MATH 22AL
Lab # 7

1 Objectives

In this LAB you will explore the following topics using MATLAB.

- What to do with Inconsistent Linear System
- Minimizing the error in solving Inconsistent Linear system
- Orthogonal Projections.
- Least square solution.
- Normal Equation
- Polynomial Interpolation.

2 What to turn in for this lab

Please save and submit your MATLAB session.
Important: Do not Change the name of the Variables that you supposed to type in MATLAB, type as you are asked.

3 Recording and submitting your work

The following steps will help you to record your work and save and submit it successfully.

- Open a terminal window.
  - In Computer LAB (2118 MSB) click on terminal Icon at the bottom of the screen
  - In Windows OS, Use Putty
  - In MAC OS, Use terminal window of MAC.
- Start a MATLAB Session that is :
  - Type "textmatlab" Press Enter
- Enter your information that is :
  - Type "diary LAB7.text"
  - Type "% First Name:" then enter your first name
  - Type "% Last Name:" then enter your Last name
  - Type "% Date:" then enter the date
  - Type "% Username:" then enter your Username for 22AL account
- Do the LAB that is :
  - Follow the instruction of the LAB.
  - Type needed command in MATLAB.
  - All commands must be typed in front of MATLAB Command " >> "..
- Close MATLAB session Properly that is :
  - When you are done or if you want to stop and continue later do the following:
  - Type "save" Press Enter
  - Type "diary off" Press Enter
  - Type "exit" Press Enter
- Edit Your Work before submitting it that is :
  - Use pico or editor of your choice to clean up the file you want to submit:
  - in command line of pine type "pico LAB6.text"
  - Delete the errors or insert missed items.
  - Save using "ˆ o= control key then o"
  - Exit using "ˆ x= control key then x"
- Send your LAB that is :
  - Type "ssh point" : Press enter
  - Type submitm22al LAB7.text

4 Background, Reading Part : Introduction

Probably you know how to construct a curve through specified points. For example how to construct a line passing through two given points, or a parabola passing through three points. In general you can construct a polynomial of degree n that passes through $n+1$ specified points. Such a polynomial is called **Interpolating polynomial.**

What happens if you have for example more than two points and you want to represent your data with a straight line? In cases like this we face an inconsistent system of linear equations $Ax = b$. Instead of solving $Ax = b$ we try to find an x such that Ax is good approximation of b.

5 Background, Reading Part : Origin of inconsistent systems?

Inconsistent systems arise often in applications. Scientists try to find a functional relationship between variables. They collect data, which usually involve experimental error and by studying this data or from other findings, they suggest a mathematical model for it, “functional relationship between variables.”

Because of measurement error, usually a polynomial that passes through all the given points is not the a true representation of the relationship between variable. A lower degree polynomial may represent the relationship better. To find coefficients of such polynomial the method introduced in LAB 5 is useful. But most of the times this method gives us an inconsistent system, which requires an approximation. In this laboratory we discuss such approximation, called **“Least Square Solution”.**

5.1 Background, Reading Part : Example 1

Suppose that you know two variables x and y have linear relationship that is $y = mx + c$. Suppose in an experiment you obtained the following data:

$$\begin{array}{lcl} x_1 & = & \frac{5}{2} \\ x_2 & = & 3 \\ x_3 & = & \frac{3}{2} \\ x_4 & = & 1 \end{array} \qquad \begin{array}{lcl} y_1 & = & 2 \\ y_2 & = & \frac{9}{2} \\ y_3 & = & 2 \\ y_4 & = & 1 \end{array}$$

If you want to fit a line with equation $y = mx + b$ through these points. As you learned in Laboratory 5, you need to form the following system of linear equations.

$$\begin{array}{lclcl} \frac{5}{2}m & + & b & = & 2 \\ 3m & + & b & = & \frac{9}{2} \\ \frac{3}{2}m & + & b & = & 2 \\ 1m & + & b & = & 1 \end{array}$$

This is an overdetermined system. That is it has more equation than unknowns. Over determined systems usually are inconsistent.
Writing the matrix equation for this linear system we get

$$\begin{bmatrix} \frac{5}{2} & 1 \\ 3 & 1 \\ \frac{3}{2} & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{9}{2} \\ 2 \\ 1 \end{bmatrix}$$

Find the reduced row echelon form of the augmented matrix.

$$\left[\begin{array}{cc|c} \frac{5}{2} & 1 & 2 \\ 3 & 1 & \frac{9}{2} \\ \frac{3}{2} & 1 & 2 \\ 1 & 1 & 1 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ 0 & 0 & 0 \end{array}\right]$$

As you see the rank of rref of the augmented matrix is 3 while the rank of the coefficient matrix is 2. Therefore the system is inconsistent. So there is no x such that $Ax = b$ or $Ax - b = 0$.

Since $Ax - b$ is non-zero how about forcing it to be as small as possible. So, the goal is to find an approximation, Ax, for b or to minimize $d = Ax - b$?

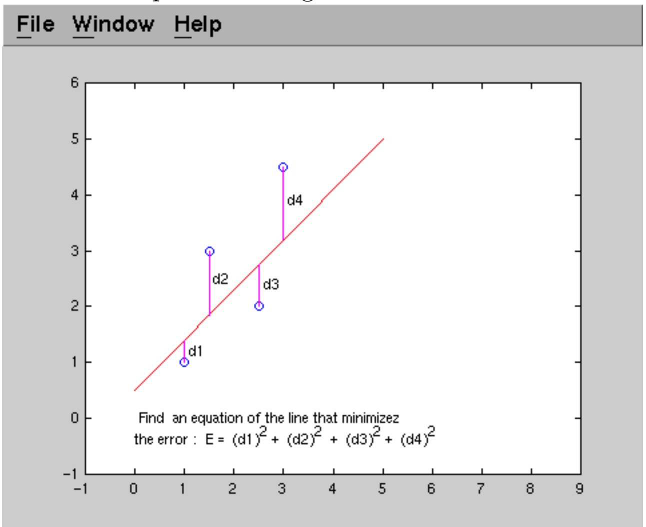
Restating the question:

How could we find an x such that Ax is as close as possible to b?

6 Background, Reading Part : Minimizing Ax-b

Let A be an $m \times n$ matrix. Suppose that $Ax = b$ is an inconsistent system, we are interested in finding an x such that Ax is as close as possible to b.

Lets first look at the Example 1. There are several ways to make your line "close" to given points, depending how we define "closeness". Usual way is to add the square of $d_1, d_2 d_3 \ldots d_n$, then minimize the sum of squares. See figure bellow.



This method is called " **least square Approximation** ".

We may also think that Ax in a $m \times 1$ vector and b is another $m \times 1$ vector. We want to minimize $Ax - b$. Assume that our vector space is an Inner Product Space with the usual Euclidean inner product, minimizing $(Ax - b)$ translates to minimizing the distance $||Ax - b||$ between the two vectors Ax and b. Note that $||Ax - b||$ is the length of the vector $Ax - b$.

One thing which helps understanding the procedure of minimizing the distance between Ax and b is the fact that Ax is a vector in column space of A. (Why?)

Since $Ax = b$ is not consistent, b is not in the column space of A. So we are looking for a vector, Ax, in the column space of A which is closest to the vector b. It can be proved (see your text book) that such a vector is the orthogonal projection of b onto the column space of A. Now if Ax is the orthogonal projection of b onto $col(A)$ then $Ax - b$ is orthogonal to $col(A)$. (why?) That is $Ax - b$ is in $null(A^t)$. So

$$A^t(b - Ax) = 0$$

or

$$A^t Ax = A^t b.$$

This system of linear equations is consistent (Why?) and it is called **Normal System**. Moreover if the column of A are linearly independent, then $A^t A$ is invertible and

$$x = (A^t A)^{-1}(A^t)b$$

is the unique solution of the system $A^tAx = A^tb$.
If columns of A are orthonormal then $A^tA = I$ and we can easily compute $x = A^tb$, and the closest vector p in the column space of A to b is $p = AA^tb$.

6.1 Background, Reading Part : Orthogonal projection of b onto W

Let b be a vector in R^m and W be a subspace of R^m spanned by the vectors

$$w_1, w_2, w_3, \dots w_n$$

. To find orthogonal projection of b onto W denoted by $Proj_W b$ form a matrix A whose columns are the vectors

$$w_1, w_2, w_3, \dots w_n$$

then solve the normal system

$$A^t Ax = A^t b.$$

The solution vector x is the orthogonal projection of b onto W. Orthogonal Projection of R^m onto W

Let W be a subspace of R^m spanned by the basis vectors

$$w_1, w_2, w_3, \dots w_n.$$

To find orthogonal projection of R^m onto W denoted by $[P]$ form a matrix A whose columns are the vectors

$$w_1, w_2, w_3, \dots w_n$$

then transformation

$$P = A(A^t A)^{-1} A^t$$

is called **the orthogonal projection of R^m onto W.**

7 Background, Reading Part : Least Square Lines

Assume that y is a linear function of x that is $y = mx + b$ and experimental data $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ is given. We want to find the parameters m and b such that the line $y = mx + b$ be as " close " as possible to the given points.

Set $\quad x = \begin{bmatrix} m \\ b \end{bmatrix} \qquad A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \qquad b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

solve the linear system $A^t A x = A^t b$.

Start Typing in MATLAB

7.1 Example 2

Find the equation $y = mx + c$ of the **least square line** that best fits the data from Example1:

$$\begin{array}{lcl} x_1 & = & \frac{5}{2} \\ x_2 & = & 3 \\ x_3 & = & \frac{3}{2} \\ x_4 & = & 1 \end{array} \qquad \begin{array}{lcl} y_1 & = & 2 \\ y_2 & = & \frac{9}{2} \\ y_3 & = & 2 \\ y_4 & = & 1 \end{array}$$

Note: This line is called **least-squares line** and the coefficients m and c are called **regression coefficients.**

Solution :

Steps : First form your matrix A, then solve the linear system $A^tAx = A^tb$.

i) Enter the data matrix D as:

Type :

$$D = [5/2\ \ 3\ \ 3/2\ \ 1; 1\ \ 1\ \ 1\ \ 1\ ;\ 2\ \ 9/2\ \ 2\ \ 1\ ]'$$

Type :

$$A = D(:, [1\ \ 2])$$

Type :

$$b = D(:, 3)$$

ii) Find the augmented matrix of the system $A^tAx = A^tb$

Type :

$$AG = [A' * A \quad A' * b]$$

Then find the rref of the augmented matrix

Type :

$$RRAG = rref(AG)$$

This should give you $m = 7/5$ and $c = -17/40$

8 Example 3

Find equation of a polynomial of degree 3, $\mathbf{y = a_3x^3 + a_2x^2 + a_1x + a_0}$ that best fits the following data.

$$\begin{array}{rcrcrcr} x_1 & = & 2 & \quad & y_1 & = & 7 \\ x_2 & = & 1 & & y_2 & = & -2 \\ x_3 & = & 0 & & y_3 & = & -7 \\ x_4 & = & -2 & & y_4 & = & -27 \\ x_5 & = & 5 & & y_4 & = & 110 \\ x_6 & = & -1 & & y_4 & = & -20 \end{array}$$

Solution :

Note: The method used to fit a straight line to data points in Example 1 can be easily generalized to a polynomial of a given degree

By plugging the given six points in the equation of the polynomial, we get a system of six linear equations. By solving this linear system we find the coefficients of the third degree polynomial $y = a_3x^3 + a_2x^2 + a_1x + a_0$.

The linear system $Ax = b$ would be

$$\begin{bmatrix} x_1^3 & x_1^2 & x_1 & 1 \\ x_2^3 & x_2^2 & x_2 & 1 \\ x_3^3 & x_3^2 & x_3 & 1 \\ x_4^3 & x_4^2 & x_4 & 1 \\ x_5^3 & x_5^2 & x_5 & 1 \\ x_6^3 & x_6^2 & x_6 & 1 \end{bmatrix} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix}$$

Steps : First form your matrix A, then solve the linear system $A^tAx = A^tb$.

To distinguish matrices of this example with the previous ones we call the coefficient matrix A1.

To form your matrix $A1$

Type :

$$D = [2\ 1\ 0\ \ -2\ 5\ \ -1;\ \ 1\ 1\ 1\ 1\ 1\ 1; 7\ \ -2\ \ -7\ \ -27\ 110\ \ -20]'$$

Type :

$$A1 = [D(:,1).\hat{}\,3\ \ \ D(:,1).\hat{}\,2\ \ \ D(:,1)\ \ D(:,2)]$$

Note : $D(:,1).\hat{}\,3$ has two parts:

- $D(:,1)$ Choosing the first column of D
- $.\hat{}\,3$ raising each entry to the third power

To enter your matrix b

Type :

$$b1 = D(:, 3)$$

Solve the linear system $(A1)^t(A1)x = (A1)^t(b1)$
First form the augmented matrix
Type :

$$AG1 = [(A1)' * (A1) \quad (A1)' * b1]$$

Then find the rref of the augmented matrix:
Type :

$$RRAG1 = rref(AG1)$$

This should give you
$a_3 = 252/347$,
$a_2 = -21/233$,
$a_1 = 2439/394$,
$a_0 = -3301/344$.

9 Exercise:

1. Enter matrices $A2$ and $b2$ by typing
Type :

$$A2 = [1\ \ 2\ \ 3; 4\ \ 5\ \ 6; 7\ \ 8\ \ 9; 3\ \ 2\ \ 4; 6\ \ 5\ \ 4; 9\ \ 8\ \ 7]$$

$$b2 = [1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1]'$$

a.) Show that $(A2)x = b2$ is inconsistent. Show this in two ways:
a1.) By finding
Type :

$$RRA2 = rref([A2\quad b2])$$

and
a2.) Finding both
Type :

$$rankA2b = rank([A2\quad b2])$$

and
Type :

$$rankA2 = rank(A2)$$

Problem 1 continues:

b.) **Show that the system $(A2)^t(A2)x = (A2)^t b2$ is a consistent system and find a solution for $(A2)^t(A2)x = (A2)^t b2$.**

Type :

$$N1A2 = [A2' \star A2 \;\; A2' \star b2]$$

Type :

$$RRN1A2 = rref(N1A2)$$

To get the solution

Type :

$$X2 = RRN1A2(:,4)$$

c.) **Find $||A2 \star X2 - b2||$ the norm of the vector $A2 \star X2 - b2$ by typing $norm(A2 \star X2 - b2)$ where $X2$ is your solution from part (b).**

Type :

$$E2 = A2 \star X2 - b2$$

Type :

$$NOR2 = norm(E2)$$

i) **Find a random vector, say $Z2$, in the column space of $A2$ by typing**

Type :

$$Z2 = A2 * [1 \;\; 2 \;\; -1]'$$

ii) **Compute $||Z2 - b2||$ by typing**

Type :

$$NORZ2 = norm(Z2 - b2)$$

Compare this value with $||A2 \star X2 - b2||$ which one is smaller?

Type :

% NORZ2 is smaller than NOR2

or Type :

% NORZ2 is larger than NOR2

Problem 1 continues:

iii) Find two other vectors, $Z3,$ and $Z4$, in column space of $A2$ as we found $Z2$ in part(ii),

Type :

$Z3 = A2 \star (10 \star rand(3,1))$

Type :

$Z4 = A2 \star (10 \star rand(3,1))$

Find the norm of $||Z3 - b2||$ **and** $||Z4 - b2||$.

Type :

$$NORZ3 = norm(Z3 - b2)$$

Type :

$$NORZ4 = norm(Z4 - b2)$$

Compare the results with $A2 \star X2 - b2$.

Type :

$$DNORM12 = norm(Z2 - b2) - norm(A2 \star X2 - b2)$$

Type :

$$DNORM32 = norm(Z3 - b2) - norm(A2 \star X2 - b2)$$

No matter what vector you choose in column space of $A2$, the distance of that vector from $b2$ will be larger than distance of $A2 \star X2$ from $b2$. Think if you can prove this (challenging).

Problem 1 continues:

d) Is $A2^tA2$ invertible?

You may check this by one of the following ways:

i)) Finding $det(A2^tA2)$ **Note: Recall** $det(C) \neq 0 \iff C \; is \; invertible$

Type :

$$DETA2TA2 = det(A2' \star A2)$$

ii)) Finding $rref(A^tA)$. **Note: Recall** $rref(C) = I \iff C \; is \; invertible$

Type :

$$RRA2TA2 = rref(A2' \star A2)$$

e) Are the columns of $A2$ linearly independent?

Type :

$$RREA2 = rref(A2)$$

Explain Enter your answer after typing %.

f) Are the rows of $A2$ are linearly independent?

Explain Enter your answer after typing %.

g) Find a basis for the column space of $A2$.

Enter your answer after typing %.

Problem 1 continues:

h) Find a basis for the row space of $A2$.

Enter your answer after typing %.

i) How many solutions does $A2 \star x = 0$ have?

Enter your answer after typing %.

j) What is the number of solutions of $A2^t A2 \star x = 0$?

Enter your answer after typing %.

k) Find a basis for the range of the linear transformation defined by $A2$.

Note: **T_{A2} is defined to be a linear transformation which maps any vector x to $A2 \star x$. That is $T_{A2} = A2 \star x$. Also the range of the Linear transformation represented by $A2$ is the same as the column space of $A2$.**

l) Find a basis for the $null(T_{A2})$.

Enter your answer after typing %.

m) Find nullity of $A2$, T_{A2} and $A2^t A2$.

Enter your answer after typing %.

n) Find $rank(A2)$, $rank(A2^t)$, $rank(T_{A2})$ and $rank(A2^t A2)$.

2) Recall the following theorem: If B is an $m \times n$ matrix, then the following are equivalent:
i) Columns of B are linearly independent. **ii) $B^t B$ is an invertible matrix.**

a) Determine if the following vectors are linearly independent.
$v_1 = (1, 3, -2, 0, 2)$
$v_2 = (2, 6, -5, -2, 4)$
$v_3 = (0, 0, 5, 10, 0)$
$v_4 = (2, 6, 0, 8, 4)$
(Note: form a matrix $B1$ whose columns are the vectors v_1, v_2, v_3, v_4, then find $det(B1^t B1)$.
Type :
$B1 = [1\ 3\ \ -2\ 0\ 2; 2\ 6\ \ -5\ \ -2\ 4; 0\ 0\ 5\ 10\ 0; 2\ 6\ 0\ 8\ 4]'$

Type :
$B1TB1 = B1' \star B1$

Type :
$DETB1TB1 = det(B1TB1)$

If $B1' \star B1$ is not invertible
Type :
$\%B1TB1\ is\ not\ invertible$

If $B1' \star B1$ is invertible
Type :
$\%INVB1TB1 = inv(B1TB1)$

b) Determine if the following vectors are linearly independent.
$v_1 = (0, 0, -2, 0, 7, 12)$
$v_2 = (2, 4, -10, 6, 12, 28)$
$v_3 = (2, 4, -5, 6, -5, -1)$
Repeat the steps in part(a) above:
Form a matrix $B2$ whose columns are the vectors v_1, v_2, v_3, then find $det(B2^t B2)$.
Type :
$B2TB2 = B2' \star B2$

Type :
$DETB2TB2 = det(B2TB2)$

If $B2' \star B2$ is not invertible
Type :
$\%B2TB2\ is\ not\ invertible$

If $B2' \star B2$ invertible
Type :
$\%INVB2TB2 = inv(B2TB2)$

3. Find an equation of the line that passes through the points $(3,4)$ and $(1,2)$.

a.) Using the least square solution.

Type :

Slope1= Enter slope of the line

Intercept1= Enter the y-intercept

b.) Using the regular algebraic way: Use pen and paper.

Type :

Slope2= Enter slope of the line

Intercept2= Enter the y-intercept

4. Consider the following set of points:
$(3,4),(1,2),(-1,1),(6,5),(7,9)$. **Find a polynomial of degree two** $y = a_2x^2 + a_1x + a_0$, **that best fits the points given above.**
Enter this points in a matrix :
Type :
$E4 = [\ 3\ \ 1\ \ -1\ \ 6\ \ 7;4\ \ 2\ \ 1\ \ 5\ \ 9;1\ \ 1\ \ 1\ \ 1\ \ 1]'$

To create a matrix of x-values
Type :
$X = E4(:,1)$

To create a matrix of y-values
Type :
$Y = E4(:,2)$

Create a 5×3 **matrix** $A4$
Type :
$A4 = [X.\hat{}2\ \ X\ \ E4(:,3)]$

The first column is the x -values raised to second power.
The second column is x-values.
The Third column is ones.
Now in $A4x = b$ **the vector** x **is**
$$x = \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix}$$
and b is the vector of y-values.
Type :
$b = Y$

Then find the solution to the normal system $A4^tA4X = A4^tb$.
To solve this system you may find $rref([A4^tA4\ \ A4^tb]$
Type :
$RRA4AGUM = rref([A4' \star A4\ \ A4' \star b])$

Enter Your Answers for a_2, a_1 **and** a_0 **as**
Type :
$a2 =$ **Your answer for** a_2

Type :
$a1 =$ **Your answer for** a_1

Type :
$a0 =$ **Your answer for** a_0

ii) Find a polynomial of degree three, $y = b_3x^3 + b_2x^2 + b_1x + b_0$ that best fits the points $(3,4),(1,2),(-1,1),(6,5),(7,9)$ (the same points as were given above).

Follow the steps in part(i) and examples. Define the matrix B4 and solve the equation.

Type :

$RRB4AGUM = rref([B4' \star B4 \quad B4' \star b])$

Enter Your Answers for b_3, b_2, b_1 and b_0 as

Type :

$b_3 =$ **Your answer for** b_3

Type :

$b_2 =$ **Your answer for** b_2

Type :

$b_1 =$ **Your answer for** b_1

Type :

$b_0 =$ **Your answer for** b_0

iii) Find a polynomial of degree four, $y = c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0$ that best fits the points $(3,4),(1,2),(-1,1),(6,5),(7,9)$ (the same points as were given above).

Follow the steps in part(i) and examples. Define the matrix C4 and solve the equation.

Type :

$RRC4AGUM = rref([C4' \star C4 \quad C4' \star b])$

Enter Your Answers for c_4, c_3, c_2, c_1 and c_0 as

Type :

$c_4 =$ **Your answer for** c_4

Type :

$c_3 =$ **Your answer for** c_3

Type :

$c_2 =$ **Your answer for** c_2

Type :

$c_1 =$ **Your answer for** c_1

Type :

$c_0 =$ **Your answer for** c_0

iv) Find a polynomial of degree five $y = d_5x^5 + d_4x^4 + d_3x^3 + d_2x^2 + d_1x + d_0$ that best fits the points given above.

Follow the steps in part(i) and examples. Define the matrix D4 and solve the equation.

Type :

$RRD4AGUM = rref([D4' \star D4 \quad D4' \star b])$

Enter Your Answers for d_5, d_4, d_3, d_2, d_1 and d_0 as

Type :

$d_5 =$ **Your answer for** d_5

Type :

$d_4 =$ **Your answer for** d_4

Type :

$d_3 =$ **Your answer for** d_3

Type :

$d_2 =$ **Your answer for** d_2

Type :

$d_1 =$ **Your answer for** d_1

Type :

$d_0 =$ **Your answer for** d_0