Lab #10

1 What to submit ?

There are two parts for this lab toiy need to save both parts together as LAB10.text
1. While reading about complex numbers each section has activities indicated by type . Please follow the instructions on these parts.
2. At the end of the lab there is a summery and an Exercises section. Please review summery and do the Excercises.
Submit your diary using "submitm22al"

Start MATLAB, ...
Type: diary LAB10.text
Type: % your last name
Type: % your First name
Type: % your login name (username)
Type: % date

2 Introduction

In introductory Linear Algebra we work with real numbers only. If we replace the Real Numbers with Complex numbers almost everything will work the same. We can have a linear system with complex coefficients, matrices with complex entries, and complex vector spaces. We can find Determinant, Eigenvalues and Eigenvectors of a complex matrix.

Matlab does arithmetic of complex numbers automatically.

Objective of this section is to introduce Complex numbers in Linear Algebra. We will do this by introducing Complex numbers and its arithmetics, then few examples of using complex numbers in Linear Algebra, will follow by few new definitions related to complex matrices.

3 Basics of complex numbers

3.1 Introduction

Consider the following questions:
1. Find all numbers x, such that $x^2 - 4 = 0$.
Type: P1= [1 0 -4]
Type: S1 = roots(P1)

2. Find all numbers x, such that $x^2 + 1 = 0$.
Type: P1= [1 0 1]
Type: S1 = roots(P1)

Real numbers 2, and -2 will satisfy the first equation, but there is no real number such that $x^2 + 1 = 0$. We need a new number (it is called imaginary number) say i whose square is -1. So, we write $i = \sqrt{-1}$.

3.1.1 History:

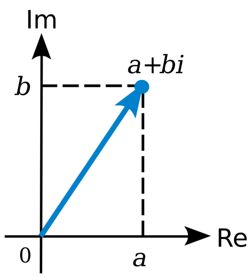
The Italian mathematician Gerolamo Cardano (1501-1576) encountered complex numbers during his attempts to find solutions to cubic equations. Cardano called them "fictitious" numbers. You may find more info about history of development of complex numbers. A short history of Complex numbers can be found at " http://www.math.uri.edu/~merino/spring06/mth562/ShortHistoryComplexNumbers2006.pdf"
We want to be able to add this number $i = \sqrt{-1}$ to an other real number say 3, the result will be shown as $3 + i$. Or we might want to multiply i with 5 then add the result to 3 to get $3 + 5i$.
In general every number of the form $a + ib$ is called a complex number. Real numbers can be put in a one-to-one correspondence with the points on the x-axis. We can think of the correspondence of the complex numbers with the points on the xy-plane.

3.2 Real and Imaginary part of a Complex Number

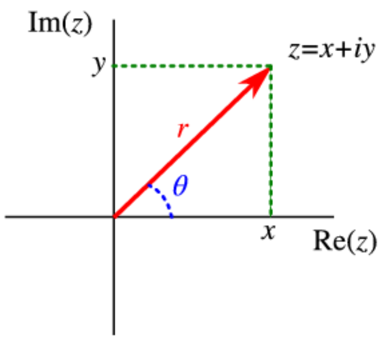
A complex number c can be written as $c = a + bi$ where a and b are real numbers.
a is called the real part of c and b is called the imaginary part of c.

3.3 Complex numbers as vectors Geometric Representation

A complex number $c = a + ib$ corresponds to a point in the xy-plane. If we call the $x-$axis as **Real axis** and the $y-$ axis as imaginary axis the new plane will be called **complex plane.**
Every complex number $c = a + ib$ corresponds to a point (a, b) in the Complex plane.



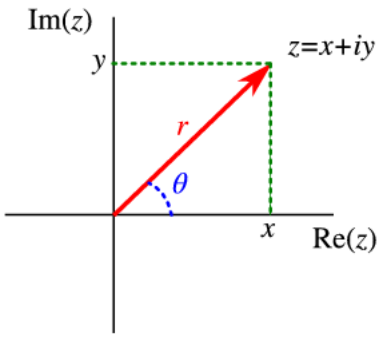
Also every complex number $c = a + ib$ can be viewed as a vector.



If $\mathbf{v} = (\mathbf{a}, \mathbf{b})$ represents complex number $c = a + ib$, then length of this vector is called **absolute value** or **modulus** of the complex number $c = a + ib$.
Type: c1 = 3+ 4i
Type: c2 = -5 +7i
Type: r1 = abs(c1)
Type: a = angle(c1)
Type: z= r1.*exp(i*a)

3.3.1 Polar Representation of Complex numbers

A complex number in rectangular form $c = a + ib = xi + iy$ can also be represent in polar form or trigonometric form.



Simple connection between Polar and cartesian or rectangular coordinate form of representation of a point. Points in rectangular form are represented as $(a, b) = (x, y)$ and in polar coordinates as (r, θ), where $x = a = r\cos\theta$ and $y = b = r\sin\theta$. So, $c = a + ib$ can be represent as

$$c = r\cos\theta + ri\sin\theta = r(\cos\theta + i\sin\theta) = rcis(\theta)$$

The angle θ is called the argument of c denoted by $arg(c) = \theta$ and
r is called the **modulus** or **absolute value** or **magnitude** of c.
Polar coordinates representation is very useful when we multiply or divide complex numbers.
Entering complex number in MATLAB:
Type: c3 = -1+ 4i
Type: c4 = -2 -3i
Type: c5 = 5
Type: c6 = 2i
Finding magnitude and argument of a given complex number.

Type: a3 = angle(c3)
Type: a4 = angle(c4)
Type: a5 = angle(c5)
Type: a6 = angle(c6)

Type: r3 = abs(c3)
Type: r4 = abs(c4)
Type: r5 = abs(c5)
Type: r6 = abs(c6)

Given magnitude and argument of a complex number, one can construct the rectangular form of it using the following:
Type: z= r3.*exp(i*a3)
Type: z= r4.*exp(i*a4)

Exercise 1: Write the given complex number in polar form as $c = r(\cos\theta + i\sin\theta) = rcis(\theta)$:
a.) $c10 = -1 - i$
b.) $c11 = 7 + 2i$
c.) $c12 = 3 - i$.

Exercise 2: Write the given complex number in rectangular coordinates (recall that the angels are in radians) :
a.) $c13 = 3cis(\frac{\pi}{4}) = 3(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})$
b.) $c14 = 6cis(\pi)$
c.) $c15 = 3cis(4.2)$

3.4 Sum, Difference and Product of Complex Numbers

If $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$ then the **sum** of c_1 and c_2 is

$$\boxed{c_1 + c_2 = (a_1 + a_2) + i(b_1 + b_2)}$$

and their **difference** is

$$c_1 - c_2 = (a_1 - a_2) + i(b_1 - b_2)$$

.
The **product** of $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$ in Rectangular coordinates is given by

$$\boxed{c_1c_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)}$$

The **product** of $c_1 = r_1(\cos\theta_1 + i\sin\theta)$ and $c_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ in Polar coordinates is given by

$$\boxed{c_1c_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]}$$

Type: $c10 = -1 - i$
Type: $c11 = 7 + 2i$
Type: $c12 = 3 - i$

Type: $z_1 = c10 + c11$
Type: $z_2 = c10 - c11$
Type: $z_3 = c10 * c11$

Type: imag(c11)
Type: real(c11)

Exercise 3: Let $c16 = 3cis(0.7)$ and $c17 = 2cis(1.2)$ (recall that the angels are in radians)
:
a.) Find a complex number $z16 = rcis(\theta)$ such that $(z16)(c16) = 1cis(0)$
b.) Find a complex number $z17 = rcis(\theta)$ such that $(z17)(c17) = 1cis(0)$
c.) Find a complex number $z18 = rcis(\theta)$ such that $(z18)(rcis(\theta) = 1cis(0)$

3.5 Conjugate of complex number and Division of Complex Numbers

3.5.1 Conjugate of Complex Number

Conjugate of $c = a + ib$ is defined to be the complex number $\overline{c} = a - ib$.

Basic properties.
1. $\overline{\overline{c}} = c$
2. $\overline{c + d} = \overline{c} + \overline{d}$
3. $\overline{cd} = \overline{c}\overline{d}$
4. $\overline{c} = c$ if and only if c is a real number.
5. $c\overline{c} = a^2 + b^2$ is a nonnegative real number and $c\overline{c} = 0$ if and only if $c = 0$.

Type: $z21 = 3 - 5i$
Type: $z22 = 2 + 9i$
Type: $z23 = 4 - 7i$
Type: $z24 = conj(z21)$
Type: $z25 = z21 + z22$
Type: $z26 = conj(z25)$
Type: $z27 = z21 * z22$
Type: $z28 = conj(z21) * conj(z22)$
Type: $z29 = conj(z27)$
Type: $z30 = conj(z28)$

3.5.2 Division of Complex Numbers

We can divide a complex number by another complex number .

$$\boxed{\frac{c_1}{c_2} = \frac{a+ib}{a'+ib'} = \frac{c_1\bar{c}_2}{c_2\bar{c}_2} = (\frac{aa'+bb'}{a'^2+b'^2}) - (\frac{ab'+a'b}{a'^2+b'^2})i}$$

Type: $z21 = 3 - 5i$
Type: $z22 = 2 + 9i$
Type: $z23 = 4 - 7i$
Type: $z24 = 1/z21$
Type: $z25 = 1/z22$
Type: $z26 = 1/z23$
Type: $z27 = z21/z22$
Type: $z28 = z22 * conj(z22)$
Type: $z29 = z21 * conj(z22)$
Type: $z30 = z21 * conj(z22)/z22 * conj(z22)$

4 Matrices with Complex Entries:

By a **complex matrix** we mean a matrix whose entries are complex numbers. Matrix operations such as addition, subtraction, multiplication and scalar multiplication is the same for complex matrices as they defined for matrices with real entries. We define few new types of matrices and conjugate of a complex matrix as follow:

4.1 Conjugate of a matrix

Conjugate of a complex matrix A is denoted by $\overline{A}$ and obtained by replacing each entry of the matrix A by its conjugate. That is $\overline{A} = (\overline{a_{ij}})$

Example:

$$A = \begin{bmatrix} 2-3i & -4+7i \\ 9i & -12 \end{bmatrix}$$

$$\overline{A} = \begin{bmatrix} \overline{2-3i} & \overline{-4+7i} \\ \overline{9i} & \overline{-12} \end{bmatrix} = \begin{bmatrix} 2+3i & -4-7i \\ -9i & -12 \end{bmatrix}$$

4.2 Properties of Conjugate of a matrix

- $\overline{\overline{A}} = A$
- $\overline{A+B} = \overline{A} + \overline{B}$
- $\overline{AB} = \overline{AB}$
- $\overline{A^T} = \overline{A}^T$
- If k is a real number, then $\overline{kA} = k\overline{A}$
- If c is a complex number, then $\overline{cA} = \overline{c}\,\overline{A}$
- If A is invertible, then $\overline{A^{-1}} = \left(\overline{A}\right)^{-1}$

Type: $A = [1+i \;\; 2; i \;\; 2+i]$
Type: $conj(A)$
Type: $A * conj(A)$
Type: $inv(A)$
Type: $det(A)$
Type: $[V \;\; W] = eig(A)$

4.3 Hermitian matrix

A square matrix A is called **Hermitian** if

$$\overline{A^T} = A$$

Example:

The following matrix is hermitian because $\overline{A^T} = A$

Type the following matrix in MATLAB

$$A = \begin{bmatrix} 2 & -4+7i \\ -4-7i & 3 \end{bmatrix}$$

Type: A'

You should see :

$$\overline{A^T} = \overline{\begin{bmatrix} 2 & -4-7i \\ -4+7i & 3 \end{bmatrix}} = \begin{bmatrix} 2 & -4+7i \\ -4-7i & 3 \end{bmatrix} = A$$

It can be proved that the eigenvalues of a hermitian matrix are real numbers.
Type: [W V] =eig(A)

Since a hermitian matrix with real entries is symmetric, we can conclude that the eigenvalues of a symmetric matrix with real enteries are real numbers.

In MATLAB the command htranspose(A)
returns the Hermitian transpose A^H of the matrix A (the complex conjugate of the transpose of A).
Type: [W V] =eig(A)

4.4 Unitary matrix

An square matrix A is called **Unitary** if

$$\left(\overline{A^T}\right)A = A\left(\overline{A^T}\right) = I_n$$

That is A is Unitary if and only if $A^{-1} = \left(\overline{A^T}\right)$

If A is a unitary matrix, then the columns of A form an orthonormal set in $\mathbf{C}^n$ with Euclidean inner product.

Complex Euclidean inner product of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \cdots \mathbf{v} = u_1\overline{v_1} + u_2\overline{v_2} + \cdots + u_n\overline{v_n}$$

If A is a unitary matrix, then the rows of A form an orthonormal set in $\mathbf{C}^n$ with Euclidean inner product.

Recall that a if A is unitary and real matrix then it is called **orthogonal** .

$$A = \begin{bmatrix} \frac{1}{\sqrt{3}} - \frac{i}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} + \frac{i}{\sqrt{3}} \end{bmatrix}$$

Type: A*(conj(A)')

4.5 Normal matrix

An square matrix A is called **Normal** if

$$(\overline{A^T})A = A(\overline{A^T})$$

Note that every Hermitian matrix A is normal.

Theorem:

If A is square matrix with complex entries, then the following are equivalent:

a) A is unitarily diagonalizable. That is there is a Unitary matrix P such that $P^{-1}AP$ is diagonal.

b) A has an orthonormal set of n eigenvectors.

c) A is normal.

Recall that if A is unitary and real matrix then it is called **orthogonal** .

Enter the following matrix in MATLAB:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Show that this matrix is not unitary.
Type: conj(A)
Show that this matrix is not hermitian.
Type: (conj(A)')*A
Type: A*(conj(A)')
Show that A is normal
Type: A*(conj(A)')
Type: (conj(A)') *A

5 Summery: MATLAB and Complex Numbers:

To start you may type : Format rat

i =sqrt(-1)

To enter a complex number type : **c1= 4+5i**

To find the conjugate of c1 type : **conj(c1)**

You may ask for the real part and Imaginary part of a complex number by typing
real(c1) **imag(c1)**

To enter a complex matrix, enter it in the same way that you enter a real matrix:

A= [2+i 3-5i -2; -7i 6+7i 55-12i]

This is what you will see

A =	2	+	1i	3	-	5i	-2			
	0	-	7i	6	+	7i	55	-	12i	

You may find the real part, imaginary part or the conjugate of a matrix by typing **real(A)** **imag(A)** **conj(A)**

To find the conjugate transpose of a matrix type : **conj(A)'**

The commands det(A), rref(A), roots(p) , poly(A), eig(A) work in the same way that they work for real matrices. Note that A' will provide the conjugate transpose of A.

Examples:

Type

B=[2+3i 3-i 5-7i; 9+i 5+i 2-i; 2-i 3-4i 4+i]

To find determinant of B type:	**det(B)**
To find the conjugate transpose of B type	**B'**
To find coefficients of the characteristic polynomial of B type	**poly(B)**
To find reduced row echelon form of B type	**rref(B)**
To find the inverse of B type	**inv(B)**
To find eigenvalues of B type	**eig(B)**
To find eigenvalues and eigenvectors of B type	**[V D] = eig(B)**

6 Exercise:

1. Let $c1 = 3 + 7i$, $c2 = 2 - i$, $c3 = 5$ and $c4 = -9i$

a) Compute the following a) c1 + c2

b) c1c3 (note you need to type c1*c2)

c) c1 - c3

d) $\frac{c1}{c2}$

e) $\overline{(\frac{c1}{c2})}$

f) $\overline{(5c2 - 3c1)}$

g) $\overline{c2}(3c1)$

h) $\frac{\overline{c2}}{3c1}$

i) $\overline{(\frac{c2 - ic3}{c4 + 3c1})}$

2. You may enter a polynomial $p(x) = 6x^3 + 3x^2 - 5x + 7$ as

$p = [6 \quad 3 \quad -5 \quad 7]$

to find the roots of p(x) you can type

roots(p)

Find all roots of $p(x) = 0$

i) $x^2 + 5x - 20 = 0$

ii) $7x^2 + 5x + 20 = 0$

iii) $x^2 + 20 = 0$

iv) $x^5 + 1 = 0$

vi) $x^5 - 1 = 0$

vii) $x^5 + 6x^4 - 9x^3 - 2x^2 + x - 1 = 0$

3. Enter the following matrices in MATLAB and the compute the following parts a-h.

$$\begin{array}{llllllllll}
A = [& 2 & 5+2i & 3-i; & 5-2i & 7 & 4+3i; & 3+i & 4-3i & 1 \quad] \\
B = [& 3 & 1+i & i; & 1-i & 1 & 3; & -i & 3 & 1] \\
C = [& 1+i & 15+i & 1-4i; & 3i & 5-i & 2+5i; 4i & -3+i & 2-7i] &
\end{array}$$

i) Which one of the matrices A, B and C are Hermisian ?

ii) Which one of the matrices A, B and C are Normal ?

iii) Which one of the matrices A, B and C are Unitary ?

iv) compute AB.

v) compute $A + \overline{B}$

vi) compute $A + B$.

vii) compute $A^{-1}\overline{B}$

viii) Compute $D = (A + \overline{A})/2$ then compute $E = A - \overline{A})/2i$