0.1 Notes

Green typewriter text represents comments you must type. Each comment is worth one point.

Blue typewriter text represents commands you must type. Each command is worth one point. MATLAB is case-sensitive, so commands are case-sensitive as well. Beware of typos! Also, do not copy and paste commands. Special characters are sometimes used (e.g. the left quote character) which MATLAB does not like.

1 Objectives

In this lab, you will explore the following topics using MATLAB:

- Matrix inverses.
- Properties of invertible matrices.
- $LU$ factorization.

2 Header

Enter your information:

 COMMENT: % [your first name]
 COMMENT: % [your last name]
 COMMENT: % [the date, in any format]
3 Matrix inverses

3.1 Theory

Let $A$ be a square matrix. The inverse of $A$ (if it exists) is denoted by $A^{-1}$ and is defined to be a matrix such that $AA^{-1} = A^{-1}A = I$ ($I$ is the identity matrix).

The following facts about matrix inverses appear in Mathematics 22A. Please review them before doing the lab, especially if your 22A class has not covered them yet.

- Not all square matrices have inverses. For example, the zero matrix has no inverse.
- A square matrix with an inverse is called invertible or nonsingular.
- A square matrix with no inverse is called non-invertible or singular.
- The inverse exists if and only if elimination produces $n$ pivots.
- The inverse of an invertible matrix is unique—a matrix cannot have two different inverses.
- If $A$ is invertible, the system of equations $Ax = b$ has only one solution: $x = A^{-1}b$.
- If there exists a nonzero vector $x$ such that $Ax = 0$, then $A$ is non-invertible.
- If $A$ and $B$ are invertible matrices, then so is $AB$: $(AB)^{-1} = B^{-1}A^{-1}$.
- A $2 \times 2$ matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$.
- A diagonal matrix is invertible if and only if no diagonal entries are zero.
- A triangular matrix is invertible if and only if no diagonal entries are zero.
- The right-inverse of a square matrix is always equal to the left-inverse: if $BA = I$ and $AC = I$ then $B = C$.
- If a matrix doesn’t have $n$ pivots, elimination will lead to a zero row.
- An invertible matrix can’t have a zero row!
- The Gauss-Jordan method solves $AA^{-1} = I$ to find the $n$ columns of $A^{-1}$.
3.2 Basic properties of the inverse

Type: \[ A = \begin{bmatrix} 2 & 1 & -1 \\ -4 & 1 & 4 \\ 5 & -6 & -8 \end{bmatrix} \]
Type: inv(A)
Type: inv(inv(A))

Comment: % [an explanation of what happened in the previous line]
Type: rref(A)

What kind of matrix is rref(A)?

Comment: % [the type of matrix that AR is]

Is this true for all invertible matrices?

If all invertible matrices have this kind of rref, If not,
Type: s3a1 = 'yes' Type: s3a1 = 'no'

Define a matrix B as follows.

Type: B(:,1) = A(:,1)
Type: B(:,2) = A(:,2)
Type: B(:,3) = A(:,1) + A(:,2)

Note that the first two columns of B are equal to the first two columns of A, while the last column of B is the sum of the first two columns of A. Compute the reduced-row echelon form of B.

Type: rref(B)

Do you see a row of zeros?

Type: inv(B)

Does the inverse of B exist?

If B is invertible, If B is singular,
Type: s3a2 = 'yes' Type: s3a2 = 'no'

3.3 The inverse of a diagonal matrix

Type: \[ C = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \]
Type: inv(C)
Type: format rat
Type: inv(C)
Type: \[ D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 5 \end{bmatrix} \]
Type: inv(D)
Type: \[ E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix} \]

Without using MATLAB, guess the inverse of E.

Type your answer into MATLAB, starting with EI =

Type: E*EI

If you did not get an identity matrix, try again until you get it right.
3.4 The Inverse of a Block Matrix

Type: \( F = \begin{bmatrix} 6 & 5 & 0 & 0 \\ 7 & 6 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 4 & 3 \end{bmatrix} \)

Type: \( \text{inv}(F) \)

Do you see anything interesting about \( \text{inv}(F) \)?

Comment: \( % [\text{something interesting about } \text{inv}(F)] \)

Type: \( \text{inv([6 5; 7 6])} \)
Type: \( \text{inv([3 2; 4 3])} \)

Comment: \( % [\text{the relationship of the last two inverses with } \text{inv}(F)] \)

Type: \( G_1 = 5 * \text{eye}(4) - 1 \)
Type: \( \text{inv}(G_1) \)
Type: \( G_2 = 6 * \text{eye}(5) - 1 \)
Type: \( \text{inv}(G_2) \)
Type: \( G_3 = 3 * \text{eye}(2) - 1 \)
Type: \( \text{inv}(G_3) \)

Generalize your observation: Let \( I_n \) and \( O_n \) be the \( n \times n \) identity matrices and matrices of ones, respectively. The inverse of \((n + 1)I_n - O_n\) will be in the form \( k(I_n + O_n) \) for some \( k \) dependent upon \( n \). What is \( k \)?

Comment: \( % [\text{the value of } k, \text{ in terms of } n] \)

3.5 The Inverse of a Matrix Product

Type: \( A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 3 & 0 \\ -1 & -2 & 1 \end{bmatrix} \)
Type: \( B = \begin{bmatrix} 6 & 3 & -5 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix} \)

Type: \( AB = A \times B \)
Type: \( \text{ABI} = \text{inv}(A \times B) \)
Type: \( \text{ABI2} = \text{inv}(A) \times \text{inv}(B) \)
Type: \( \text{ABI3} = \text{inv}(B) \times \text{inv}(A) \)

If \( A \) and \( B \) are invertible matrices, is it always true that \( (AB)^{-1} = A^{-1}B^{-1} \)?

If it is true, \( \text{Type: } s3a3 = 'yes' \)
If it is false, \( \text{Type: } s3a3 = 'no' \)

If \( A \) and \( B \) are invertible matrices, is it always true that \( (AB)^{-1} = B^{-1}A^{-1} \)?

If it is true, \( \text{Type: } s3a4 = 'yes' \)
If it is false, \( \text{Type: } s3a4 = 'no' \)

If \( A \) and \( B \) are matrices, is it always true that \( (AB)^{-1} = B^{-1}A^{-1} \)?

If it is true, \( \text{Type: } s3a5 = 'yes' \)
If it is false, \( \text{Type: } s3a5 = 'no' \)

Is there a multiplicative group of matrices (a set of matrices \( G \) such that every matrix in the group has an inverse, and the set is closed under multiplication) in which for any
given matrices $A$ and $B$ in that group, $(AB)^{-1} = A^{-1}B^{-1}$?

If such a group exists, If no such group exists,
Type: s3a6 = ‘yes’ Type: s3a6 = ‘no’

4 Using Gauss-Jordan Elimination to Invert Matrices

You can find the inverse of an invertible $n \times n$ matrix $A$ by finding the reduced-row echelon form of $K = [A \ I]$. If $A$ is invertible, then $\text{rref}(K)$ will be $[I \ A^{-1}]$.

Type: $A = \begin{bmatrix} 2 & 7; 1 & 3 \end{bmatrix}$
Type: $I = \text{eye}(2)$
Type: $\text{AIR} = \text{rref}([A \ I])$
Type: $\text{AI} = \text{AIR}(; , 3: \text{end})$
Type: $\text{inv}(A)$

Note that $AI = A^{-1}$.

Type: $B = \begin{bmatrix} 2 & 1 & 0; 1 & 2 & 1; 0 & 1 & 2 \end{bmatrix}$
Type: $\text{BIR} = \text{rref}([B \ \text{eye}(3)])$
Type: $\text{BI} = \text{BIR}(; , 4: \text{end})$
Type: $\text{inv}(B)$

Note that $BI = B^{-1}$.

Type: $C = \begin{bmatrix} 1 & 3 & -5; 3 & 2 & 1; 0 & 1 & 2 \end{bmatrix}$
Type: $\text{CIR} = \text{rref}([C \ \text{eye}(3)])$
Type: $\text{CI} = \text{CIR}(; , 4: \text{end})$
Type: $\text{inv}(C)$

Note that $CI = C^{-1}$. 
5 LU Factorization

5.1 Theory

Using Gaussian elimination, we can express any square matrix as the product of a permutation matrix (which reorders the rows and columns), a lower triangular matrix, and an upper triangular matrix: \( A = PLU \). If \( A \) is invertible, usually we choose the lower triangular matrix \( L \) to have a diagonal consisting of ones. If we want both \( L \) and \( U \) to have diagonals consisting of ones, then we need a diagonal matrix in the middle; the decomposition becomes \( A = PLDU \), which is unique.

One of the applications of \( LU \) decomposition is solving linear systems of the form \( Ax = b \). Substituting \( PLU \) for \( A \), we have

\[
PLU x = b.
\]

This can be rewritten (\( P^{-1}b \) is easy to compute) as

\[
L(Ux) = P^{-1}b.
\]

Note that \( Ux \) is a column vector. Since \( L \) is triangular, the system \( Ly = P^{-1}b \) can be solved easily for \( y \); \( Ux = y \) can then be solved for \( x \).

5.2 LU Factorization in MATLAB

To get the \( LU \) decomposition of a matrix in MATLAB, use the \texttt{lu} command. There are two versions of \texttt{lu}: one with two outputs \([L U]\) (\( L \) is a permutation of a lower triangular matrix), and one with three outputs \([L U P]\).

\begin{verbatim}
Type: M = [2 1 0; 1 3 1; 0 1 2]
Type: [L U] = lu(M)
Type: P * L * U
The output of that last command should be equal to M.
Type: N = [1 3 0; 3 11 4; 0 4 9]
Type: [L U P] = lu(N)
Type: [L U] = lu(N)
Note that L is a permuted lower triangular matrix.
\end{verbatim}

End of Lab 4