1. (10 points) Find the general solution of the following differential equation:

\[ \frac{dy}{dx} = ky^2 \]

Make sure to solve for \( y \).

We separate variables, and then integrate:

\[
\int \frac{1}{y^2} \, dy = \int k \, dx \\
\frac{-1}{y} = kx + C \\
y = \frac{-1}{kx + C}
\]
2. (10 points) Find the particular solution to the differential equation:

\[ \frac{dx}{dt} = 3(x + 1) \]

Given that \( x \) is 2 when \( t \) is 0. You may assume that \( x \geq 0 \) for all time. Be sure to write your solution as a function of \( x \).

We first find the general solution by separating variables, and then integrating:

\[
\int \frac{1}{x + 1} \, dx = \int 3 \, dt
\]

\[
\log(x + 1) = 3t + C
\]

\[
x + 1 = e^{3t+C}
\]

\[
x = Ce^{3t} - 1
\]

We now plug in our initial condition to solve for \( C \). We know that \( x = 2 \) when \( t = 0 \), so \( x(0) = 2 \). This gives:

\[
2 = Ce^{3(0)} - 1 = C - 1
\]

\[
C = 3
\]
3. **(10 points)** Find the general solution to the differential equation:

\[
\frac{dy}{dt} + y = 12t^3
\]

Make sure to solve for \( y \).

We begin by writing this equation in its standard form:

\[
y' + \frac{1}{t}y = 12t^2
\]

We can now see that this is a first order linear differential equation, and so we can use the standard method to solve them. We identify that \( P(t) = \frac{1}{t} \), and \( Q(t) = 12t^2 \). We can then calculate the integrating factor as:

\[
u(t) = e^{\int P(t) \, dt} = e^{\int \frac{1}{t} \, dt} = e^{\log(t)} = t
\]

We can then solve the equation using the general solution for these kinds of differential equations:

\[
y(t) = \frac{1}{u(t)} \int Q(t) u(t) \, dt
\]

\[
= \frac{1}{t} \int (12t^2)(t) \, dt
\]

\[
= 12 \frac{1}{t} \int t^3 \, dt
\]

\[
= \frac{12}{t} \left( \frac{t^4}{4} + C \right)
\]

\[
= 3t^3 + \frac{C}{t}
\]
4. (10 points) Newton’s Law of Cooling States that the rate of change in the temperature $T$ of an object is proportional to the difference between the temperature of the object $T$ and the surrounding temperature $T_0$. This can be expressed by the differential equation:

$$\frac{dT}{dt} = k(T - T_0)$$

David is baking a cake in a kitchen that has an ambient temperature of 75°F. The buzzer dings telling him that the cake is done at noon, so he removes the cake from the oven, where it was baking at a temperature of 375°F, and places it on the cooling rack. After 1 hour, the cake has cooled to 275°F. David is having some friends over at 6:30PM, and he really wants to feed them the delicious cake, but he doesn’t want to burn their mouths. Will the cake have reached the safe, tasty temperature of 100°F by the time David’s friends arrive? Note: to facilitate calculations, assume that $\log(2/3) = -0.4$, and that $\log(1/12) = -2.4$. 
We begin by finding the general solution, which we do by separating variables and then integrating:

\[
\frac{dT}{T - T_0} = k \, dt
\]

\[
\int \frac{dT}{T - T_0} = \int k \, dt
\]

\[
\log(T - T_0) = kt + C
\]

\[
T - T_0 = e^{kt+C}
\]

\[
T = Ce^{kt} + T_0
\]

Note that we know that \(T_0 = 75\), so:

\[
T = Ce^{kt} + 75
\]

First we solve for \(C\). To do so, we note that the initial temperature is 375°F, so \(T(0) = 375\). Hence:

\[
375 = Ce^{k(0)} + 75
\]

\[
= C + 75
\]

\[
C = 300
\]

Hence the equation for \(T\) as a function of time is now:

\[
T = 300e^{kt} + 75
\]

We now need to find \(k\). We do this by noting that we know that \(T(1) = 275\), so:

\[
275 = 300e^{k(1)} + 75
\]

\[
200 = 300e^k
\]

\[
e^k = \frac{2}{3}
\]

\[
k = \log\left(\frac{2}{3}\right)
\]

We now want to find how long it takes for the cake to get to 100°F:

\[
100 = 300e^{\log\left(\frac{2}{3}\right)t} + 75
\]

\[
25 = 300e^{-0.4t}
\]

\[
-0.4t = \log\left(\frac{1}{12}\right)
\]

\[
= -2.4
\]

\[
t = -2.4 / -0.4
\]

\[
= 6
\]

So the cake will be ready to eat at 6pm, with a solid half of an hour to spare.
5. **(10 points)** A chemical reaction converts substance $A$ into substance $B$ at a rate proportional to $e^a$, where $a$ is the amount of substance $A$ present. After 1 hour, only 50% of substance $A$ remains. After how many hours will only 10% of substance $A$ remain? You do NOT need to simplify your answer.

The differential equation should be:

$$\frac{da}{dt} = -ke^a$$

This is a separable equation so we just separate the variables and then solve:

$$\frac{da}{e^a} = -k \int dt$$

$$e^{-a} = -k(t + C)$$

$$e^{-a} = kt + C$$

$$a = -\log(kt + C)$$

Let’s assume that $a = 1$ when we start (note that this doesn’t actually matter! try a different initial value of $a$, you’ll get the same thing.) So $a(0) = 1$:

$$1 = -\log(C)$$

$$C = 1/e$$

We then know that $a(1) = 0.5$, since half of the initial amount is gone after 1 hour. Hence:

$$0.5 = -\log(k + 1/e)$$

$$e^{-0.5} = k + 1/e$$

$$k = 1/\sqrt{e} - 1/e$$

We want to solve for when 10% is left, so we want to solve $a(t) = 0.1$ for $t$. This gives:

$$0.1 = -\log((1/\sqrt{e} - 1/e)t + 1/e)$$

$$e^{-0.1} = (1/\sqrt{e} - 1/e)t + 1/e$$

$$t = \frac{e^{-0.1} - 1/e}{1/\sqrt{e} - 1/e}$$

$$t \approx 2.25$$

So 10% of the initial amount of substance $A$ will be left after approximately 2.25 hours. Of course, on an exam, you’d never simplify the final quantity, so I would expect your final answer to be:

$$t = \frac{e^{-0.1} - 1/e}{1/\sqrt{e} - 1/e} \text{ hours}$$
6. (10 points)

(a) Sketch the points \( a = (-2, 6, 2) \) and \( b = (4, -4, -2) \) on two different sets of axes.

(b) Find the distance between the points \( a \) and \( b \).

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}
\]
\[
= \sqrt{(-2 - 4)^2 + (6 - (-4))^2 + (2 - (-2))^2}
\]
\[
= \sqrt{36 + 100 + 16}
\]
\[
= \sqrt{152}
\]

(c) Find the midpoint between the points \( a \) and \( b \).

\[
m = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)
\]
\[
= \left( \frac{-2 + 4}{2}, \frac{6 + (-4)}{2}, \frac{2 + (-2)}{2} \right)
\]
\[
= (1, 1, 0)
\]
7. **(10 points)** Let $S$ be the surface given by \( \frac{x^2}{9} - \frac{y^2}{4} + \frac{z^2}{9} = 1 \).

(a) Find the $xy$ trace of $S$, and classify it.

Set $z = 0$ to get:

\[
\frac{x^2}{9} - \frac{y^2}{4} = 1
\]

This is the equation of an hyperbola.

(b) Find the $xz$ trace of $S$, and classify it.

Set $y = 0$ to get:

\[
\frac{x^2}{9} + \frac{z^2}{9} = 1
\]

This is an ellipse (a circle of radius 3, actually).

(c) Find the $yz$ trace of $S$, and classify it.

Set $x = 0$ to get:

\[
-\frac{y^2}{4} + \frac{z^2}{9} = 1
\]

This is an hyperbola.

(d) Classify $S$ as one of the types of quadratic surfaces we learned about in class.

This is a hyperboloid of one sheet. It should be clear from the traces that this is an hyperboloid. It might help you to think about why it has only one sheet and not two.
8. (12 points)

(a) Evaluate the function \( f(x, y) = x^2 + \sqrt{y} \) at the points \((1, 4), (7, t^4), \) and \((t, x)\).

\[
\begin{align*}
  f(1, 4) &= 1^2 + \sqrt{4} \\
           &= 3 \\
  f(7, t^4) &= 7^2 + \sqrt{t^4} \\
             &= 49 + t^2 \\
  f(t, x) &= t^2 + \sqrt{x}
\end{align*}
\]

(b) Find the domain and range of the following function \( f(x, y) = \frac{1}{x + 4y} \).

The domain is all points such that \( x + 4y \neq 0 \).
The range is all of \( \mathbb{R} \), i.e. \( (-\infty, \infty) \).

(c) Find the domain and range of the following function \( f(x, y) = \log(x - y + 2z) \).

The domain is all points such that \( x - y + 2z > 0 \).
The range is all of \( \mathbb{R} \), i.e. \( (-\infty, \infty) \).

(d) Find the domain and range of the following function \( f(x, y) = \sqrt{9 - x^2 - y^2} \).

The domain is all points such that \( 9 - x^2 - y^2 \geq 0 \).
The range is all of \( (0, -\infty) \).
9. **(8 points)** For each of the following functions, calculate $f_x$ and $f_y$.

(a) $f(x, y) = y^2 e^{-x^2 y}$

$$
\begin{align*}
  f_x &= y^2 e^{-x^2 y} (-2xy) \\
  &= -2xy^3 e^{-x^2 y} \\
  f_y &= 2ye^{-x^2 y} + y^2 e^{-x^2 y}(-x^2) \\
  &= ye^{-x^2 y} (2 - x^2 y)
\end{align*}
$$

(b) $f(x, y) = \sqrt{x^3 + y^2}$

$$
\begin{align*}
  f_x &= (1/2)(x^3 + y^2)^{-1/2} (3x^2) \\
  &= \frac{3x^2}{2 \sqrt{x^3 + y^2}} \\
  f_y &= (1/2)(x^3 + y^2)^{-1/2} (2y) \\
  &= \frac{y}{\sqrt{x^3 + y^2}}
\end{align*}
$$
10. **(10 points)** Find all relative extrema of the following functions and classify as a maxima, minima, or saddlepoint.

(a) \( f(x, y) = 12x + 12y - xy - x^2 - y^2 \)

We begin by finding the first partials:

\[
\begin{align*}
  f_x &= 12 - y - 2x \\
  f_y &= 12 - x - 2y
\end{align*}
\]

And then set them to 0:

\[
\begin{align*}
  0 &= 12 - y - 2x \\
  0 &= 12 - x - 2y
\end{align*}
\]

We solve the first equation for \( y \) to get \( y = 12 - 2x \), and plug in to the second equation:

\[
\begin{align*}
  0 &= 12 - x - 2(12 - 2x) \\
  &= 12 - x - 24 + 4x \\
  &= -12 + 3x \\
  x &= 4
\end{align*}
\]

Then \( y = 12 - 2x = 12 - 2(4) = 4 \). So \((4, 4)\) is a critical point. These are the only critical points. Make sure you understand why! To classify this critical point, we do the second derivative test. We first find the various second derivatives:

\[
\begin{align*}
  f_{xx} &= -2 \\
  f_{yy} &= -2 \\
  f_{xy} &= -1
\end{align*}
\]

So then:

\[
\begin{align*}
  d &= f_{xx}f_{yy} - (f_{xy})^2 \\
  &= (-2)(-2) - (-1)^2 \\
  &= 4 - 1 \\
  &= 3 \\
  &\geq 0
\end{align*}
\]

Since \( d \geq 0 \), and \( f_{xx} < 0 \), then \((4, 4)\) is a maximum.
(b) \( f(x,y) = x^2 - 3xy - y^2 \)

We’ll just find all the partials to start with:

\[
\begin{align*}
    f_x &= 2x - 3y \\
    f_y &= -3x - 2y \\
    f_{xx} &= 2 \\
    f_{yy} &= -2 \\
    f_{xy} &= -3
\end{align*}
\]

For a change of pace, I’m going to multiply the equations obtained by setting \( f_x = 0 \) and \( f_y = 0 \) respectively by 3 and 2 to get:

\[
\begin{align*}
    0 &= 6x - 9y \\
    0 &= -6x - 2y
\end{align*}
\]

Adding these together, we see that \(-11y = 0\), or \( y = 0 \), from which we can immediately conclude that \( x = 0 \). Please convince yourself that this is the only critical point (on the test, you will have to convince me).

Evaluating \( d \) at \((0,0)\) gives:

\[
\begin{align*}
    d &= (2)(-2) - (-3)^2 \\
    &= -4 - 9 \\
    &= -13
\end{align*}
\]

Since \( d(0,0) < 0 \), then \((0,0)\) is a saddle point.