

# MATH 22AL

## Lab # 5

### 1 Objectives

In this LAB you will explore the following topics using MATLAB.

- Linear combination of vectors.
- Spanning set of a vector space
- Linearly independent set of vectors.
- Basis for a vector space.

**Important:** Do not Change the name of the Variables that you supposed to type in MATLAB, type as you are asked.

## 2 Recording and submitting your work

The following steps will help you to record your work and save and submit it successfully.

- **Open a terminal window.**
  - In Computer LAB (2118 MSB) click on terminal Icon at the bottom of the screen
  - In Windows OS, Use Putty
  - In MAC OS, Use terminal window of MAC.
- **Start a MATLAB Session** that is :
  - Type "textmatlab" Press Enter
- **Enter your information** that is :
  - Type "diary LAB5.text"
  - Type "% First Name:" then enter your first name
  - Type "% Last Name:" then enter your Last name
  - Type "% Date:" then enter the date
  - Type "% Username:" then enter your Username for 22AL account
- **Do the LAB** that is :
  - Follow the instruction of the LAB.
  - Type needed command in MATLAB.
  - All commands must be typed in front of MATLAB Command " >> "..
- **Close MATLAB session Properly** that is :
  - When you are done or if you want to stop and continue later do the following:
  - Type "save" Press Enter
  - Type "diary off" Press Enter
  - Type "exit" Press Enter
- **Edit Your Work before submitting it** that is :
  - Use pico or editor of your choice to clean up the file you want to submit:
  - in command line of pine type "pico LAB5.text"
  - Delete the errors or insert missed items.
  - Save using "^ o= control key then o"
  - Exit using "^ x= control key then x"
- **Send your LAB** that is :
  - Type "ssh point" : Press enter
  - Type submitm22al LAB5.text

### 3 Background Linear Combinations (Reading Materials)

**Definition :**

Let  $a_1, a_2, \dots, a_n$  be scalars and  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  be vectors in  $R^n$ .

the vector  $\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n$ , is called a *Linear Combination* of the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ .

**EXAMPLE 1:** Let  $\mathbf{v}_1 = (1, 2, 0)$ ,  $\mathbf{v}_2 = (0, -1, 2)$ , then

$$\mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 = 2(1, 2, 0) - 3(0, -1, 2) = (2, 7, -6)$$

So,  $\mathbf{w}$  is linear combination of the vectors  $\mathbf{v}_1, \mathbf{v}_2$ .

The vector equation  $2(1, 2, 0) - 3(0, -1, 2) = (2, 7, -6)$  can be written as

$$2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ -6 \end{bmatrix}$$

the left hand side can be represented as a matrix multiplication,

$$\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \mathbf{w} \end{bmatrix}$$

**In General** if a vector  $\mathbf{w}$  is a linear combination of the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  then the vector equation

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n$$

can be expressed in matrix notation as

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \mathbf{w}$$

Note that the columns of the matrix  $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}$  are the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ .

**EXAMPLE 2:** Suppose you want to determine if a given vector  $\mathbf{w} = (2, 7, -6)$  is a linear combination of the vectors  $\mathbf{v}_1 = (1, 2, 0)$ , and  $\mathbf{v}_2 = (0, -1, 2)$ . This means find (if they exist) scalars  $x_1$  and  $x_2$  so that  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{w}$ ? Substitute for  $\mathbf{v}_1$  and  $\mathbf{v}_2$ . We need to answer this question:

Can we find scalars  $x_1$  and  $x_2$  such that

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ -6 \end{bmatrix}$$

This is equivalent to solving the following linear system  $Ax = w$ :

$$\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ -6 \end{bmatrix}$$

which represents the following system of linear equations:

$$\begin{cases} 1x_1 + 0x_2 = 2 \\ 2x_1 + (-1)x_2 = 7 \\ 0x_1 + 2x_2 = -6 \end{cases}$$

Recall that a linear system  $AX = \mathbf{w}$  is consistent (has a solution) if and only if  $\text{rref}(A)$  and  $\text{rref}([A | \mathbf{w}])$  have the same number of nonzero rows or ( $\text{rank}(A) = \text{rank}([A | \mathbf{w}])$ ).

So to find out if  $\mathbf{w} = (2, 7, -6)$  is a linear combination of the vectors  $\mathbf{v}_1 = (1, 2, 0)$ , and  $\mathbf{v}_2 = (0, -1, 2)$ , we need to determine the number of the non-zero rows of

$$\text{rref} \left( \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 2 \end{bmatrix} \right)$$

and the augmented matrix

$$\text{rref} \left( \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 7 \\ 0 & 2 & -6 \end{bmatrix} \right)$$

**Recall** The number of the non-zero rows of a matrix is called the **rank** of that matrix.

**Using MATLAB:** To determine if a given vector  $w$  is a linear combinations of the vectors  $v_1, v_2, \dots, v_n$  Use the following steps:

- Form matrix  $A$  whose columns are the vectors  $v_1, v_2, \dots, v_n$  you can do that by entering vectors
- Form the augmented matrix  $B = [A \mid w]$ ,
- Find  $rref(A)$  and  $rref(B)$
- If number of non-zero rows two row reduced forms are the same then  $w$  is a linear combination of vectors  $v_1, v_2, \dots, v_n$ . Otherwise it is not.
- This can be done in MATLAB by typing  $rank(A)$  and  $rank(B)$ .
- So if  $rank(A) = rank(B)$  then  $w$  is a linear combinations of the vectors  $v_1, v_2, \dots, v_n$ . Otherwise, it is not.

**End of reading Materials**

### Start Typing in MATLAB

**EXAMPLE 3:** Use MATLAB to determine if  $w = (2, 7, -6)$  is a linear combination of the vectors  $v_1 = (1, 2, 0)$ , and  $v_2 = (0, -1, 2)$ .

*Step 1 :* Type the following and press "return" after each line:

```
v1=[1 2 0]'  
v2=[ 0 -1 2]'  
w = [ 2 7 -6]'  
A=[v1 v2]  
B= [A w]
```

*Step 2 :* Type :

```
rank(A)  
rank(B)
```

If the ranks are equal then  $w$  is a linear combination of the vectors  $v_1, v_2$ .

**EXAMPLE 4:** Use MATLAB to express  $w = (2, 7, -6)$  as a linear combination of the vectors  $v_1 = (1, 2, 0)$ , and  $v_2 = (0, -1, 2)$ .

type :

```
rref(B) to solve the linear system  $AX = [w]$ .
```

Use a pen and paper (and `rref(B)`) to find  $c_1$  and  $c_2$  such that  $w = c_1 v_1 + c_2 v_2$ .

Enter your answers as: `c1= Your answer`

`c2 = Your answer.`

Check your work by typing:

```
w1= c1*v1 + c2*v2
```

Then type:

```
tf = isequal(w,w1)
```

If the answer is 1 they are equal if the answer is 0 they are not equal.

## 4 Spanning set of a vector space(Reading Materials)

### Definition:

Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  be vectors in the vector space  $V$ . Let  $W$  be the set of all linear combination of the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ . You can prove that  $W$  is a subspace of  $V$ . The subspace  $W$  is called **the space spanned** by the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ . The set  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is called a **spanning set** of  $W$ .

### Note:

- You should be able to prove that  $W$  is a subspace of  $V$  by showing that  $W$  is closed under addition and scalar multiplication.
- To show that the vector  $\mathbf{w}$  is in the vector space spanned by the set of vectors  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ ? one needs to show that  $\mathbf{w}$  is a linear combination of the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  in  $S$ . As we did in Example 4 above.
- To show that  $S = \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  span  $R^n$  we need to define  $A$  as  $A = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n]$ , then find the  $\text{rank}(A)$ . If  $\text{rank}(A) = n$  then  $S$  is a spanning set of  $R^n$ , otherwise  $S = \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  does not span  $R^n$ . ( Why?)

(End of Reading Materials)

### Start Typing in MATLAB

#### Use MATLAB:

1.) Determine whether the vectors  $\mathbf{v}_1 = (1, 1, 2)$ ,  $\mathbf{v}_2 = (1, 0, 1)$  and  $\mathbf{v}_3 = (1, 1, 2)$  span the vector space  $R^3$

Type:

```
A1 = [1 1 2; 1 0 1; 1 1 2]'
```

You should have vectors in columns of the matrix.

```
AR1 = rref(A1)
```

If you get an identity matrix, the vectors span  $R^3$ .

```
A1n = rank(A1)
```

If the rank is 3 the vectors span  $R^3$ .

```
A1n = rank(AR1)
```

If the rank is 3 the the vectors span  $R^3$ .

2.) Determine whether the vectors  $\mathbf{b} = (6, 11, 6)$  is in the spanning subspace of the vectors in the set  $S = \mathbf{v}_1 = (2, 1, 4)$ ,  $\mathbf{v}_2 = (1, -1, 3)$ ,  $\mathbf{v}_3 = (3, 2, 5)$ .

Type:

```
A2 = [ 2  1  4; 1 -1  3; 3  2  5]'
```

```
B2 = [ 2  1  4; 1 -1  3; 3  2  5; 6 11  6]'
```

```
r1 = rref(A2)
```

```
r2 = rref(B2)
```

```
r3 = rank(A2)
```

```
r4 = rank(B2)
```

If you decide that  $\mathbf{b} = (6, 11, 6)$  is in spanning subspace of vectors in  $S$ ,

type:

```
ANSA1 = 1
```

Otherwise

type:

```
ANSA1 = 0
```



## 5 Linearly independent vectors(Reading Materials)

### Definition

Let  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  be a set of vectors in the vector space  $V$ . The set  $S$  is a linearly independent set if none of the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  is expressible as a linear combination of the other vectors in  $S$ . This is equivalent to the following:

- Vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  are linearly independent if and only if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_n\mathbf{v}_n = \mathbf{0}$$

has only trivial solution. That is the only solution is  $x_1 = x_2 = \dots = x_n = 0$ .

- Let  $A$  be the matrix whose columns are vectors in  $S$ , so

$$A = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n]$$

then  $S$  is linearly independent if and only if  $\text{rref}(A)$  looks like  $\begin{bmatrix} I \\ 0 \end{bmatrix}$

- If  $S$  is a linearly independent set then the solution to  $AX = 0$  is

$$\begin{array}{rcl} x_1 & & = 0 \\ x_2 & & = 0 \\ & \ddots & \vdots \\ & & x_n = 0 \end{array}$$

- A special case: If  $A$  is a square matrix.  $AX = 0$  has only the trivial solution if and only if  $A$  is invertible if and only if  $\det(A) \neq 0$
- So the columns of a square matrix  $A$  are linearly independent only if  $\det(A) \neq 0$ .

(End of Reading Materials)

**Start Typing in MATLAB**

**Use MATLAB:**

1.) Determine if the vectors  $\mathbf{v}_1 = (2, -1, 2, 3)$ ,  $\mathbf{v}_2 = (1, 2, 5, -1)$ ,  $\mathbf{v}_3 = (7, -1, 5, 8)$  are linearly independent vectors in  $R^4$ .

Type:

$L1 = [2 \ -1 \ 2 \ 3; 1 \ 2 \ 5 \ -1; 7 \ -1 \ 5 \ 8]'$

$LR1 = rref(L1)$

If you decide that  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are linearly independent

type:

$ANSL1 = 1$

Otherwise

type:

$ANSL1 = 0$

2.) Determine if the vectors  $\mathbf{v}_1 = (2, -1, 0, 3)$ ,  $\mathbf{v}_2 = (1, 2, 5, -1)$ ,  $\mathbf{v}_3 = (7, -1, 5, 8)$  are linearly independent vectors in  $R^4$ .

Type:

$L2 = [2 \ -1 \ 0 \ 3; 1 \ 2 \ 5 \ -1; 7 \ -1 \ 5 \ 8]'$

$LR2 = rref(L2)$

If you decide that  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are linearly independent

type:

$ANSL2 = 1$

Otherwise

type:

$ANSL2 = 0$

3.) If the vectors in problem 2 above are linearly dependent, write  $\mathbf{v}_1$  as linear combination of two other vectors.

Type:

$L3 = [2 \ -1 \ 0 \ 3; 1 \ 2 \ 5 \ -1; 7 \ -1 \ 5 \ 8]'$

$LR3 = rref(L3)$

Solve  $L3 \ X = 0$  to get  $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$  then solve it for  $v_1$ . (need pen and paper !!!! )

Your answer should look like  $v_1 = av_2 + bv_3$ .

If you calculated  $a$  and  $b$ ,

type:

$ANSa = [a]$

Enter the numerical value of  $a$  inside the bracket for  $a$

type:

$ANSb = [b]$

Enter the numerical value of  $b$  inside the bracket for  $b$

## 6 BASIS (Reading Materials)

### Definition

A set of vectors in a vector space  $V$  form a basis for  $V$  if the set is linearly independent AND spans  $V$ .

If you have a set of  $n$  vectors in an  $n$ -dimensional vector space  $V$  to determine if they form a basis for  $V$  put them in columns of a matrix  $BA$ . Matrix  $BA$  should be an  $n$  by  $n$  matrix. Find  $\det(BA)$  if  $\det(BA) \neq 0$  then the columns of  $BA$  are **linearly independent** and they **span** the vector space  $V$ , that is they form a basis for  $V$ .

### Note:

- A set of three vectors in  $R^4$  can not form a basis for the 4-dimensional vector space  $R^4$ . (Why?)

You do not have to type your answer, review your book if you are not sure about the answer.

- A set of three vectors in  $R^2$  can not form a basis for the 2-dimensional vector space  $R^2$ . (Why?)

You do not have to type your answer, review your book if you are not sure about the answer.

(End of Reading Materials)

### Start Typing in MATLAB

#### Use MATLAB:

1.) Determine if the vectors  $\mathbf{v}_1 = (2, -1, 2, 3)$ ,  $\mathbf{v}_2 = (1, 2, 5, -1)$ ,  $\mathbf{v}_3 = (7, -1, 5, 8)$  form a basis for  $R^4$ .

Type:

```
BA1 = [2 -1 2 3; 1 2 5 -1; 7 -1 5 8]'
```

```
BAR1 = rref(BA1)
```

If you decide that  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  form a basis for  $R^4$ ,

type:

```
ANBA1= 1
```

Otherwise

type:

```
ANBA1= 0
```

2.) Determine if the vectors  $\mathbf{v}_1 = (1, 2, 3)$ ,  $\mathbf{v}_2 = (2, 9, 0)$ ,  $\mathbf{v}_3 = (3, 3, 4)$  form a basis for  $R^3$ .

Type:

```
BA2 = [1 2 3; 2 9 0; 3 3 4]'
```

```
BAR2 = rref(BA2)
```

If you decide that  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  form a basis for  $R^3$ ,

type:

```
ANBA2= 1
```

Otherwise

type:

```
ANBA2= 0
```

3.) Determine if the vectors  $\mathbf{v}_1 = (2, 4, 6)$ ,  $\mathbf{v}_2 = (2, 9, 0)$ ,  $\mathbf{v}_3 = (3, 3, 4)$  form a basis for  $R^3$ .

Type:

```
BA3 = [2 4 6; 2 9 0; 3 3 4]'
```

```
DBA3 = det(BA3)
```

If you decide that  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  form a basis for  $R^3$ ,

type:

```
ANBA3= 1
```

Otherwise

type:

```
ANBA3= 0
```

Start Typing in MATLAB

Use MATLAB:

## 7 EXERCISE:

Type the names as they are

I. Use MATLAB to determine whether  $w$  is in the span of the given vectors  $v_1, v_2, \dots, v_n$ . If your answer is yes, write  $w$  as a linear combination of the vectors  $v_1, v_2, \dots, v_n$  and enter the coefficients as entries of the matrix as instructed:

1.)  $v_1 = (1, 2, 3, 4)$ ,  $v_2 = (-1, 0, 1, 3)$ ,  $v_3 = (0, 5, -6, 8)$  and  $w = (3, -6, 17, 11)$ .

( Note you need enter the vectors in MATLAB as columns of a matrix EXM1.)

If you decided that  $w = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

type

EXA1=[ $c_1$   $c_2$   $\dots$   $c_n$ ]      Example: if  $w = 2v_1 - 3v_2 + 5v_3$  enter EX1 = [2 -3 5]

2.)  $v_1 = (1, 2, 3, 4)$ ,  $v_2 = (-1, 0, 1, 3)$ ,  $v_3 = (0, 5, -6, 8)$  and  $w = (0, -6, 17, 11)$ .

( Note you need enter the vectors in MATLAB as columns of a matrix EXM2.)

If you decided that  $w = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

type

EXA2=[ $c_1$   $c_2$   $\dots$   $c_n$ ]      Example: if  $w = 2v_1 - 3v_2 + 5v_3$  enter EX1 = [2 -3 5]

3.)  $v_1 = (1, 2, 3, 4)$ ,  $v_2 = (-1, 0, 1, 3)$ ,  $v_3 = (0, 5, -6, 8)$ ,  $v_4 = (1, 15, -12, 8)$  and  $w = (0, -6, 17, 11)$ .

( Note you need enter the vectors in MATLAB as columns of a matrix EXM3.)

If you decided that  $w = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

type

EXA3=[ $c_1$   $c_2$   $\dots$   $c_n$ ]      Example: if  $w = 2v_1 - 3v_2 + 5v_3$  enter EX1 = [2 -3 5]

4.)  $v_1 = (1, 2, 3, 4)$ ,  $v_2 = (-1, 0, 1, 3)$ ,  $v_3 = (0, 5, -6, 8)$ ,  $v_4 = (0, -8, 16, -9)$  and  $w = (0, -6, 17, 11)$ .

( Note you need enter the vectors in MATLAB as columns of a matrix EXM4.)

If you decided that  $w = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

type

EXA4=[ $c_1$   $c_2$   $\dots$   $c_n$ ]      Example: if  $w = 2v_1 - 3v_2 + 5v_3$  enter EX1 = [2 -3 5]

Start Typing in MATLAB

Use MATLAB:

## 8 EXERCISE:

Type the names as they are

5.) Let EXRA15 be a random 4 x 6 matrix. (Type

EXRA5= round(10\*rand(4,6) ) )

a.) The columns of EXRA5 are vectors in  $R^4$ . Do they form a basis for  $R^4$ ?

If you decided that the columns of EXRA5 form a basis for  $R^4$

type

EXA5a=1

otherwise type

EXA5a=0

b.) The columns of EXRA5 are vectors in  $R^4$ . Are they linearly independent vectors in  $R^4$ ?

If you decided that the columns of EXRA5 are linearly independent vectors in  $R^4$

type

EXA5b=1

otherwise type

EXA5b=0

c.) The columns of EXRA5 are vectors in  $R^4$ . Do they span  $R^4$ ?

If you decided that the columns of EXRA5 span  $R^4$

type

EXA5c=1

otherwise type

EXA5c=0

This last part is only for your own practice you do not have to report on this part.

6. ) Do the same test in problem 5 above, on a 4 x 2 matrix and a 4 x 4 matrix. See if you can decide and answer the questions in parts a, b and c of the problem above, about this matrix.

7.) Now try the same test on random matrix of size 6 x 7, 6 x 1, and 6 x 6.

8.) How many vectors do you think you need to make a basis for a n-dim space.

9.) What seems to happen if you have less vectors than the dimension of the vector space?

10.) What seems to happen if you have more vectors than the dimension of the vector space?