\\
&\qquad + 6b^2c^8C_{1,3}(1,2,5) + 30b^2c^8C_{1,3}(1,3,4) + c^{10}C_{1,3}(1,4,5)\\
&= 900b^2c^8 + 540c^{10}
\end{aligned}$$

$$\begin{aligned}
\widetilde{M}_{1,3}(2,2,6;b,c) &= 84b^6c^4C_{1,3}(1,1,2) + 60b^4c^6C_{1,3}(1,1,4) + 4b^2c^8C_{1,3}(1,1,6)\\
&\qquad + 80b^4c^6C_{1,3}(1,2,3) + 24b^2c^8C_{1,3}(1,2,5) + 15b^4c^6C_{1,3}(2,2,2)\\
&\qquad + 15b^2c^8C_{1,3}(2,2,4) + c^{10}C_{1,3}(2,2,6)\\
&= 1200b^2c^8 + 480c^{10}
\end{aligned}$$

$$\begin{aligned}
\widetilde{M}_{1,3}(2,3,5;b,c) &= 105b^6c^4C_{1,3}(1,1,2) + 30b^4c^6C_{1,3}(1,1,4) + 115b^4c^6C_{1,3}(1,2,3)\\
&\qquad + 9b^2c^8C_{1,3}(1,2,5) + 10b^2c^8C_{1,3}(1,3,4) + 30b^4c^6C_{1,3}(2,2,2)\\
&\qquad + 15b^2c^8C_{1,3}(2,2,4) + 10b^2c^8C_{1,3}(2,3,3) + c^{10}C_{1,3}(2,3,5)\\
&= 1050b^2c^8 + 480c^{10}
\end{aligned}$$

$$\begin{aligned}
\widetilde{M}_{1,3}(2,4,4;b,c) &= 112b^6c^4C_{1,3}(1,1,2) + 16b^4c^6C_{1,3}(1,1,4) + 128b^4c^6C_{1,3}(1,2,3)\\
&\qquad + 16b^2c^8C_{1,3}(1,3,4) + 36b^4c^6C_{1,3}(2,2,2) + 12b^2c^8C_{1,3}(2,2,4)\\
&\qquad + 16b^2c^8C_{1,3}(2,3,3) + c^{10}C_{1,3}(2,4,4)\\
&= 960b^2c^8 + 480c^{10}
\end{aligned}$$

$$\begin{aligned}
\widetilde{M}_{1,3}(3,3,4;b,c) &= 126b^6c^4C_{1,3}(1,1,2) + 9b^4c^6C_{1,3}(1,1,4) + 132b^4c^6C_{1,3}(1,2,3)\\
&\qquad + 6b^2c^8C_{1,3}(1,3,4) + 54b^4c^6C_{1,3}(2,2,2) + 9b^2c^8C_{1,3}(2,2,4)\\
&\qquad + 30b^2c^8C_{1,3}(2,3,3) + c^{10}C_{1,3}(3,3,4)\\
&= 900b^2c^8 + 468c^{10}
\end{aligned}$$

Case $(g,v) = (2,2)$**.**

For $n_1 + n_2 \leq 12$, we may compute:

$$\begin{aligned}
\widetilde{M}_{2,2}(1,1;b,c) &= c^2C_{2,2}(1,1)\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\widetilde{M}_{2,2}(1,2;b,c) &= 2bc^2C_{2,2}(1,1)\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\widetilde{M}_{2,2}(1,3;b,c) &= 3b^2c^2C_{2,2}(1,1) + c^4C_{2,2}(1,3)\\
&= 0\\
\widetilde{M}_{2,2}(2,2;b,c) &= 4b^2c^2C_{2,2}(1,1) + c^4C_{2,2}(2,2)\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\widetilde{M}_{2,2}(1,4;b,c) &= 4b^3c^2C_{2,2}(1,1) + 4bc^4C_{2,2}(1,3)\\
&= 0\\
\widetilde{M}_{2,2}(2,3;b,c) &= 6b^3c^2C_{2,2}(1,1) + 2bc^4C_{2,2}(1,3) + 3bc^4C_{2,2}(2,2)\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\widetilde{M}_{2,2}(1,5;b,c) &= 5b^4c^2C_{2,2}(1,1) + 10b^2c^4C_{2,2}(1,3) + c^6C_{2,2}(1,5)\\
&= 0\\
\widetilde{M}_{2,2}(2,4;b,c) &= 8b^4c^2C_{2,2}(1,1) + 8b^2c^4C_{2,2}(1,3) + 6b^2c^4C_{2,2}(2,2) + c^6C_{2,2}(2,4)\\
&= 0\\
\widetilde{M}_{2,2}(3,3;b,c) &= 9b^4c^2C_{2,2}(1,1) + 6b^2c^4C_{2,2}(1,3) + 9b^2c^4C_{2,2}(2,2) + c^6C_{2,2}(3,3)\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\widetilde{M}_{2,2}(1,6;b,c) &= 6b^5c^2C_{2,2}(1,1) + 20b^3c^4C_{2,2}(1,3) + 6bc^6C_{2,2}(1,5)\\
&= 0\\
\widetilde{M}_{2,2}(2,5;b,c) &= 10b^5c^2C_{2,2}(1,1) + 20b^3c^4C_{2,2}(1,3) + 2bc^6C_{2,2}(1,5)\\
&\quad + 10b^3c^4C_{2,2}(2,2) + 5bc^6C_{2,2}(2,4)\\
&= 0
\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{2,2}(3,4;b,c) &= 12b^5c^2C_{2,2}(1,1) + 16b^3c^4C_{2,2}(1,3) + 18b^3c^4C_{2,2}(2,2)\\ &\quad + 3bc^6C_{2,2}(2,4) + 4bc^6C_{2,2}(3,3)\\ &= 0\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{2,2}(1,7;b,c) &= 7b^6c^2C_{2,2}(1,1) + 35b^4c^4C_{2,2}(1,3) + 21b^2c^6C_{2,2}(1,5)\\ &\quad + c^8C_{2,2}(1,7)\\ &= 0\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{2,2}(2,6;b,c) &= 12b^6c^2C_{2,2}(1,1) + 40b^4c^4C_{2,2}(1,3) + 12b^2c^6C_{2,2}(1,5)\\ &\quad + 15b^4c^4C_{2,2}(2,2) + 15b^2c^6C_{2,2}(2,4) + c^8C_{2,2}(2,6)\\ &= 0\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{2,2}(3,5;b,c) &= 15b^6c^2C_{2,2}(1,1) + 35b^4c^4C_{2,2}(1,3) + 3b^2c^6C_{2,2}(1,5)\\ &\quad + 30b^4c^4C_{2,2}(2,2) + 15b^2c^6C_{2,2}(2,4) + 10b^2c^6C_{2,2}(3,3)\\ &\quad + c^8C_{2,2}(3,5)\\ &= 0\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{2,2}(4,4;b,c) &= 16b^6c^2C_{2,2}(1,1) + 32b^4c^4C_{2,2}(1,3) + 36b^4c^4C_{2,2}(2,2)\\ &\quad + 12b^2c^6C_{2,2}(2,4) + 16b^2c^6C_{2,2}(3,3) + c^8C_{2,2}(4,4)\\ &= 0\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{2,2}(1,8;b,c) &= 8b^7c^2C_{2,2}(1,1) + 56b^5c^4C_{2,2}(1,3) + 56b^3c^6C_{2,2}(1,5)\\ &\quad + 8bc^8C_{2,2}(1,7)\\ &= 0\end{aligned}$$

$$
\begin{aligned}
\widetilde{M}_{2,2}(2,7;b,c) &= 14b^7c^2C_{2,2}(1,1) + 70b^5c^4C_{2,2}(1,3) + 42b^3c^6C_{2,2}(1,5) \\
&\quad + 2bc^8C_{2,2}(1,7) + 21b^5c^4C_{2,2}(2,2) + 35b^3c^6C_{2,2}(2,4) \\
&\quad + 7bc^8C_{2,2}(2,6) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{2,2}(3,6;b,c) &= 18b^7c^2C_{2,2}(1,1) + 66b^5c^4C_{2,2}(1,3) + 18b^3c^6C_{2,2}(1,5) \\
&\quad + 45b^5c^4C_{2,2}(2,2) + 45b^3c^6C_{2,2}(2,4) + 3bc^8C_{2,2}(2,6) \\
&\quad + 20b^3c^6C_{2,2}(3,3) + 6bc^8C_{2,2}(3,5) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{2,2}(4,5;b,c) &= 20b^7c^2C_{2,2}(1,1) + 60b^5c^4C_{2,2}(1,3) + 4b^3c^6C_{2,2}(1,5) \\
&\quad + 60b^5c^4C_{2,2}(2,2) + 40b^3c^6C_{2,2}(2,4) + 40b^3c^6C_{2,2}(3,3) \\
&\quad + 4bc^8C_{2,2}(3,5) + 5bc^8C_{2,2}(4,4) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{2,2}(1,9;b,c) &= 9b^8c^2C_{2,2}(1,1) + 84b^6c^4C_{2,2}(1,3) + 126b^4c^6C_{2,2}(1,5) \\
&\quad + 36b^2c^8C_{2,2}(1,7) + c^{10}C_{2,2}(1,9) \\
&= 189c^{10}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{2,2}(2,8;b,c) &= 16b^8c^2C_{2,2}(1,1) + 112b^6c^4C_{2,2}(1,3) + 112b^4c^6C_{2,2}(1,5) \\
&\quad + 16b^2c^8C_{2,2}(1,7) + 28b^6c^4C_{2,2}(2,2) + 70b^4c^6C_{2,2}(2,4) \\
&\quad + 28b^2c^8C_{2,2}(2,6) + c^{10}C_{2,2}(2,8) \\
&= 168c^{10}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{2,2}(3,7;b,c) &= 21b^8c^2C_{2,2}(1,1) + 112b^6c^4C_{2,2}(1,3) + 63b^4c^6C_{2,2}(1,5) \\
&\quad + 3b^2c^8C_{2,2}(1,7) + 63b^6c^4C_{2,2}(2,2) + 105b^4c^6C_{2,2}(2,4) \\
&\quad + 21b^2c^8C_{2,2}(2,6) + 35b^4c^6C_{2,2}(3,3) + 21b^2c^8C_{2,2}(3,5) \\
&\quad + c^{10}C_{2,2}(3,7) \\
&= 147c^{10}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{2,2}(4,6;b,c) &= 24b^8c^2C_{2,2}(1,1) + 104b^6c^4C_{2,2}(1,3) + 24b^4c^6C_{2,2}(1,5) \\
&\quad + 90b^6c^4C_{2,2}(2,2) + 105b^4c^6C_{2,2}(2,4) + 6b^2c^8C_{2,2}(2,6) \\
&\quad + 80b^4c^6C_{2,2}(3,3) + 24b^2c^8C_{2,2}(3,5) + 15b^2c^8C_{2,2}(4,4) \\
&\quad + c^{10}C_{2,2}(4,6) \\
&= 156c^{10}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{2,2}(5,5;b,c) &= 25b^8c^2C_{2,2}(1,1) + 100b^6c^4C_{2,2}(1,3) + 10b^4c^6C_{2,2}(1,5) \\
&\quad + 100b^6c^4C_{2,2}(2,2) + 100b^4c^6C_{2,2}(2,4) + 100b^4c^6C_{2,2}(3,3) \\
&\quad + 20b^2c^8C_{2,2}(3,5) + 25b^2c^8C_{2,2}(4,4) + c^{10}C_{2,2}(5,5) \\
&= 165c^{10}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{2,2}(1,10;b,c) &= 10b^9c^2C_{2,2}(1,1) + 120b^7c^4C_{2,2}(1,3) + 252b^5c^6C_{2,2}(1,5) \\
&\quad + 120b^3c^8C_{2,2}(1,7) + 10bc^{10}C_{2,2}(1,9) \\
&= 1890bc^{10}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{2,2}(2,9;b,c) &= 18b^9c^2C_{2,2}(1,1) + 168b^7c^4C_{2,2}(1,3) + 252b^5c^6C_{2,2}(1,5) \\
&\quad + 72b^3c^8C_{2,2}(1,7) + 2bc^{10}C_{2,2}(1,9) + 36b^7c^4C_{2,2}(2,2) \\
&\quad + 126b^5c^6C_{2,2}(2,4) + 84b^3c^8C_{2,2}(2,6) + 9bc^{10}C_{2,2}(2,8) \\
&= 1890bc^{10}
\end{aligned}
$$

$$\begin{aligned}
\widetilde{M}_{2,2}(3,8;b,c) &= 24b^9c^2C_{2,2}(1,1) + 176b^7c^4C_{2,2}(1,3) + 168b^5c^6C_{2,2}(1,5)\\
&\quad + 24b^3c^8C_{2,2}(1,7) + 84b^7c^4C_{2,2}(2,2) + 210b^5c^6C_{2,2}(2,4)\\
&\quad + 84b^3c^8C_{2,2}(2,6) + 3bc^{10}C_{2,2}(2,8) + 56b^5c^6C_{2,2}(3,3)\\
&\quad + 56b^3c^8C_{2,2}(3,5) + 8bc^{10}C_{2,2}(3,7)\\
&= 1680bc^{10}\\
\widetilde{M}_{2,2}(4,7;b,c) &= 28b^9c^2C_{2,2}(1,1) + 168b^7c^4C_{2,2}(1,3) + 84b^5c^6C_{2,2}(1,5)\\
&\quad + 4b^3c^8C_{2,2}(1,7) + 126b^7c^4C_{2,2}(2,2) + 231b^5c^6C_{2,2}(2,4)\\
&\quad + 42b^3c^8C_{2,2}(2,6) + 140b^5c^6C_{2,2}(3,3) + 84b^3c^8C_{2,2}(3,5)\\
&\quad + 4bc^{10}C_{2,2}(3,7) + 35b^3c^8C_{2,2}(4,4) + 7bc^{10}C_{2,2}(4,6)\\
&= 1680bc^{10}\\
\widetilde{M}_{2,2}(5,6;b,c) &= 30b^9c^2C_{2,2}(1,1) + 160b^7c^4C_{2,2}(1,3) + 36b^5c^6C_{2,2}(1,5)\\
&\quad + 150b^7c^4C_{2,2}(2,2) + 225b^5c^6C_{2,2}(2,4) + 10b^3c^8C_{2,2}(2,6)\\
&\quad + 200b^5c^6C_{2,2}(3,3) + 80b^3c^8C_{2,2}(3,5) + 75b^3c^8C_{2,2}(4,4)\\
&\quad + 5bc^{10}C_{2,2}(4,6) + 6bc^{10}C_{2,2}(5,5)\\
&= 1770bc^{10}
\end{aligned}$$

$$\begin{aligned}
\widetilde{M}_{2,2}(1,11;b,c) &= 11b^{10}c^2C_{2,2}(1,1) + 165b^8c^4C_{2,2}(1,3) + 462b^6c^6C_{2,2}(1,5)\\
&\quad + 330b^4c^8C_{2,2}(1,7) + 55b^2c^{10}C_{2,2}(1,9) + c^{12}C_{2,2}(1,11)\\
&= 10395b^2c^{10} + 5313c^{12}\\
\widetilde{M}_{2,2}(2,10;b,c) &= 20b^{10}c^2C_{2,2}(1,1) + 240b^8c^4C_{2,2}(1,3) + 504b^6c^6C_{2,2}(1,5)\\
&\quad + 240b^4c^8C_{2,2}(1,7) + 20b^2c^{10}C_{2,2}(1,9) + 45b^8c^4C_{2,2}(2,2)\\
&\quad + 210b^6c^6C_{2,2}(2,4) + 210b^4c^8C_{2,2}(2,6) + 45b^2c^{10}C_{2,2}(2,8)\\
&\quad + c^{12}C_{2,2}(2,10)\\
&= 11340b^2c^{10} + 4830c^{12}
\end{aligned}$$

$$
\begin{aligned}
\widetilde{M}_{2,2}(3,9;b,c) &= 27b^{10}c^2C_{2,2}(1,1) + 261b^8c^4C_{2,2}(1,3) + 378b^6c^6C_{2,2}(1,5) \\
&\quad + 108b^4c^8C_{2,2}(1,7) + 3b^2c^{10}C_{2,2}(1,9) + 108b^8c^4C_{2,2}(2,2) \\
&\quad + 378b^6c^6C_{2,2}(2,4) + 252b^4c^8C_{2,2}(2,6) + 27b^2c^{10}C_{2,2}(2,8) \\
&\quad + 84b^6c^6C_{2,2}(3,3) + 126b^4c^8C_{2,2}(3,5) + 36b^2c^{10}C_{2,2}(3,7) \\
&\quad + c^{12}C_{2,2}(3,9) \\
&= 10395b^2c^{10} + 4725c^{12}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{2,2}(4,8;b,c) &= 32b^{10}c^2C_{2,2}(1,1) + 256b^8c^4C_{2,2}(1,3) + 224b^6c^6C_{2,2}(1,5) \\
&\quad + 32b^4c^8C_{2,2}(1,7) + 168b^8c^4C_{2,2}(2,2) + 448b^6c^6C_{2,2}(2,4) \\
&\quad + 168b^4c^8C_{2,2}(2,6) + 6b^2c^{10}C_{2,2}(2,8) + 224b^6c^6C_{2,2}(3,3) \\
&\quad + 224b^4c^8C_{2,2}(3,5) + 32b^2c^{10}C_{2,2}(3,7) + 70b^4c^8C_{2,2}(4,4) \\
&\quad + 28b^2c^{10}C_{2,2}(4,6) + c^{12}C_{2,2}(4,8) \\
&= 10080b^2c^{10} + 4760c^{12}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{2,2}(5,7;b,c) &= 35b^{10}c^2C_{2,2}(1,1) + 245b^8c^4C_{2,2}(1,3) + 112b^6c^6C_{2,2}(1,5) \\
&\quad + 5b^4c^8C_{2,2}(1,7) + 210b^8c^4C_{2,2}(2,2) + 455b^6c^6C_{2,2}(2,4) \\
&\quad + 70b^4c^8C_{2,2}(2,6) + 350b^6c^6C_{2,2}(3,3) + 245b^4c^8C_{2,2}(3,5) \\
&\quad + 10b^2c^{10}C_{2,2}(3,7) + 175b^4c^8C_{2,2}(4,4) + 35b^2c^{10}C_{2,2}(4,6) \\
&\quad + 21b^2c^{10}C_{2,2}(5,5) + c^{12}C_{2,2}(5,7) \\
&= 10395b^2c^{10} + 4795c^{12}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{M}_{2,2}(6,6;b,c) &= 36b^{10}c^2C_{2,2}(1,1) + 240b^8c^4C_{2,2}(1,3) + 72b^6c^6C_{2,2}(1,5) \\
&\quad + 225b^8c^4C_{2,2}(2,2) + 450b^6c^6C_{2,2}(2,4) + 30b^4c^8C_{2,2}(2,6) \\
&\quad + 400b^6c^6C_{2,2}(3,3) + 240b^4c^8C_{2,2}(3,5) + 225b^4c^8C_{2,2}(4,4) \\
&\quad + 30b^2c^{10}C_{2,2}(4,6) + 36b^2c^{10}C_{2,2}(5,5) + c^{12}C_{2,2}(6,6) \\
&= 10620b^2c^{10} + 4770c^{12}
\end{aligned}
$$

Case $(g, v) = (3, 1)$**.**

For $n \leq 14$, we may compute:

$$\widetilde{M}_{3,1}(1; b, c) = 0$$

$$\begin{aligned}\widetilde{M}_{3,1}(2; b, c) &= c^2 C_{3,1}(2) \\ &= 0\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{3,1}(3; b, c) &= 3bc^2 C_{3,1}(2) \\ &= 0\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{3,1}(4; b, c) &= 6b^2c^2 C_{3,1}(2) + c^4 C_{3,1}(4) \\ &= 0\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{3,1}(5; b, c) &= 10b^3c^2 C_{3,1}(2) + 5bc^4 C_{3,1}(4) \\ &= 0\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{3,1}(6; b, c) &= 15b^4c^2 C_{3,1}(2) + 15b^2c^4 C_{3,1}(4) + c^6 C_{3,1}(6) \\ &= 0\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{3,1}(7; b, c) &= 21b^5c^2 C_{3,1}(2) + 35b^3c^4 C_{3,1}(4) + 7bc^6 C_{3,1}(6) \\ &= 0\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{3,1}(8; b, c) &= 28b^6c^2 C_{3,1}(2) + 70b^4c^4 C_{3,1}(4) + 28b^2c^6 C_{3,1}(6) \\ &\quad + c^8 C_{3,1}(8) \\ &= 0\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{3,1}(9; b, c) &= 36b^7c^2 C_{3,1}(2) + 126b^5c^4 C_{3,1}(4) + 84b^3c^6 C_{3,1}(6) \\ &\quad + 9bc^8 C_{3,1}(8) \\ &= 0\end{aligned}$$

$$\begin{aligned}
\widetilde{M}_{3,1}(10;b,c) &= 45b^8c^2C_{3,1}(2) + 210b^6c^4C_{3,1}(4) + 210b^4c^6C_{3,1}(6) \\
&\quad + 45b^2c^8C_{3,1}(8) + c^{10}C_{3,1}(10) \\
&= 0 \\
\widetilde{M}_{3,1}(11;b,c) &= 55b^9c^2C_{3,1}(2) + 330b^7c^4C_{3,1}(4) + 462b^5c^6C_{3,1}(6) \\
&\quad + 165b^3c^8C_{3,1}(8) + 11bc^{10}C_{3,1}(10) \\
&= 0 \\
\widetilde{M}_{3,1}(12;b,c) &= 66b^{10}c^2C_{3,1}(2) + 495b^8c^4C_{3,1}(4) + 924b^6c^6C_{3,1}(6) \\
&\quad + 495b^4c^8C_{3,1}(8) + 66b^2c^{10}C_{3,1}(10) + c^{12}C_{3,1}(12) \\
&= 1485c^{12} \\
\widetilde{M}_{3,1}(13;b,c) &= 78b^{11}c^2C_{3,1}(2) + 715b^9c^4C_{3,1}(4) + 1716b^7c^6C_{3,1}(6) \\
&\quad + 1287b^5c^8C_{3,1}(8) + 286b^3c^{10}C_{3,1}(10) + 13bc^{12}C_{3,1}(12) \\
&= 19305bc^{12} \\
\widetilde{M}_{3,1}(14;b,c) &= 91b^{12}c^2C_{3,1}(2) + 1001b^{10}c^4C_{3,1}(4) + 3003b^8c^6C_{3,1}(6) \\
&\quad + 3003b^6c^8C_{3,1}(8) + 1001b^4c^{10}C_{3,1}(10) + 91b^2c^{12}C_{3,1}(12) \\
&\quad + c^{14}C_{3,1}(14) \\
&= 135135b^2c^{12} + 56628c^{14}
\end{aligned}$$

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