Write legibly and neatly. Show all your work and simplify answers for full credit.
1. (10 points) Find the general solution of the following differential equation:

\[ \frac{dy}{dx} = 8xy \]

Make sure to solve for \( y \).

We separate variables:

\[
\frac{dy}{y} = 8x \, dx
\]

\[
\int \frac{1}{y} \, dy = \int 8x \, dx
\]

\[
\log y = 8 \frac{x^2}{2} + C
\]

\[
y = e^{4x^2}
\]
2. (12 points) Find the particular solution to the differential equation:

\[
\frac{dx}{dt} = 1 - \frac{x}{t + 8},
\]
given that \(x\) is 5 when \(t\) is 2.

We first rearrange this to the standard form for a first order linear differential equation:

\[
x' + x \frac{1}{t + 8} = 1
\]

And hence identify \(P(t) = \frac{1}{t+8}\) and \(Q(t) = 1\). We can then calculate the integrating factor \(u(t)\) as follows:

\[
u(t) = e^{\int P(t) dt} = e^{\int \frac{1}{t+8} dt} = e^{\log(t+8)} = t + 8
\]

The solution \(x(t)\) is then given by the formula:

\[
x(t) = \frac{1}{t+8} \int (t + 8) dt = \frac{1}{t+8} \left( \frac{(t + 8)^2}{2} + C \right)
\]

\[
= \frac{t + 8}{2} + \frac{C}{t + 8}
\]

This is a perfectly acceptable form of the general solution; there are many others. To find the particular solution, we simply plug in 5 for \(x\) and 2 for \(t\), and then solve for \(C\):

\[
5 = \frac{2 + 8}{2} + \frac{C}{2+8}
\]

\[
= 5 + \frac{C}{10}
\]

\[
C = 0
\]

Note that your value of \(C\) may be different depending on the form of your general solution. However, for everyone, the particular solution should then simplify to:

\[
x(t) = t/2 + 4
\]
3. (10 points) Find the general solution to the differential equation:

\[ yy' - xy^2 = x \]

Make sure to solve for \( y \).

This one we want to solve by separation of variables. It’s actually not a first order linear differential equation (can you figure out why?), so the method of integrating factors is not applicable. To separate, we have to do a little bit of algebra:

\[
yy' = x + xy^2
\]

\[
y \frac{dy}{dx} = x(1 + y^2)
\]

\[
y \frac{1}{1 + y^2} dy = x dx
\]

\[
\int y \frac{1}{1 + y^2} dy = \int x dx
\]

This can be solved via a u-substitution, though I won’t show the work (see me in office hours if you’re having trouble with this kind of integral.)

\[
\frac{1}{2} \log(1 + y^2) = \frac{x^2}{2} + C
\]

\[
\log(1 + y^2) = x^2 + C
\]

\[
1 + y^2 = Ce^{x^2}
\]

\[
y = \sqrt{Ce^{x^2} - 1}
\]
4. **(5 points)** The rate of increase in sales (S) of a product is proportional to the current level of sales and inversely proportional to the square of the time (t). What differential equation represents this relationship? Circle the correct answer, no justification is necessary.

(a) \[ \frac{dS}{dt} = kSt^2 \]

(b) \[ \frac{dS}{dt} = k \frac{t^2}{S} \]

(c) \[ \frac{dS}{dt} = k \frac{S}{t^2} \]

(d) \[ \frac{dS}{dt} = k \frac{t}{S} \]

(e) None of the above.

The answer is (c).

5. **(5 points)** A population of \( P \) rabbits breeds at a rate proportional to the current population of rabbits squared; is eaten by wolves at a rate which is proportional to the square root of the current population, and dies of non-wolf related incidents at a rate proportional to the current population. Let \( b \) be the proportionality constant associated with breeding, \( w \) be the proportionality constant associated with wolf-death, and \( d \) be the proportionality constant associated with non-wolf death. Assume that all proportionality constants are positive. Write a differential equation that models this population. **YOU DO NOT NEED TO SOLVE THE EQUATION.**

\[ \frac{dP}{dt} = bP^2 - w\sqrt{P} - dP \]
6. **(12 points)** During a chemical reaction, a substance $A$ is converted into another substance at a rate proportional to the square of the amount of $A$. There is initially 100 grams of substance $A$ present, and 50 grams remain after 1 hour. A differential equation that models this process is:

$$\frac{dA}{dt} = -kA^2$$

Where $k > 0$.

(a) Solve the differential equation for $A$ as a function of time. Make sure to find the values of *all* constants.

We separate variables and integrate:

$$\frac{dA}{A^2} = -k \, dt$$

$$\int \frac{dA}{A^2} = \int -k \, dt$$

$$-\frac{1}{A} = -kt + C$$

$$A = \frac{1}{kt + C}$$

To solve for $C$, we use the fact that $A(0) = 100$:

$$100 = \frac{1}{k(0) + C}$$

$$= \frac{1}{C}$$

$$C = 1/100$$

To solve for $k$, we use the fact that $A(1) = 50$:

$$50 = \frac{1}{k(1) + 1/100}$$

$$1/50 = k + 1/100$$

$$k = 1/100$$

Hence $A$ as a function of time is given by:

$$A(t) = \frac{1}{t/100 + 1/100}$$

$$= \frac{100}{t + 1}$$
(b) How much time needs to pass for there to be only 10 grams of substance $A$ remaining?

We just plug in 10 for $A$ and solve for $t$:

\[
10 = \frac{100}{t + 1} \\
(t + 1) = 10 \\
\therefore t = 9
\]
7. (8 points)

(a) Sketch the plane given by the equation:

\[ x + 3y + 2z = 6 \]

Please label your axes.

(b) Let \( a \) be the point given by the intersection of the above plane and the \( x \) axis. What is \( a \)? (give your answer in \((x, y, z)\) coordinates.)

On the \( x \) axis, \( y = z = 0 \). To find \( x \), we just plug those values into the equation and see that \( x = 6 \). Hence \( a = (6, 0, 0) \).

(c) Let \( b = (2, -3, 0) \). Find the distance between \( a \) and \( b \).

\[
d = \sqrt{(6 - 2)^2 + (0 - (-3))^2 + (0 - 0)^2} \\
= \sqrt{16 + 9} \\
= 5
\]

(d) Find the midpoint between the points \( a \) and \( b \).

\[
m = \left( \frac{6 + 2}{2}, \frac{0 + (-3)}{2}, \frac{0 + 0}{2} \right) \\
= \left( 4, \frac{-3}{2}, 0 \right)
\]
8. **(6 points)** Find the radius and the center of the sphere given by the following equation:

\[ x^2 - 4x + y^2 + 2y + z^2 = 11 \]

In order to do this, we put the equation in standard form by completing the square:

\[
(x^2 - 4x + 4) + (y^2 + 2y + 1) + z^2 = 11 + 4 - 1 \\
(x - 2)^2 + (y + 1)^2 + z^2 = 16
\]

From which we can identify that the radius is 4 and the center is \((2, -1, 0)\).
9. **(8 points)** Let $S$ be the surface given by the equation:

$$2x^2 + 4y^2 - z^2 = 2$$

(a) Find the xy-trace and classify as a parabola, hyperbola, or ellipse.

The trace is:

$$2x^2 + 4y^2 = 2$$

Which is an ellipse.

(b) Find the yz-trace and classify as a parabola, hyperbola, or ellipse.

The trace is:

$$4y^2 - z^2 = 2$$

Which is a hyperbola.

(c) Find the xz-trace and classify as a parabola, hyperbola, or ellipse.

The trace is:

$$2x^2 - z^2 = 2$$

Which is a hyperbola.

(d) Classify as one of the six kinds of quadratic surfaces we learned about in class.

Hyperboloid of one sheet.
10. **(6 points)** For the following six equations, on the line following the equation, place both the letter of the corresponding graph, and the name of the type of surface (e.g., ‘elliptic cone’).

\[
\begin{align*}
    x^2 - 2y^2 + 2z^2 &= -1 \quad &\text{c, hyperboloid of two sheets} \\
    2z^2 &= y^2 - x^2 \quad &\text{e, elliptic cone} \\
    2x^2 &= -y - z^2 \quad &\text{f, elliptic paraboloid} \\
    z &= x^2 - y^2 \quad &\text{a, hyperbolic paraboloid} \\
    2x^2 + y^2 - z^2 &= 2 \quad &\text{b, hyperboloid of one sheet} \\
    2x^2 + 2y^2 + z^2 &= 2 \quad &\text{d, ellipsoid}
\end{align*}
\]
11. **(8 points)** Please read the instructions carefully - they differ on each part.

(a) Evaluate the function \( f(x, y) = x^3 + \ln(x + \sqrt{y}) \) at the points \((0, e^2)\).

\[
f(0, e^2) = 0 + \log \left( 0 + \sqrt{e^2} \right) \\
= 0 + \log(e) \\
= 1
\]

(b) Find the domain and range of the function \( f(x, y) = \frac{1}{|x - y|} \).

\[
D = \{(x, y) : x \neq y\} \\
R = (0, \infty)
\]

(c) Find and sketch the domain of the function \( f(x, y) = \sqrt{1-x^2-y^2} \).

\[
D = \{(x, y) : x^2 + y^2 \leq 1\}
\]
12. **(10 points)** Please read the instructions carefully - they differ on each part.

(a) Calculate $f_x$ and $f_y$ for the following function:

$$f(x, y) = 2e^{x^2y^3}$$

$$f_x = 2e^{x^2y^3} (2xy^3)$$
$$= 4xy^3 e^{x^2y^3}$$

$$f_y = 2e^{x^2y^3} (x^2 3y^3)$$
$$= 6x^2y^3 e^{x^2y^3}$$

(b) Show that $f_{xy} = f_{yx}$ for the following function:

$$f(x, y) = x \ln(xy^2)$$

$$f_x = \log(xy^2) + x \frac{1}{xy^2} y^2$$
$$= \log(xy^2) + 1$$

$$f_{xy} = \frac{1}{xy^2} 2xy$$
$$= \frac{2}{y}$$

$$f_y = x \frac{1}{xy^2} (2xy)$$
$$= \frac{2x}{y}$$

$$f_{yx} = \frac{2}{y}$$

So we see that $f_{xy} = f_{yx}$. 
13. **(10 points)** For the given function, find the \((x, y)\) coordinates of all critical points. Classify all critical points as minima, maxima, or saddle points. You must justify your classifications.

\[
f(x, y) = 5y + xy - x^2 + y^2
\]

To find the critical points, we need to find both first partials:

\[
f_x = y - 2x
\]

\[
f_y = 5 + x + 2y
\]

We then set these both to 0 and find all \((x, y)\) that satisfy the two resulting equations. The first equation, \(0 = y - 2x\), gives us that \(y = 2x\). We can plug this into the second equation:

\[
0 = 5 + x + 2(2x)
\]

\[
-5 = 5x
\]

\[
x = -1
\]

To find that \(x = -1\). Since \(y = 2x\), then \(y = -2\). So \((-1, -2)\) is a critical point. The derivatives are always defined, so this point is our only critical point. To classify it, we can use the second derivative test. Fortunately, the second derivatives are easy to take, and give us that \(f_{xx} = -2\), \(f_{yy} = 2\), and that \(f_{xy} = 1\). We can then evaluate the magical quantity \(d\) as follows:

\[
d = f_{xx}f_{yy} - f_{xy}^2
\]

\[
= (-2)(2) - 1
\]

\[
= -5
\]

\[
< 0
\]

Since \(d < 0\), then there is a saddle point at the point \((-1, -2)\).